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Another proof for the rigidity of Clifford minimal hypersurfaces of $\mathbf{S}^n$

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Abstract

Let $M \subset \mathbf{S}^n$ be a minimal hypersurface, and let us denote by A the shape operator of M. In this paper we give an alternative proof of the theorem that states that if $|A|^2 = n-1$, then M is a Clifford minimal hypersurface.

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1 Introduction and preliminaries

Let M be a minimal hypersurface of $\mathbf{S}^n \subset \mathbb{R}^{n+1}$. For every $x \in M$ we will denote by T_xM the tangent space of M at x and by $A_x : T_xM \to T_xM$ the shape operator. Notice that if $\nu : M \to \mathbb{R}^{n+1}$ is a normal unit vector field along M, i.e., for every $x \in M$, $\nu(x)$ is perpendicular to the vector x and to the vector space T_xM, then $A_x(v) = -d\nu_x(v) = -\beta'(0)$ where $\beta(t) = \nu(\alpha(t))$ and $\alpha(t)$ is any smooth curve in M such that $\alpha(0) = x$ and $\alpha'(0) = v$. It can be shown that the linear map $A_x : T_xM \to T_xM$ is symmetric, therefore it has $n-1$ real eigenvalues $\kappa_1(x), \dots, \kappa_{n-1}(x)$. These eigenvalues are known as the principal curvatures of M at x. The mean curvature $H(x)$ of M at x is the average of the principal curvatures. M is said to be minimal if $H(x) = 0$ for every $x \in M$. The norm of the shape operator is defined by the equation $\|A\|^2 = \kappa_1^2 + \cdots + \kappa_{n-1}^2$.

1.1 Examples: The equators and the Clifford hypersurfaces:

Let $v \in \mathbb{R}^{n+1}$ be a unit fixed vector. Let us define

$$\mathbf{S}^{n-1}(v) = \{x \in \mathbf{S}^n : \langle x, v\rangle = 0\}.$$

Clearly, $\mathbf{S}^{n-1}(v)$ is a hypersurface of $\mathbf{S}^n$. In this case the map $\nu\colon \mathbf{S}^{n-1}(v) \to \mathbb{R}^{n+1}$ given by $\nu(p) = v$ is a normal unit vector field along $\mathbf{S}^{n-1}(v)$. Therefore $A_p : T_pM \to T_pM$ is the zero linear map, and $\kappa_1(p) = \cdots = \kappa_{n-1}(p) = 0$ for all $p \in M$ and M is minimal. These examples are called *equators*. It is not difficult to show that the equators are the only minimal hypersurfaces with $\|A\|^2 : M \to \mathbb{R}$ identically zero.

Given any integer $k \in \{1, \dots, n-2\}$, let us define $l = (n-1) - k$ and

$$M_{kl} = \left\{ (x, y) \in \mathbb{R}^{k+1} \times \mathbb{R}^{l+1} : \|x\|^2 = \frac{k}{n-1} \quad \text{and} \quad \|y\|^2 = \frac{l}{n-1} \right\}.$$

It is not difficult to see that for any $(x,y)\in M$

$$T_{(x,y)}M_{kl}=\left\{(v,w)\in\mathbb{R}^{k+1}\times\mathbb{R}^{l+1}:\langle x,v\rangle=0\quad\text{and}\quad\langle w,y\rangle=0\right\}.$$

Therefore the map $\nu: M_{kl}\to\mathbb{R}^{n+1}$ given by

$$\nu(x,y)=\left(\sqrt{\frac{l}{k}}x,\ -\sqrt{\frac{k}{l}}y\right)$$

is a normal unit vector field along M. Notice that the vectors in $T_{(x,y)}M_{kl}$ of the form $(v,\mathbf{0})$ define a k dimensional space. A direct computation, using the expression for ν, gives us that if $(v,\mathbf{0})\in T_{(x,y)}M_{kl}$, then, $A_{(x,y)}(v,\mathbf{0})=-\sqrt{\frac{l}{k}}(v,\mathbf{0})$. Therefore $-\sqrt{\frac{l}{k}}$ is an eigenvalue of $A_{(x,y)}$ with multiplicity k. In the same way we can show that $\sqrt{\frac{k}{l}}$ is an eigenvalue of $A_{(x,y)}$ with multiplicity l. Now, we have that the mean curvature $H(x,y)=k\left(-\sqrt{\frac{l}{k}}\right)+l\left(\sqrt{\frac{k}{l}}\right)=0$. We also have that

$$\|A_{(x,y)}\|^2=k\left(-\sqrt{\frac{l}{k}}\right)^2+l\left(\sqrt{\frac{k}{l}}\right)^2=l+k=n-1.$$

