Sturm-Liouville boundary conditions for a second order ODE

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Abstract
We study the semilinear second order ODE \( u'' + g(t, u) = 0 \) under the following Sturm-Liouville boundary condition \( au(0) + bu'(0) = u_0 \) and \( cu(T) + du'(T) = u_T \). We obtain solutions by topological methods. Moreover, we show that a solution may be constructed recursively, under appropriate conditions.

Keywords: Sturm-Liouville boundary conditions - Topological methods
MSC(2000): 34B15

1 Introduction
We study the semilinear second order problem

\[
\begin{align*}
    u'' + g(t, u) &= 0 \\
    au(0) + bu'(0) &= u_0 \\
    cu(T) + du'(T) &= u_T
\end{align*}
\]

with \( g : [0, T] \times \mathbb{R} \to \mathbb{R} \) continuous, and \( ad - bc \neq 0 \). Problems of this kind have been considered since the fifties by, among others, Ehrmann [4] and Struwe [7] using shooting arguments, and by Bahri-Berestycki [1], Rabinowitz [6], using critical point theory. In the nineties, Capietto, Henrard, Mawhin and Zanolin [2], [3] applied the Leray-Schauder continuation method for a nonlinearity of the type \( g = g_1(u) + p(t, u, u') \), where \( g_1 \) is superlinear and \( p \) satisfies a linear growth condition.

Throughout the paper, we shall assume that all the eigenvalues \( \{\lambda_n\}_{n \in \mathbb{N}} \) of the problem

\[
    -u'' = \lambda u, \quad au(0) + bu'(0) = cu(T) + du'(T) = 0
\]

are non-negative. Writing \( u = \gamma e^{rt} + \delta e^{-rt} \) as a possible eigenfunction (corresponding to an eigenvalue \( \lambda = -r^2 \)), it is easy to verify that the previous non-negativity assumption is equivalent to the following condition:

\[
    (a + br)(c - dr) \neq (a - br)(c + dr)e^{2rT} \quad \text{for } r > 0.
\]

If furthermore \( ad - bc + acT \neq 0 \), then \( \lambda_1 > 0 \), and the problem is called non-resonant. On the other hand, if \( ad - bc + acT = 0 \), then \( \lambda_1 = 0 \). This
situation corresponds to the resonant case, for which a simple computation shows that the corresponding (normalized) eigenfunction $\varphi_1$ is given by

$$\varphi_1(t) = \left( \frac{a^2T^3}{3} - abT^2 + b^2T \right)^{-1/2} (b - at). \quad (3)$$

We shall prove the existence of solutions of (1) by topological methods. More precisely, for the non-resonant case we obtain in section 2.1 an existence result under a linear growth condition on $g$ using Schauder’s fixed point theorem. On the other hand, we shall prove the existence of at least one solution when $g$ is sub quadratic and satisfies the one-sided growth condition

$$\frac{g(t, u) - g(t, v)}{u - v} \leq \gamma < \lambda_1. \quad (4)$$

We recall that the first eigenvalue can be computed by the Rayleigh quotient:

$$\lambda_1 = \inf_{u \in E} \frac{-\int_0^T u''(t)u(t)dt}{\int_0^T u^2(t)dt} \quad (5)$$

with $E = \{ u \in H^2(0, T) : au(0) + bu'(0) = cu(T) + du'(T) = 0 \}$.

In section 2.2 we shall embed problem (1) in a family $(1)_\sigma$ of problems with a parameter $\sigma \in [0, 1]$. Thus, starting at a solution $u_\sigma$ for some $\sigma < 1$ we shall define recursively a sequence which converges to a solution of $(1)_{\sigma + \varepsilon}$ for some appropriate step $\varepsilon$. In particular, when $\varepsilon$ does not depend on $u_\sigma$, we obtain recursively solutions for $0 = \sigma_0 < \sigma_1 < \ldots < \sigma_N = 1$, which gives a solution of the original problem. Finally, in section 3 we obtain solutions for the resonant case under the so-called Landesman-Lazer type conditions.

**Remark 1.1.** For simplicity, we consider only the case $g = g(t, u)$, although the methods presented in this paper can be extended to the non-variational case $g = g(t, u, u')$.

