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On the induced MO-mappings between arcs and simple closed curves

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Abstract

Let $f : X \to Y$ be a mapping between continua. We say that f is an MO-mapping if there are a monotone mapping $m : Z \to Y$ and an open mapping $o : X \to Z$ such that $f = m \circ o$. Given an MO-mapping $f : [0, 1] \to [0, 1]$, the induced mapping $C(f)$ is studied. Also, we prove that if $f : S^1 \to S^1$ is an MO-mapping, then $C(f)$ is an MO-mapping.

Keywords: continua, hyperspaces of continua, induced mappings, monotone mappings, MO-mappings, open mappings.

MSC(2000): 54B20, 54E40, 54F15

Resumen

Sea $f : X \to Y$ una función continua entre continuos. Diremos que f es MO si existen una función monótona $m : Z \to Y$ y una función abierta $o : X \to Z$ tales que $f = m \circ o$. Dada una función MO $f : [0, 1] \to [0, 1]$, estudiaremos la función inducida $C(f)$. También, probaremos que si $f : S^1 \to S^1$ es una función MO, entonces $C(f)$ es siempre una función MO.

Palabras y frases claves: continuos, hiperespacios de continuos, funciones inducidas, funciones monótonas, funciones MO, funciones abiertas.

1 Introduction

A continuum is a nonempty, compact, connected and metric space. For a continuum X we denote by $C(X)$ the hyperspace of all subcontinua of X. Given a mapping $f : X \to Y$ between continua X and Y, we define the induced mapping $C(f) : C(X) \to C(Y)$, by $C(f)(A) = f(A)$ [9, (0.49), p.18].

In [4], the following is asked:

Question 1.1. *Is it true that if $f : X \to Y$ is an MO-mapping, then so is $C(f) : C(X) \to C(Y)$?*

J. J. Charatonik and W. J. Charatonik showed an open mapping $f : [0, 1] \to [0, 1]$ such that the induced mapping $C(f)$ is not an MO-mapping [2, Proposition 9, p.249]. This answers the Question 1.1 in the negative. Also, in [2], the authors give a condition for the openness of $f : [0, 1] \to [0, 1]$, to imply that $C(f)$ is an MO-mapping and conversely. In this paper, we give a collection of MO-mappings (no necessarily open mappings) such that the induced mappings are MO-mappings.

We also show that if $f : S^1 \to S^1$ is an MO-mapping, then the induced mapping $C(f) : C(S^1) \to C(S^1)$ is also an MO-mapping. Therefore, we give a positive answer to Question if f is defined between simple closed curves.

2 Preliminaries

If (X, d) is a metric space, then given $a \in X$ and $\epsilon > 0$, the open ball about a of radius ϵ is denoted by $B_d(a, \epsilon)$. A *continuum* is a nonempty, compact, connected and metric space. We say that a continuum X is a *simple closed curve* if X is homeomorphic to $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. A *mapping* is assumed to be a continuous function. If $f : X \to Y$ is a mapping between continua and A is a subset of X, then $f|_A$ denotes the restriction of f to A.

Remark 2.1. *In this paper every mapping will be assumed surjective and defined between nondegenerate continua.*

Definition 2.2. *Let $f : X \to Y$ be a mapping between continua. We say that f is open if f maps every open set in X onto an open set in Y; f is called monotone provided that $f^{-1}(y)$ is connected for each y in Y. We say that f is an MO-mapping if there are a monotone mapping $m : Z \to Y$ and an open mapping $o : X \to Z$ such that $f = m \circ o$.*

The reader may find information about MO-mappings in [8].

Definition 2.3. *Two mappings f and g are topologically equivalent provided that there exist homeomorphisms h_1 and h_2 such that $f = h_1 \circ g \circ h_2$.*

The following obvious remark will be used later.

