Factoring certain decic polynomials

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Abstract
This paper presents a simple method for decomposing and synthesizing certain decic polynomials.

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1 Introduction
A reducible polynomial in a given field is the one, which can be decomposed into polynomials of lower degree with coefficients in that field [1]. If these polynomials (of lower degree) are of fourth-degree or less, the reducible polynomial automatically becomes solvable (in radicals of course). Thus all solvable polynomials are reducible (in complex field, C, in general), however all reducible polynomials are not solvable.

In this paper, we describe a simple method to decompose a reducible decic polynomial in C into a product of two quintic polynomial factors. However no attempt is made to further decompose the quintics as this is dealt elsewhere [2]. A procedure to synthesize such decic polynomials is given in the end.

2 The proposed method
We know that an N-th degree polynomial, \( x^N + \sum_{i=0}^{N-1} a_i x^i \), can be converted to:
\( u^N + \sum_{i=0}^{N-2} b_i u^i \), using the substitution, \( x = u - (a_{N-1}/N) \). Therefore without loss of generality we consider the following decic polynomial (to be decomposed):

\[
p(x) = x^{10} + \sum_{i=0}^{8} a_i x^i
\]

where \( a_i \) are real coefficients. Consider another decic polynomial, \( q(x) \), which is in the form of difference of two squares as shown below:

\[
q(x) = (x^5 + b_3 x^3 + b_2 x^2 + b_1 x + b_0)^2 - (c_2 x^2 + c_1 x + c_0)^2
\]

where \( b_0, b_1, b_2, b_3 \) are coefficients of quintic polynomial, and \( c_0, c_1, c_2 \) are coefficients of quadratic polynomial in the above expression. Note that these coefficients are unknowns to be determined. Observe that the polynomial \( q(x) \) can be
decomposed into product of two quintic polynomial factors as shown below.

\[
q(x) = [x^5 + b_3x^3 + (b_2 - c_2)x^2 + (b_1 - c_1)x + (b_0 - c_0)] [x^5 + b_3x^3 + (b_2 + c_2)x^2 + (b_1 + c_1)x + (b_0 + c_0)]
\] (3)

Our objective is to represent the given decic polynomial \( p(x) \) in the form of (2), so that \( p(x) \) can be decomposed and expressed as shown in (3). This can be achieved if the coefficients of \( p(x) \) are made equal to that of \( q(x) \). However notice that the coefficients of \( q(x) \) are not explicitly written. Hence the expression (2) is expanded and rearranged in descending powers of \( x \) as shown below.

\[
q(x) = x^{10} + 2b_1x^8 + 2b_2x^7 + (2b_1 + b_3^2)x^6 + 2(b_0 + b_2b_3)x^5 + (2b_1b_3 + b_2^2 - c_2^2)x^4 + 2(b_0b_3 + b_1b_2 - c_1c_2)x^3 + (b_1^2 - c_1^2 + 2b_0b_2 - 2c_0c_2)x^2 + 2(b_0b_1 - c_0c_1)x + b_0^2 - c_0^2
\] (4)

Now equating the coefficients of \( p(x) \) and \( q(x) \) [shown in (4)], we obtain nine equations in seven unknowns \( (b_0, b_1, b_2, b_3, c_0, c_1, c_2) \), as given below.

\[
2b_3 = a_8
\] (5)

\[
2b_2 = a_7
\] (6)

\[
2b_1 + b_3^2 = a_6
\] (7)

\[
2(b_0 + b_2b_3) = a_5
\] (8)

\[
2b_1b_3 + b_2^2 - c_2^2 = a_4
\] (9)

\[
2(b_0b_3 + b_1b_2 - c_1c_2) = a_3
\] (10)

\[
b_1^2 - c_1^2 + 2b_0b_2 - 2c_0c_2 = a_2
\] (11)

\[
2(b_0b_1 - c_0c_1) = a_1
\] (12)

\[
b_0^2 - c_0^2 = a_9
\] (13)