## 2 The non-resonant case

In this section we study the non-resonant case, in which condition

$$ad - bc + acT \neq 0 \quad (6)$$

holds. In section 2.1 we establish two existence results by topological methods, and in section 2.2 we define an iterative scheme that converges to a solution of (1).
2.1 Solutions by fixed point methods

We shall define a fixed point operator in order to obtain solutions of (1) by topological methods, under the assumption $ad - bc + acT \neq 0$. In this case, for any $\theta \in L^2(0, T)$ there exists a unique solution of the problem

$$u'' = \theta, \quad au(0) + bu'(0) = cu(T) + du'(T) = 0$$

given by the integral formula

$$u(t) = \int_0^T G(t, s)\theta(s)ds,$$

where $G$ is the following Green function:

$$G(t, s) = \frac{(b - at)(c(T - s) + d)}{ad - bc + acT} + \max\{t - s, 0\}.$$

Thus, the solutions of (1) can be regarded as fixed points of the operator $T$ given by

$$Tu(t) = \alpha t + \beta - \int_0^T G(t, s)g(s, u(s))ds,$$

where

$$\alpha = \frac{au_T - cu_0}{ad - bc + acT}, \quad \beta = \frac{(cT + d)u_0 - bu_T}{ad - bc + acT}.$$

Thus we obtain:

**Theorem 2.1.** Let (2) and (6) hold, and assume that $|g(t, u)| \leq k|u| + l$, with $k < \lambda_1$. Then problem (1) admits at least one solution.

**Proof.** From the assumption on $g$, it follows that $T : L^2(0, T) \to L^2(0, T)$ is well defined. Furthermore, by Arzelá-Ascoli’s Theorem we deduce that $T$ is compact. Moreover, from the Rayleigh quotient (5) we get, for fixed $\tilde{u}$:

$$\|Tu - T\tilde{u}\|_{L^2} \leq \frac{1}{\lambda_1}\|(Tu - T\tilde{u})''\|_{L^2} = \frac{1}{\lambda_1}\|g(\cdot, u) - g(\cdot, \tilde{u})\|_{L^2} \leq \frac{k}{\lambda_1}\|u\|_{L^2} + s$$

for some constant $s \geq 0$. Thus, for $R$ large enough we conclude that $T(B_R(0)) \subset B_R(0)$, and the proof follows from Schauder’s Fixed Point theorem.

**Theorem 2.2.** Let (2) and (6) hold. Further, assume that $g$ satisfies (4), and that $|g(t, u)| \leq k|u|^r + l$ for some constants $k, l$ and some $r < 2$. Then problem (1) admits a unique solution.
Proof. From the assumptions, if \( u \in L^2(0,T) \) then \( g(\cdot, u) \in L^p(0,T) \) for some \( p > 1 \), and the operator \( T : L^2(0,T) \to L^2(0,T) \) given by (7) is well defined. Moreover, if \( u = \sigma T u \) for some \( \sigma \in [0,1] \), then

\[
S_\sigma u := u'' + \sigma g(t, u) = 0, \quad au(0) + bu'(0) = \sigma u_0, \quad cu(T) + du'(T) = \sigma u_T.
\]

Let \( \tilde{u} \in H^2(0,T) \) satisfy \( a\tilde{u}(0) + b\tilde{u}'(0) = \sigma u_0, c\tilde{u}(T) + d\tilde{u}'(T) = \sigma u_T \). Then:

\[
\|S_\sigma u - S_\sigma \tilde{u}\|_{L^2} \geq \int_0^T (S_\sigma u - S_\sigma \tilde{u})(u - \tilde{u})dt \\
\geq \lambda_1 \|u - \tilde{u}\|_{L^2}^2 - \int_0^T (g(t, u) - g(t, \tilde{u}))(u - \tilde{u})dt \geq (\lambda_1 - \gamma) \|u - \tilde{u}\|_{L^2}^2.
\]

It follows that

\[
\|u - \tilde{u}\|_{L^2} \leq \frac{1}{\lambda_1 - \gamma} \|S_\sigma u - S_\sigma \tilde{u}\|_{L^2} = \frac{1}{\lambda_1 - \gamma} \|S_\sigma \tilde{u}\|_{L^2}.
\]

Thus, if we fix \( z \in H^2(0,T) \) such that \( az(0) + bz'(0) = u_0, cz(T) + dz'(T) = u_T \), then setting \( \tilde{u} = \sigma z \) we obtain:

\[
\|u - \sigma z\|_{L^2} \leq \frac{\sigma}{\lambda_1 - \gamma} \|z'' + g(\cdot, \sigma z)\|_{L^2} \leq C
\]

for some constant \( C \) independent of \( \sigma \). This implies that all solutions of the problem \( u = \sigma T u \) satisfy \( \|u\|_{L^2} \leq M \) for some constant \( M \), and the existence of a fixed point of \( T \) follows from the Leray-Schauder theorem (see e.g. [5]).