Remark 2.4. *If f is an MO-mapping and f is topologically equivalent to g, then g is also an MO-mapping.*

Given a continuum X, we consider *the hyperspace of subcontinua of* X, denoted by $C(X)$, defined by:

$$C(X) = \{A \subset X : A \text{ is a subcontinuum of } X\}.$$

$C(X)$ is topologized with the Hausdorff metric H, defined by:

$$H(A, B) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\},$$

for each $A, B \in C(X)$. It is known that the collection of sets $\langle U_1, U_2, ..., U_l \rangle$ form a base on $C(X)$ (*Vietoris topology*, see [6, p.3]), where $U_1, U_2, ..., U_l$ are open sets in X and:

$$\langle U_1, U_2, ..., U_l \rangle = \{A \in C(X) : A \subset \cup_{i=1}^{l} U_i \text{ and } A \cap U_i \neq \emptyset \text{ for each } i\}.$$

Let $f : X \to Y$ be a mapping between continua. The mapping $C(f) : C(X) \to C(Y)$ given by $C(f)(A) = f(A)$ for each $A \in C(X)$, is called the *induced mapping between the hyperspaces* $C(X)$ *and* $C(Y)$ [6, p.106].

3 MO-mappings between arcs

We begin the section with a simple result. The following proposition shows a condition to obtain an induced MO-mapping.

Proposition 3.1. *Let f be an MO-mapping between continua such that $f = m \circ o$ where m is monotone and o is an open mapping. If $C(o)$ is an MO-mapping, then $C(f)$ is an MO-mapping.*

Proof. Let m_1 and o_1 be monotone and open mappings, respectively, such that $C(o) = m_1 \circ o_1$. It is not difficult to show that $C(m \circ o) = C(m) \circ C(o)$. Thus, $C(f) = C(m) \circ C(o) = C(m) \circ m_1 \circ o_1$. Since m is monotone, $C(m)$ is monotone [5, Theorem 3.2, p.241]. Furthermore, the composition of monotone mappings is monotone [8, 5.1, p.29]. Therefore, $C(f)$ is an MO-mapping. □

The following corollary is a consequence of Proposition 3.1 and [2, Proposition 8, p.248].

Corollary 3.2. *Let T be the tent mapping; i.e., $T : [0,1] \to [0,1]$ is an open mapping defined by:*

$$T(t) = \begin{cases} 2t, & \text{if } t \in [0,1/2]; \\ 2-2t, & \text{if } t \in [1/2,1]. \end{cases}$$

If m is any monotone mapping, then $C(m \circ T)$ is an MO-mapping.

The idea of the following theorem was taken from [2, Theorem 10, p.250].

Theorem 3.3. *Let $f : [0,1] \to [0,1]$ be an MO-mapping such that $f = m \circ o$, where $m : [0,1] \to [0,1]$ is monotone, such that either $|m^{-1}(m(0))| = 1$ or $|m^{-1}(m(1))| = 1$, and $o : [0,1] \to [0,1]$ is open. Then the induced mapping $C(f)$ is an MO-mapping if and only if o is topologically equivalent to either the identity or the tent mapping.*

Proof. Let $o : [0,1] \to [0,1]$ be an open mapping and let $m : [0,1] \to [0,1]$ be a monotone mapping, such that $f = m \circ o$. Since o is an open mapping, there are $0 = a_0 < a_1 < a_2 < \cdots < a_n = 1$ such that $o|_{[a_i,a_{i+1}]} : [a_i, a_{i+1}] \to [0,1]$ is a homeomorphism, for each $i \in \{0,1,...,n-1\}$, by [11, 1.3, p.184]. Assume that $o(0) = 0$ and $m^{-1}(m(0)) = \{0\}$.

Suppose that $C(f)$ is an MO-mapping. We prove that $n \in \{1,2\}$ and so, o is topologically equivalent to either the identity or the tent mapping.

Suppose the contrary and take $0 = a_0 < a_1 < a_2 < a_3 \leq 1$ such that $o|_{[0,a_1]}, o|_{[a_1,a_2]}$ and $o|_{[a_2,a_3]}$ are homeomorphisms, where $o(0) = o(a_2) = 0$ and $o(a_1) = o(a_3) = 1$.