Notice that from (5) \( b_3 \) is evaluated as: \( b_3 = a_8/2 \); from (6) we obtain \( b_2 = a_7/2 \); using the value of \( b_3 \) in (7), \( b_1 \) is determined as: \( b_1 = [a_6 - (a_8^2/4)]/2 \); and using the values of \( b_2 \) and \( b_3 \) in (8), we obtain \( b_0 \) as: \( b_0 = [a_5 - (a_7a_8/2)]/2 \). Similarly using the values of \( b_1, b_2, \) and \( b_3 \) in (9), we obtain two values of \( c_2 \) as: \( c_2 = \pm a_9 \), where \( a_9 \) is given by:

\[
a_9 = \sqrt{(a_6a_8/2) + (a_7^2/4) - (a_8^3/8) - a_4}
\] (14)

Choosing \( c_2 = a_9, c_1 \) and \( c_0 \) are determined from (10) and (11) as: \( c_1 = a_{10} \) and \( c_0 = a_{11} \); where \( a_{10} \) and \( a_{11} \) are given by:

\[
a_{10} = \frac{a_8[a_5 - (a_7a_8/2)] + a_7[a_6 - (a_8^2/4)] - 2a_3}{4a_9}
\] (15)
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\[ a_{11} = \frac{\left[ a_6 - \left( \frac{a_8^2}{4} \right) \right]^2 + 2a_7 \left[ a_5 - \left( \frac{a_7a_8}{2} \right) \right] - 4a_{10}^2 - 4a_2}{8a_9} \] (16)

Since all the unknowns in (2) are determined, the given decic polynomial \( p(x) \) can be represented by \( q(x) \) and it can be decomposed into product of two quintic polynomials as shown in (3). Does this mean any decic polynomial [which is expressed in ‘reduced form’ as given in (1)] can be decomposed? No; the coefficients of the given decic have to satisfy certain conditions in order that it becomes decomposable. In the next section we shall derive those conditions.

3 Conditions on coefficients

Notice that we have not yet used the equations (12) and (13). Using the values (determined earlier) for \( b_0, b_1, c_0 \) and \( c_1 \) in expression (12), we obtain an expression for \( a_1 \) in terms of coefficients, \( a_2 \) to \( a_8 \), as shown below.

\[ a_1 = \frac{1}{2} \left[ a_6 - \left( \frac{a_8^2}{4} \right) \right] \left[ a_5 - \left( \frac{a_7a_8}{2} \right) \right] - 2a_{10}a_{11} \] (17)

Similarly, use of values for \( b_0 \) and \( c_0 \) in (13) results in an expression for \( a_0 \) as shown below.

\[ a_0 = \frac{1}{4} \left[ a_5 - \left( \frac{a_7a_8}{2} \right) \right]^2 - a_{11}^2 \] (18)

Thus the expressions, (17) and (18), form the conditions for the coefficients to satisfy so that the given decic polynomial [\((1)\)] is decomposable. In fact these expressions [(17) and (18)] can be used for synthesis of decomposable decic polynomials described here. For this purpose the real coefficients, \( a_2 \) to \( a_8 \), are chosen first (arbitrarily), and then \( a_1 \) and \( a_0 \) are determined using (17) and (18). Notice that the coefficients \( a_0 \) and \( a_1 \) are also real, no matter whether \( a_9 \) [of expression (14)] is real or imaginary. In the numerical example given below, we illustrate the synthesis procedure for the decic and then decompose it.

4 Numerical example

Consider the polynomial:

\[ p(x) = x^{10} + 2x^8 + 2x^7 + 3x^6 + 4x^5 - x^4 - 4x^3 + 7x^2 + a_1x + a_0 \]

where \( a_0 \) and \( a_1 \) are required to be determined for synthesizing the desired reducible decic. Using (14), (15), and (16), we determine \( a_9 \) (= \( c_2 \)), \( a_{10} \) (= \( c_1 \)), and \( a_{11} \) (= \( c_0 \)) as: 2, 2, and - 2. Using these values in (17) and (18), the coefficients \( a_1 \) and \( a_0 \) are obtained as: 10 and - 3. Having synthesized the decic, now we proceed to decompose it. We determine \( b_3, b_2, b_1, \) and \( b_0 \) as: 1, 1, 1, and 1. Using these values along with the values of \( c_0, c_1, \) and \( c_2, \) the decomposed decic as given in (3) is expressed as:

\[ p(x) = (x^5 + x^3 - x^2 - x + 3)(x^5 + x^3 + 3x^2 + 3x - 1) \]
5 Conclusions
We have described a simple method to decompose decic polynomials, whose coefficients satisfy certain conditions. These conditions are derived.

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References


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