Finally, if \( u \) and \( \tilde{u} \) are solutions of (1), then \( S_1 u = S_1 \tilde{u} = 0 \). As before,

\[
\|u - \tilde{u}\|_{L^2} \leq \frac{1}{\lambda_1 - \gamma} \|S_1 u - S_1 \tilde{u}\|_{L^2} = 0.
\]

\[\square\]

### 2.2 An iterative procedure for problem (1)

In what follows of this section we shall embed problem (1) in a family of problems

\[
(1)_\sigma \begin{cases}
  u''(t) + \sigma g(t, u) = 0 \\
  au(0) + bu'(0) = u_0 \\
  cu(T) + du'(T) = u_T.
\end{cases}
\]

Starting at a solution \( u_\sigma \) for \( \sigma < 1 \) we shall define recursively a sequence that converges to a solution of \( (1)_{\sigma + \epsilon} \) for some step \( \epsilon \leq 1 - \sigma \).

As a basic assumption, we shall assume that \( g \) is \( C^2 \) with respect to \( u \), and \( \frac{\partial g}{\partial u} \leq \gamma < \lambda_1 \). In particular, note that (4) holds.
Let \( u_\sigma \) be a solution of \((1)_\sigma \) and consider the sequence \( \{ u_n \} \subset H^2(0, T) \) given recursively by \( u_1 = u_\sigma \), and \( u_{n+1} \) the unique solution of the linear problem:

\[
\begin{aligned}
&\frac{d^2}{dt^2} u_{n+1} + (\sigma + \varepsilon) \left( g(t, u_n) + \frac{\partial g}{\partial u}(t, u_n) (u_{n+1} - u_n) \right) = 0 \\
&au_{n+1}(0) + bu'_{n+1}(0) = u_0 \\
cu_{n+1}(T) + du'_{n+1}(T) = u_T.
\end{aligned}
\]

(8)

From the Fredholm alternative for linear operators (and also as a particular case of Theorem 2.2) sequence \( \{ u_n \} \) is well defined. Moreover, if \( u_n \to u \) in the \( L^2 \)-norm, then it is easy to see that \( u \) is a solution of \((1)_{\sigma + \varepsilon} \).

Let \( z_n = u_{n+1} - u_n \), then for \( n \geq 2 \) we have:

\[
z_n'' + (\sigma + \varepsilon) \frac{\partial g}{\partial u}(t, u_n) z_n = -(\sigma + \varepsilon) [g(t, u_n) - g(t, u_{n-1}) - \frac{\partial g}{\partial u}(t, u_{n-1})(u_n - u_{n-1})]
\]

\[
= -\frac{1}{2} (\sigma + \varepsilon) \frac{\partial^2 g}{\partial u^2}(t, \xi) z_{n-1}^2
\]

for some mean value \( \xi(t) \) between \( u_n(t) \) and \( u_{n-1}(t) \). Then, for some constant \( \mu \) (independent of \( \sigma \)):

\[
\| z_n \|_{H^1} \leq \mu \left\| \frac{d^2}{dt^2} + (\sigma + \varepsilon) \frac{\partial g}{\partial u}(\cdot, u_n) z_n \right\|_{L^2} \leq \frac{\mu}{2} \left\| \frac{\partial^2 g}{\partial u^2}(\cdot, \xi) z_{n-1}^2 \right\|_{L^2} \leq C_n \| z_{n-1} \|^2_{H^1}
\]

for some constant \( C_n \). In particular, if \( \frac{\partial^2 g}{\partial u^2} \) is bounded, we may consider \( C_n = C := \frac{\mu \nu}{2} \| \frac{\partial^2 g}{\partial u^2} \|_{L^\infty} \) for every \( n \), where \( \nu \) is the constant of the imbedding \( H^1(0, T) \hookrightarrow L^4(0, T) \). On the other hand,

\[
z_1'' + (\sigma + \varepsilon) \frac{\partial g}{\partial u}(t, u_1) z_1 = -u_1'' - (\sigma + \varepsilon) g(t, u_1) = -\varepsilon g(t, u_1),
\]

whence \( \| z_1 \|_{H^1} \leq \mu \varepsilon \| g(\cdot, u_1) \|_{L^2} \). Thus we obtain:

**Theorem 2.3.** Assume that (2) and (6) hold, and let \( u_1 = u_\sigma \) be a solution of \((1)_\sigma \) for some \( \sigma \in [0, 1) \). Furthermore, assume that \( \frac{\partial g}{\partial u} \leq \gamma < \lambda_1 \) for some constant \( \gamma \), and that \( \frac{\partial^2 g}{\partial u^2} \) is bounded. Then the iterative scheme defined by (8) converges to a solution of \((1)_{\sigma + \varepsilon} \), provided that \( \mu \varepsilon C \| g(\cdot, u_\sigma) \|_{L^2} < 1 \), with \( C \) and \( \mu \) as before.

**Proof.** From the previous computations, we deduce that

\[
\| z_{n+1} \|_{H^1} \leq C^{2n-1} \| z_1 \|^2_{H^1} \leq \frac{1}{C} \left( \mu \varepsilon C \| g(\cdot, u_\sigma) \|_{L^2} \right)^{2n}.
\]

Then \( \{ u_n \} \) is a Cauchy sequence in \( H^1(0, T) \), and the proof follows. \( \square \)
Corollary 2.4. Let the assumptions of the previous theorem hold. Further, assume that \( g \) is bounded. Then the step \( \varepsilon \) in the iterative scheme defined by (8) can be chosen independently of \( \sigma \). In particular, there exists a sequence \( 0 = \sigma_0 < \sigma_1 < \ldots < \sigma_N = 1 \), with \( u_{\sigma_j} \) solution of (1)\(_{\sigma_j}\) constructed recursively from (8), and \( u_{\sigma_N} \) is a solution of (1).

3 Resonant case: Landesman-Lazer type conditions

In this section we study problem (1) for \( u_0 = u_T = 0 \) under the assumption of resonance at the first eigenvalue \( \lambda_1 = 0 \); namely, we consider the case in which the condition

\[
ad - bc + acT = 0
\]

(9)

holds. The proof of following lemma is straightforward:

Lemma 3.1. Assume that (2) and (9) hold. Let \( E \subset C^2([0,T]) \) and \( F \subset C([0,T]) \) the closed subspaces defined by

\[
E = \{ u \in C^2([0,T]) : au(0) + bu'(0) = cu(T) + du'(T) = 0, \quad \int_0^T u(t)\varphi_1(t)dt = 0 \}
\]

and \( F = \{ \theta \in C([0,T]) : \int_0^T \theta(t)\varphi_1(t)dt = 0 \} \). Then the continuous linear operator \( L : E \rightarrow F \) given by \( Lu = u'' \) is bijective, and hence an isomorphism. In particular, there exists a constant \( \gamma \) such that \( \|u\|_{C^2} \leq \gamma \|u''\|_C \) for every \( u \in E \).

In order to introduce appropriate Landesman-Lazer conditions for our problem, we shall assume that the following limits exist:

\[
\lim_{s \to \pm \infty} g(t,s\varphi_1(t)) := g^\pm(t). \quad (10)
\]

Thus, the main result of this section reads:

Theorem 3.2. Assume that (2) and (9) hold, and that the limits (10) exist. Then problem (1) for \( u_0 = u_T = 0 \) admits at least one solution, provided that one of the following conditions holds:

\[
\int_0^T g^+(t)\varphi_1(t)dt < 0 < \int_0^T g^-(t)\varphi_1(t)dt, \quad (11)
\]