Claim 3.4. *The set $\mathcal{D} = \{y \in [0,1] : |m^{-1}(y)| = 1\}$ is dense in $[0,1]$.*

Let $[a,b]$ be a subset of $[0,1]$ such that $a<b$. Since m is monotone, $m^{-1}([a,b])$ is a subcontinuum of $[0,1]$. Suppose that $\mathcal{D}\cap[a,b]=\emptyset$; i.e., $|m^{-1}(y)|>1$ for each $y\in[a,b]$. Since m is monotone, $m^{-1}(y)$ is a nondegenerate subcontinuum of $m^{-1}([a,b])$ for each $y\in[a,b]$. Hence, $m^{-1}(y)$ has nonempty interior. Notice that $\{m^{-1}(y):y\in[a,b]\}$ is an uncountable collection of pairwise disjoint subsets with nonempty interior of $m^{-1}([a,b])$. But this contradicts the separability of $m^{-1}([a,b])$. Therefore, there is a point y_0 in $[a,b]$ such that $|m^{-1}(y_0)|=1$ for each nondegenerate subset $[a,b]$ of $[0,1]$ and $\mathcal{D}$ is dense in $[0,1]$.

Let x_0 be a point of $(0,1)$ such that $m^{-1}(m(x_0))=\{x_0\}$. Since $o|_{[a_{i-1},a_i]}$ is a homeomorphism, there exists $t_i\in[a_{i-1},a_i]$ such that $o(t_i)=x_0$, for each $i\in\{1,2,3\}$.

We define $\mathcal{A}_1=C([0,t_1])$ and $\mathcal{A}_2=C([t_2,t_3])$. Observe that $\mathcal{A}_1$ and $\mathcal{A}_2$ are two nondegenerate and disjoint subcontinua of $C([0,1])$ such that $C(f)(\mathcal{A}_1)=C(f)(\mathcal{A}_2)=C([0,m(x_0)])$.

Furthermore, since $f(t_i)=x_0$, for each $i\in\{1,2,3\}$, and $m^{-1}(m(x_0))=\{x_0\}$, we have that $[0,t_1]$ and $[t_2,t_3]$ are components of $f^{-1}([0,m(x_0)])$. Thus, $\mathcal{A}_1$ and $\mathcal{A}_2$ are components of $C(f)^{-1}(C([0,m(x_0)]))$, by [1, Proposition 3.3, p.2041].

However, we know that $C(f)$ is an MO-mapping. Let $\widetilde{o}:C([0,1])\to Z$ be an open mapping and let $\widetilde{m}:Z\to C([0,1])$ be a monotone mapping, such that $C(f)=\widetilde{m}\circ\widetilde{o}$, for some continuum Z. Since $\widetilde{m}(\widetilde{o}(\mathcal{A}_1))=C([0,m(x_0)])$ and $\mathcal{A}_1$ is a component of $C(f)^{-1}(C([0,m(x_0)]))$, we have that $\mathcal{A}_1$ is a component of $(\widetilde{o})^{-1}(\widetilde{o}(\mathcal{A}_1))$.

Claim 3.5. $\widetilde{o}(\mathcal{A}_1)=\widetilde{o}(\mathcal{A}_2)$.

Since $\mathcal{A}_1$ is a component of $C(f)^{-1}(C([0,m(x_0)]))$, $\mathcal{A}_1$ is a component of $(\widetilde{o})^{-1}((\widetilde{m})^{-1}(C([0,m(x_0)])))$. Note that $(\widetilde{m})^{-1}(C([0,m(x_0)]))$ is a subcontinuum of Z. Since $\widetilde{o}$ is open, $\widetilde{o}(\mathcal{A}_1)=(\widetilde{m})^{-1}(C([0,m(x_0)]))$ [11, Theorem 7.5, p.148]. Similarly, we prove that $\widetilde{o}(\mathcal{A}_2)=(\widetilde{m})^{-1}(C([0,m(x_0)]))$. Therefore, $\widetilde{o}(\mathcal{A}_1)=\widetilde{o}(\mathcal{A}_2)$.

Let s_2 and s_3 be points in $[0,1]$ such that $t_2<s_2<a_2<s_3<t_3$. Let $r>0$ such that $2r=\min\{s_2-t_2,a_2-s_2,s_3-a_2,t_3-s_3\}$. We define $\mathcal{U}=B_H([s_2,s_3],r)$. Observe that $\mathcal{U}\subset C([t_2,t_3])=\mathcal{A}_2$ and if $A\in\mathcal{U}$, then $a_2\in A$. Thus, $0\in C(f)(A)$ for each $A\in\mathcal{U}$.