Definition 1. *We will say that $M\subset\mathbf{S}^n$ is a minimal Clifford hypersurface, if, up to a rigid motion, M is equal to M_{kl} for some k and l, i.e., M is Clifford if $M=A(M_{kl})$ for some orthogonal matrix $A\in O(n+1)$.*

1.2 The fundamental equation for the shape operator

Let us denote by $C^\infty(TM)$ the vector space of differentiable tangent vector fields on M. The covariant derivative of A is the tensor $DA: C^\infty(TM)\times C^\infty(TM)\to C^\infty(TM)$ given by $DA(V,W)=D_VA(W)-A(D_VW)$ where D is the Levi Civita connection on M. The second covariant derivative of A is the tensor $D^2A: C^\infty(TM)\times C^\infty(TM)\times C^\infty(TM)\to C^\infty(TM)$ given by

$$D^2(X,Y,Z)=D_Z(DA(X,Y))-DA(D_ZX,Y)-DA(X,D_ZY)$$

The Laplacian of A at $p\in M$ is the linear map $\Delta A_p: T_pM\to T_pM$ given by $\Delta A_p(v)=\sum_{i=1}^{n-1}D^2A_p(v,e_i,e_i)$ where $\{e_1,\dots,e_{n-1}\}$ is an orthonormal basis of T_pM.

In [4] Simons proved that if $M\subset\mathbf{S}^n$ is a minimal hypersurface, then

$$\Delta A=(n-1)A-|A|^2A.$$

As a consequence of this formula we have that

$$\Delta|A|^2=2\left(\langle\Delta A,A\rangle+|DA|^2\right)=2\left(|A|^2\left((n-1)-|A|^2\right)+|DA|^2\right),$$

and therefore we obtain,

Lemma 2. *Let M be a minimal hypersurface in $\mathbf{S}^n$ that is not an equator. We have that, $|A|^2 = n-1$ if and only if $DA \equiv 0$.*

In 1970, Lawson in [2] and independently Chern, Do Carmo and Kobayashi in [1] proved the following theorem:

Theorem 3. *Let M be a minimal hypersurface of $\mathbf{S}^n$. If for every $x \in M$, $\|A\|^2(x) = n-1$, then M must be a subset of a Clifford minimal hypersurface.*

The theorem we have just mentioned, is one of the results most frequently used results when a characterization of the Clifford hypersurfaces is needed. The reason is that the condition on the norm of the shape operator is a lot easier to verify and more likely to show up in a computation than any other property that may characterize the Clifford minimal hypersurfaces.

In this communication, we will give an alternative proof of this theorem. As one of the main differences with the previous proofs, [2] and [1], our proof does not use integration of distributions i.e. it does not use Frobenius' Theorem. The idea of this new proof relies on the following lemma, whose proof is a straightforward computation. See [3] for details.

Lemma 4. *If B is a fixed invertible $(n+1)\times(n+1)$ symmetric matrix and $M = \{x \in \mathbf{S}^n : \langle Bx, x\rangle = 0\}$ is a minimal hypersurface of $\mathbf{S}^n$, then M is a Clifford minimal hypersurface.*

2 Main result

In this section we give an alternative proof of Theorem 3. We will prove the theorem by showing that if M is a minimal hypersurface in $\mathbf{S}^n$ with $\|A\|^2 = (n-1)$, then, there exists a constant invertible symmetric matrix B_0, such that $M \subset \{x \in \mathbf{S}^n : \langle x, B_0 x\rangle = 0\}$. The following lemma was proven in [2] and [1].

Lemma 5. *Let $M \subset \mathbf{S}^n$ be a minimal hypersurface which is not and equator. If the covariant derivative of A is identically zero, i.e. if $DA(V,W) = \mathbf{0}$ for all $V, W \in C^\infty(TM)$, then, at every point $p \in M$, there are exactly two principal curvatures κ_1 and κ_2. Moreover, these principal curvatures are constant functions, they do not depend on the point p, and $\kappa_1\kappa_2 = -1$.*

Proof. For any $p_0 \in M$, Since $|A|^2(p_0) \neq 0$ and M is minimal, then $\kappa_i(p_0) \neq \kappa_j(p_0)$ for some i and j. Let V, W be tangent vector fields defined in a neighborhood of p_0 such that $|V(p)| = |W(p)| = 1$, $A_p(V(p)) = \kappa_i V(p)$ and $A_p(W(p)) = \kappa_j W(p)$, and $D_V W(p_0) = D_W V(p_0) = \mathbf{0}$. Since DA is identically zero, we have that $D_Z A(U) = A(D_Z U)$. Therefore, if $K(p_0)$ is the sectional curvature of M in the plane spanned by $V(p_0), W(p_0)$, then

$$\begin{aligned}\kappa_j(p_0)K(p_0) &= \langle D_W D_V V - D_V D_W V, A(W)\rangle(p_0)\\ &= \langle D_W D_V A(V) - D_V D_W A(V), W\rangle(p_0)\\ &= -\langle D_W D_V W - D_V D_W W, A(V)\rangle(p_0)\\ &= \kappa_i(p_0)K(p_0).\end{aligned}$$