\[
\int_0^T g^-(t)\varphi_1(t)dt < 0 < \int_0^T g^+(t)\varphi_1(t)dt. \quad (12)
\]
Proof. Let us first observe that, for $\sigma > 0$, problem
\[
\begin{aligned}
\begin{cases}
u'' + \sigma g(t, u) &= 0 \\
u(0) + bu'(0) &= cu(T) + du'(T) = 0
\end{cases}
\end{aligned}
\] (13)
is equivalent to the fixed point problem
\[
u = \langle u - g(\cdot, u), \varphi_1 \rangle \varphi_1 - \sigma K(g(\cdot, u) - \langle g(\cdot, u), \varphi_1 \rangle \varphi_1),
\] (14)
where $K : F \to E$ is the inverse of the mapping $L$ defined in Lemma 3.1, and $\langle \cdot, \cdot \rangle$ denotes the usual inner product of $L^2([0, T])$, namely $\langle \theta, \xi \rangle = \int_0^T \theta(t)\xi(t)dt$. Indeed, if $u$ is a solution of (13) then $\langle u'', \varphi_1 \rangle = \langle u, \varphi_1'' \rangle = 0$, which implies $\langle g(\cdot, u), \varphi_1 \rangle = 0$, and
\[
u - \langle u, \varphi_1 \rangle \varphi_1 = -\sigma K(g(\cdot, u)).
\]
Conversely, if $u$ solves (14) then $u'' = -\sigma [g(t, u) - \langle g(\cdot, u), \varphi_1 \rangle \varphi_1]$. Moreover, $\langle u, \varphi_1 \rangle = \langle u - g(\cdot, u), \varphi_1 \rangle$, and hence $\langle g(\cdot, u), \varphi_1 \rangle = 0$. Thus, it suffices to prove that (14) is solvable for $\sigma = 1$. On the other hand, observe that if $\sigma = 0$ then (14) is equivalent to the equalities
\[
u = k\varphi_1, \quad \langle g(\cdot, u), \varphi_1 \rangle = 0.
\]
Let $T_\sigma : C([0, T]) \to C([0, T])$ be the compact operator defined by
\[
u = \langle u - g(\cdot, u), \varphi_1 \rangle \varphi_1 - \sigma K(g(\cdot, u) - \langle g(\cdot, u), \varphi_1 \rangle \varphi_1),
\]
and consider $F_\sigma(u) = \nu - T_\sigma \nu$. We claim that $F_1(u) = 0$ for some $u$, which corresponds to a solution of the original problem. Indeed, we shall prove that
\begin{enumerate}
\item $F_\sigma(u) \neq 0$ for $\|u\|_C$ large, and $\sigma \in [0, 1]$.
\item $deg_{LS}(F_0, B_R, 0) = \pm 1$ for $R$ large enough, where $B_R \subset C([0, T])$ is the ball of radius $R$ centered at 0 and $deg_{LS}$ denotes the Leray-Schauder degree.
\end{enumerate}

We remark that once 1 and 2 are proved, the result follows from the homotopy invariance of the Leray-Schauder degree. In order to prove 1, assume first that $F_{\sigma_n}u_n = 0$, with $\|u_n\|_C \to \infty$ and $\sigma_n \in (0, 1]$. Then $u''_n + \sigma_n g(t, u_n) = 0$, and hence
\[
u = \langle u'', \varphi_1 \rangle = -\sigma_n \int_0^T g(t, u_n)\varphi_1(t)dt.
\]
On the other hand, we may write $u_n = v_n + \langle u_n, \varphi_1 \rangle \varphi_1$, and from the previous lemma
\[
u \leq \gamma \|v_n\|_C = \gamma \|u''_n\|_C \leq \gamma \|g(\cdot, u_n)\|_C \leq M.
\]
for some constant $M$. We deduce that $c_n := \langle u_n, \varphi_1 \rangle \to \infty$. Taking a subsequence, assume for example that $c_n \to +\infty$, then by dominated convergence

$$0 = \int_0^T g(t, u_n)\varphi_1(t)dt = \int_0^T g(t, v_n + c_n \varphi_1)\varphi_1(t)dt \to \int_0^T g^+(t)\varphi(t)dt \neq 0,$$

a contradiction. On the other hand, if $F_0u_n = 0$, with $\|u_n\|_C \to \infty$, then $u_n = c_n \varphi_1$ and $\int_0^T g(t, c_n \varphi_1(t))\varphi_1(t)dt = 0$. Applying dominated convergence as before, the claim follows.

Finally, we shall compute the Leray-Schauder degree $\deg_{LS}(F_0, B_R, 0)$ for $R$ large. As the range of $T_0$ is contained in $S := \text{span}\{\varphi_1\}$, it suffices to compute the Brouwer degree $\deg_B(F_0|_S, B_R \cap S, 0)$. Furthermore, $F_0|_S$ can be identified with the mapping $\phi : \mathbb{R} \to \mathbb{R}$ given by $\phi(r) = \int_0^T g(t, r\varphi_1(t))\varphi_1(t)dt$. Again, by dominated convergence we have that

$$\lim_{r \to \pm \infty} \phi(r) = \int_0^T g^\pm(t)\varphi_1(t)dt.$$

Hence, $\phi(r)\phi(-r) < 0$ for $r \gg 0$, and it follows that $\deg_B(F_0|_S, B_R \cap S, 0) = \pm 1$ for $R$ large enough.

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