We know that $\widetilde{o}(\mathcal{U})$ is open. Since $C([0,1])$ is locally connected [7, Theorem 6.1.4, p.288], each component of $(\widetilde{o})^{-1}((\widetilde{o})(\mathcal{U}))$ is open [10, 5.22 (a), p.83]. Let $\mathcal{W}$ be some component of $(\widetilde{o})^{-1}((\widetilde{o})(\mathcal{U}))$ such that $\mathcal{W}\subset\mathcal{A}_1$ (see Claim 3.5). Note that $C(f)(\mathcal{W})\subset C(f)(\mathcal{U})$. Hence, $0\in C(f)(A)$ for each $A\in\mathcal{W}$. Since $m^{-1}(m(0))=\{0\}$, $0\in A$, for every point A in $C([0,t_1])$ such that $0\in C(f)(A)$. Thus, $\mathcal{W}\subset\{A\in C([0,t_0]):0\in A\}$. But, this contradicts the fact that $Int_{C([0,1])}(\{A\in C([0,t_0]):0\in A\})=\emptyset$ [6, Example 5.1, p.33]. Therefore, $n\in\{1,2\}$. When either $o(0)=0$ and $m^{-1}(m(0))=\{1\}$, or $o(0)=1$ and $m^{-1}(m(0))\in\{\{0\},\{1\}\}$, repeat the previous argument.

The converse implication follows from Corollary 3.2 and [5, Theorem 3.2, p.241]. □

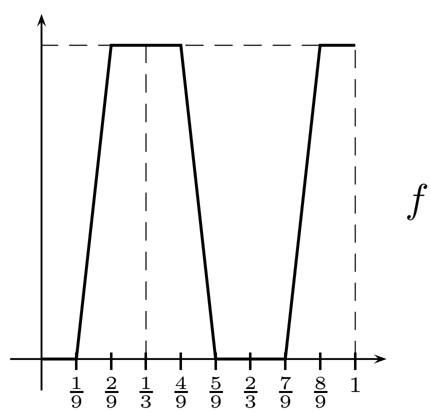
The following proposition shows that the condition: $m : [0,1] \to [0,1]$ is monotone, such that either $|m^{-1}(m(0))| = 1$ or $|m^{-1}(m(1))| = 1$, cannot be removed from Theorem 3.3.

Proposition 3.6. *There exists an MO-mapping $f : [0,1] \to [0,1]$, where $f = m \circ o$, such that $C(f)$ is an MO-mapping and o is not topologically equivalent to either the identity or the tent mapping.*

Proof. We define $f : [0,1] \to [0,1]$ by $f = m \circ o$, where:

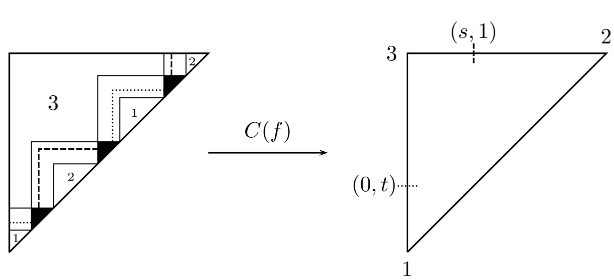
$$o(x) = \begin{cases} 3x, & \text{if } 0 \le x \le 1/3; \\ 2-3x, & \text{if } 1/3 \le x \le 2/3; \\ 3x-2, & \text{if } 2/3 \le x \le 1. \end{cases} \quad m(x) = \begin{cases} 0, & \text{if } 0 \le x \le 1/3; \\ 3x-1, & \text{if } 1/3 \le x \le 2/3; \\ 1, & \text{if } 2/3 \le x \le 1. \end{cases}$$

It is easy to check that the graph of f is the following picture:



Clearly, m is monotone, o is open and it is not topologically equivalent to either the identity or the tent mapping.

We show that $C(f)$ is an MO-mapping. Let $T = \{(x,y) \in [0,1]^2 : x \le y\}$. It is known that $\varphi : C([0,1]) \to T$ defined by $\varphi([a,b]) = (a,b)$ is a homeomorphism [6, Example 5.1, p.33]. Thus, we may represent $C(f)$ like a function from T onto T in the following way (Remark 2.4):



1. $C(f)|Z : Z \to T$ is a homeomorphism, where Z is either $C([1/9, 2/9])$, $C([4/9, 5/9])$ or $C([7/9, 8/9])$. The subset Z is represented in the picture by the solid triangles.