Notice that in the second step we used the symmetry of the shape operator and the fact that DA vanishes. In the third step we used the symmetries of the curvature tensor. Since $\kappa_i \neq \kappa_j$ then $K = 0$. By Gauss equation we get that $0 = 1 + \kappa_i(p_0)\kappa_j(p_0)$. The lemma follows from this last equation. □

Lemma 6. *Let $M \subset \mathbf{S}^n$ be a hypersurface, $\nu : M \to \mathbb{R}^{n+1}$ a normal unit vector field along M, and $w \in \mathbb{R}^{n+1}$ a fixed vector. Let us define $l_w : M \to \mathbb{R}$, $f_w : M \to \mathbb{R}$ and $w^T : M \to \mathbb{R}^{n+1}$ by $l_w(x) = \langle x, w\rangle$ and $f_w(x) = \langle \nu(x), w\rangle$ and $w^T(x) = w - \langle x, w\rangle x - \langle \nu(x), w\rangle \nu(x)$. If $x \in M$ and $v \in T_xM$, then*

$$\begin{aligned}&v(l_w) = \langle w, v\rangle = \langle w^T(x), v\rangle \quad v(f_w) = -\langle A(w^T(x)), v\rangle \quad \text{and}\\ &D_v w^T(x) = -l_w(x)v + f_w(x)A_x(v).\end{aligned}$$

Proof. Notice that $w^T(x)$ is the orthogonal projection of the vector w on T_xM and therefore it defines a tangent vector field on M. Let $\alpha : (-\epsilon, \epsilon) \to M$ be a curve such that $\alpha(0) = x$ and $\alpha'(0) = v$. we have that

$$v(l_w) = \left.\frac{dl_w(\alpha(t))}{dt}\right|_{t=0} = \left.\frac{d\langle \alpha(t), w\rangle}{dt}\right|_{t=0} = \langle \alpha'(0), w\rangle = \langle v, w\rangle.$$

Likewise,

$$\begin{aligned}v(f_w) &= \left.\tfrac{df_w(\alpha(t))}{dt}\right|_{t=0} = \left.\tfrac{d\langle \nu(\alpha(t)), w\rangle}{dt}\right|_{t=0} = \left\langle \left.\tfrac{d\nu(\alpha(t))}{dt}\right|_{t=0}, w\right\rangle\\ &= \langle d\nu_x(v), w\rangle = -\langle A_x(v), w\rangle = -\langle A_x(v), w^T(x)\rangle = -\langle A\left(w^T(x)\right), v\rangle.\end{aligned}$$

We also have that

$$\begin{aligned}D_v w^T(x) &= \left(\left.\frac{dw^T(\alpha(t))}{dt}\right|_{t=0}\right)^T\\ &= \left(\frac{d\left(w - l_w(\alpha(t))\,\alpha(t) - f_w(\alpha(t))\,\nu(\alpha(t))\right)}{dt}\right)^T\\ &= -l_w(x)v - f_w(x)d\nu_x(v) = -l_w(x)v + f_w(x)A_x(v).\end{aligned}$$

This last equation proves the lemma. □

We are now ready to prove Theorem 3.

Proof. (Theorem 3) By lemmas 2 and 5 we have that $DA \equiv 0$ and M has exactly two principal curvatures κ_1 and κ_2 in every point of M with $\kappa_1\kappa_2 = -1$. For any $x \in M$ let us consider the linear transformation $T : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ given by $T(v) = A(v)$ for any $v \in T_xM$, $T(x) = -\nu(x)$ and $T(\nu(x)) = -x + (\kappa_1 + \kappa_2)\nu(x)$. Notice that for every $x \in M$, we have decompose $\mathbb{R}^{n+1}$ as the direct sum of the three subspaces T_xM, $\{tx : t \in \mathbb{R}\}$ and $\{t\nu(x) : t \in \mathbb{R}\}$, then, we have established how T acts on each subspace; therefore, the transformation T is uniquely defined. Let $S(n+1)$ be the space of symmetric $(n+1)\times(n+1)$ matrices. For any x in M, let $B(x) \in S(n+1)$ be the matrix such that $B(x)w = T(w)$ for any $w \in \mathbb{R}^{n+1}$. Notice that if $e_1 = (1, 0, \ldots, 0) \ldots e_{n+1} = (0, \ldots, 0, 1)$ is the canonical basis of $\mathbb{R}^{n+1}$, then $B(x) = \{b_{ij}\}$ where

$$\begin{aligned} b_{ij} &= \left\langle T(e_i), e_j \right\rangle = \left\langle T\left(e_i^T + x_i x + \nu_i \nu\right), e_j \right\rangle \\ &= \left\langle A\left(e_i^T\right), e_j \right\rangle - x_i\nu_j + \nu_i(-x_j + (\kappa_1 + \kappa_2)\nu_j). \end{aligned}$$

