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Factoring certain decic polynomials

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Abstract
This paper presents a simple method for decomposing and synthesizing certain decic polynomials.

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1 Introduction

A reducible polynomial in a given field is the one, which can be decomposed into polynomials of lower degree with coefficients in that field [1]. If these polynomials (of lower degree) are of fourth-degree or less, the reducible polynomial automatically becomes solvable (in radicals of course). Thus all solvable polynomials are reducible (in complex field, C, in general), however all reducible polynomials are not solvable.

In this paper, we describe a simple method to decompose a reducible decic polynomial in C into a product of two quintic polynomial factors. However no attempt is made to further decompose the quintics as this is dealt elsewhere [2]. A procedure to synthesize such decic polynomials is given in the end.

2 The proposed method

We know that an N-th degree polynomial, $x^N + \sum_{i=0}^{N-1} a_i x^i$, can be converted to: $u^N + \sum_{i=0}^{N-2} b_i u^i$, using the substitution, $x = u - (a_{N-1}/N)$. Therefore without loss of generality we consider the following decic polynomial (to be decomposed):

$$p(x) = x^{10} + \sum_{i=0}^{8} a_i x^i \tag{1}$$

where a_i are real coefficients. Consider another decic polynomial, $q(x)$, which is in the form of difference of two squares as shown below:

$$q(x) = (x^5 + b_3x^3 + b_2x^2 + b_1x + b_0)^2 - (c_2x^2 + c_1x + c_0)^2 \tag{2}$$

where b_0, b_1, b_2, b_3 are coefficients of quintic polynomial, and c_0, c_1, c_2 are coefficients of quadratic polynomial in the above expression. Note that these coefficients are unknowns to be determined. Observe that the polynomial $q(x)$ can be decomposed into product of two quintic polynomial factors as shown below.

$$q(x) = \left[x^5 + b_3x^3 + (b_2 - c_2)x^2 + (b_1 - c_1)x + (b_0 - c_0)\right] \left[x^5 + b_3x^3 + (b_2 + c_2)x^2 + (b_1 + c_1)x + (b_0 + c_0)\right] \qquad (3)$$

Our objective is to represent the given decic polynomial $p(x)$ in the form of (2), so that $p(x)$ can be decomposed and expressed as shown in (3). This can be achieved if the coefficients of $p(x)$ are made equal to that of $q(x)$. However notice that the coefficients of $q(x)$ are not explicitly written. Hence the expression (2) is expanded and rearranged in descending powers of x as shown below.

$$q(x) = x^{10} + 2b_3x^8 + 2b_2x^7 + (2b_1 + b_3^2)x^6 + 2(b_0 + b_2b_3)x^5 + (2b_1b_3 + b_2^2 - c_2^2)x^4 + 2(b_0b_3 + b_1b_2 - c_1c_2)x^3 + (b_1^2 - c_1^2 + 2b_0b_2 - 2c_0c_2)x^2 + 2(b_0b_1 - c_0c_1)x + b_0^2 - c_0^2 \qquad (4)$$

Now equating the coefficients of $p(x)$ and $q(x)$ [shown in (4)], we obtain nine equations in seven unknowns $(b_0, b_1, b_2, b_3, c_0, c_1, c_2)$, as given below.

$$2b_3 = a_8 \qquad (5)$$

$$2b_2 = a_7 \qquad (6)$$

$$2b_1 + b_3^2 = a_6 \qquad (7)$$

$$2(b_0 + b_2b_3) = a_5 \qquad (8)$$

$$2b_1b_3 + b_2^2 - c_2^2 = a_4 \qquad (9)$$

$$2(b_0b_3 + b_1b_2 - c_1c_2) = a_3 \qquad (10)$$

$$b_1^2 - c_1^2 + 2b_0b_2 - 2c_0c_2 = a_2 \qquad (11)$$

$$2(b_0b_1 - c_0c_1) = a_1 \qquad (12)$$

$$b_0^2 - c_0^2 = a_0 \qquad (13)$$

Notice that from (5) b_3 is evaluated as:$b_3 = a_8/2$; from (6) we obtain $b_2 = a_7/2$; using the value of b_3 in (7), b_1 is determined as: $b_1 = [a_6 - (a_8^2/4)]/2$; and using the values of b_2 and b_3 in (8), we obtain b_0 as: $b_0 = [a_5 - (a_7a_8/2)]/2$. Similarly using the values of b_1, b_2, and b_3 in (9), we obtain two values of c_2 as: $c_2 = \pm a_9$, where a_9 is given by:

$$a_9 = \sqrt{(a_6a_8/2) + (a_7^2/4) - (a_8^3/8) - a_4} \qquad (14)$$

Choosing $c_2 = a_9$, c_1 and c_0 are determined from (10) and (11) as: $c_1 = a_{10}$ and $c_0 = a_{11}$; where a_{10} and a_{11} are given by:

$$a_{10} = \frac{a_8\left[a_5 - (a_7a_8/2)\right] + a_7\left[a_6 - (a_8^2/4)\right] - 2a_3}{4a_9} \qquad (15)$$

$$a_{11} = \frac{\left[a_6 - (a_8^2/4)\right]^2 + 2a_7\left[a_5 - (a_7a_8/2)\right] - 4a_{10}^2 - 4a_2}{8a_9} \tag{16}$$

Since all the unknowns in (2) are determined, the given decic polynomial $p(x)$ can be represented by $q(x)$ and it can be decomposed into product of two quintic polynomials as shown in (3). Does this mean any decic polynomial [which is expressed in 'reduced form' as given in (1)] can be decomposed? No; the coefficients of the given decic have to satisfy certain conditions in order that it becomes decomposable. In the next section we shall derive those conditions.

3 Conditions on coefficients

Notice that we have not yet used the equations (12) and (13). Using the values (determined earlier) for b_0, b_1, c_0 and c_1 in expression (12), we obtain an expression for a_1 in terms of coefficients, a_2 to a_8, as shown below.

$$a_1 = \frac{1}{2}\left[a_6 - (a_8^2/4)\right]\left[a_5 - (a_7a_8/2)\right] - 2a_{10}a_{11} \tag{17}$$

Similarly, use of values for b_0 and c_0 in (13) results in an expression for a_0 as shown below.

$$a_0 = \frac{1}{4}\left[a_5 - (a_7a_8/2)\right]^2 - a_{11}^2 \tag{18}$$

Thus the expressions, (17) and (18), form the conditions for the coefficients to satisfy so that the given decic polynomial [(1)] is decomposable. In fact these expressions [(17) and (18)] can be used for synthesis of decomposable decic polynomials described here. For this purpose the real coefficients, a_2 to a_8, are chosen first (arbitrarily), and then a_1 and a_0 are determined using (17) and (18). Notice that the coefficients a_0 and a_1 are also real, no matter whether a_9 [of expression (14)] is real or imaginary. In the numerical example given below, we illustrate the synthesis procedure for the decic and then decompose it.

4 Numerical example

Consider the polynomial:

$$p(x) = x^{10} + 2x^8 + 2x^7 + 3x^6 + 4x^5 - x^4 - 4x^3 + 7x^2 + a_1x + a_0$$

where a_0 and a_1 are required to be determined for synthesizing the desired reducible decic. Using (14), (15), and (16), we determine a_9 ($= c_2$), a_{10} ($= c_1$), and a_{11} ($= c_0$) as: 2, 2, and - 2. Using these values in (17) and (18), the coefficients a_1 and a_0 are obtained as: 10 and - 3. Having synthesized the decic, now we proceed to decompose it. We determine b_3, b_2, b_1, and b_0 as: 1, 1, 1, and 1. Using these values along with the values of c_0, c_1, and c_2, the decomposed decic as given in (3) is expressed as:

$$p(x) = (x^5 + x^3 - x^2 - x + 3)(x^5 + x^3 + 3x^2 + 3x - 1)$$

5 Conclusions

We have described a simple method to decompose decic polynomials, whose coefficients satisfy certain conditions. These conditions are derived.

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References

[1] Thangadurai, R.: Irreducibility of polynomials whose coefficients are integers, Mathematics Newsletter (RMS, India), 17(2007), No. 2, pp. 29–37.

[2] Kulkarni, R. G.: Decomposing quintics, Mathematics Teacher, 102(2009), No. 9, pp. 710–713.

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