Here $x_i : M \to \mathbb{R}$ and $\nu_i : M \to \mathbb{R}$ are the functions given by $x_i(x) = \langle x, e_i \rangle$ and $\nu_i(x) = \langle \nu(x), e_i \rangle$, and e_i^T is the orthogonal projection of e_i on T_xM. Notice that the functions x_i and ν_i are the functions l_{e_i} and f_{e_i} defined in the lemma 6.

We have that $B : M \to S(n+1)$ defines a smooth map on M. We will prove that this map is constant by showing that if $v \in T_xM$, then $v(b_{ij}) = 0$. We can assume without loss of generality that $A(v) = \kappa_1 v$ or $A(v) = \kappa_2 v$. Let us work the first case, $A(v) = \kappa_1 v$. We will use the Lemma 6 in the following computations.

$$\begin{aligned} v\left\langle A\left(e_i^T\right), e_j \right\rangle &= v\left\langle A\left(e_i^T\right), e_j^T \right\rangle \\ &= \left\langle A\left(D_v e_i^T\right), e_j^T \right\rangle + \left\langle A\left(e_i^T\right), D_v e_j^T \right\rangle \\ &= \langle A(-x_i v + \nu_i A(v)), e_j \rangle + \left\langle A\left(e_i^T\right), -x_j v + \nu_j A(v) \right\rangle \\ &= \langle A(-x_i v + \nu_i A(v)), e_j \rangle + \left\langle e_i, -x_j A(v) + \nu_j A^2(v) \right\rangle \\ &= -\kappa_1 x_i \langle v, e_j \rangle + \nu_i \kappa_1^2 \langle v, e_j \rangle - \kappa_1 x_j \langle v, e_i \rangle + \nu_j \kappa_1^2 \langle v, e_i \rangle. \end{aligned}$$

The second equality in the previous computation follows from the fact that $DA \equiv 0$.

$$\begin{aligned} v(x_i\nu_j) &= v(x_i)\nu_j + x_i v(\nu_j) \\ &= \langle e_i, v \rangle \nu_j - x_i \langle A(v), e_j \rangle \\ &= \langle e_i, v \rangle \nu_j - x_i \kappa_1 \langle v, e_j \rangle. \end{aligned}$$

$$\begin{aligned}(\kappa_1+\kappa_2)v(\nu_i\nu_j) &= (\kappa_1+\kappa_2)(v(\nu_i)\nu_j+\nu_i v(\nu_j))\\ &= -(\kappa_1+\kappa_2)\big(\langle e_i, A(v)\rangle\nu_j+\nu_i\langle A(v), e_j\rangle\big)\\ &= -(\kappa_1+\kappa_2)\kappa_1\big(\langle e_i, v\rangle\nu_j+\langle v, e_j\rangle\nu_i\big)\\ &= -\kappa_1^2\big(\langle e_i, v\rangle\nu_j+\langle v, e_j\rangle\nu_i\big)+\big(\langle e_i, v\rangle\nu_j+\langle v, e_j\rangle\nu_i\big).\end{aligned}$$

In the last equality we used the fact that $\kappa_1\kappa_2=-1$. Combining these equations we get that $v(b_{ij})=0$. A similar argument shows that $v(b_{ij})=0$ when $A(v)=\kappa_2 v$. Therefore $B(x)=B_0$ for all $x\in M$ and $M\subset M_0=\{x\in \mathbf{S}^n : \langle B_0x, x\rangle=0\}$. Since B_0 is an invertible matrix, we have that M_0 is a hypersurface. Since M is minimal then M_0 is also minimal. By lemma 4 M_0 is a Clifford minimal hypersurface. This completes the proof. □

Remark: The proof we gave of Theorem 3 actually shows that if M is a hypersurface of $\mathbf{S}^n$ (not necessarily minimal) and $DA\equiv 0$, then, either M is umbilical or $M\subset f^{-1}(0)$ where $f:\mathbb{R}^{n+1}\to\mathbb{R}$ is an homogeneous polynomial of degree 2. This is one of the advantages of the new proof in contrast with the old one.

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