2. Subsets $C([0, 1/9])$ and $C([5/9, 7/9])$ are represented by the triangles with the number 1. For every point A in $C([0, 1/9]) \cup C([5/9, 7/9])$, $C(f)(A) = \{0\}$. Hence, the triangles with the number 1 are mapped by $C(f)$ onto the vertex $(0, 0)$ indicated with 1 in the range T.

3. Subsets $C([2/9, 4/9])$ and $C([8/9, 1])$ are represented by the triangles with the number 2. For every point A in $C([2/9, 4/9]) \cup C([8/9, 1])$, $C(f)(A) = \{1\}$. Hence, the triangles with the number 2 are mapped by $C(f)$ onto the vertex $(1, 1)$ indicated with 2 in the range T.

4. The area marked with the number 3 represents the points $[x, y]$ of $C([0, 1])$ such that:

   $$(0 \le x \le 1/9 \text{ and } 2/9 \le y) \text{ or } (1/9 \le x \le 4/9 \text{ and } 5/9 \le y) \text{ or } (4/9 \le x \le 7/9 \text{ and } 8/9 \le y).$$

   It is not difficult to show that if $[x, y]$ is represented on the area 3, then $C(f)([x, y]) = [0, 1]$. Therefore, every point in the area 3 is mapped by $C(f)$ onto the vertex $(0, 1)$ indicated with the number 3 in the range T.

5. The dotted line in the picture represents the points A of $C([0, 1])$ such that $C(f)(A) = [0, t]$ for some $0 < t < 1$. Hence, if $t \in (0, 1)$ and A is such that $C(f)(A) = [0, t]$, then A belongs to:

   $$\{[x, (t+1)/9] : 0 \le x \le 1/9\} \cup \{[(5-t)/9, y] : 5/9 \le y \le (t+7)/9\} \cup \{[x, (t+7)/9] : (5-t)/9 \le x \le 7/9\}.$$

6. The dashed line in the picture represents the points A of $C([0, 1])$ such that $C(f)(A) = [s, 1]$ for some $0 < s < 1$. Hence, if $s \in (0, 1)$ and A is such that $C(f)(A) = [s, 1]$, then A belongs to:

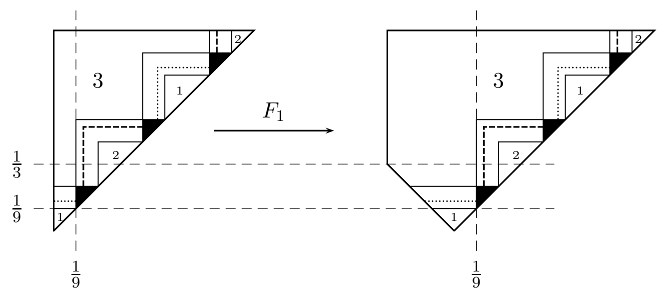
   $$\{[(s+1)/9, y] : 2/9 \le y \le (5-s)/9\} \cup \{[x, (5-s)/9] : (s+1)/9 \le x \le 4/9\} \cup \{[(s+7)/9, y] : 8/9 \le y \le 1\}.$$

Now, we define an open mapping $O : T \to N$ and a monotone mapping $M : N \to T$ such that $C(f) = M \circ O$, for some continuum N.

Let $L = \{(x, y) \in \mathbb{R}^2 : -1/3 \leq x,\ y \leq 1 \text{ and } |x| \leq y\}$ and let $F_1 : T \to L$ defined by:

$$F_1((x,y)) = \begin{cases} (x,y), & \text{if } 1/9 \leq x; \\ (2x-y,y), & \text{if } 0 \leq x \leq y \leq 1/9; \\ ((1+9y)x-y,y), & \text{if } 0 \leq x \leq 1/9 \text{ and } 1/9 \leq y \leq 1/3; \\ (4x-1/3,y), & \text{if } 0 \leq x \leq 1/9 \text{ and } 1/3 \leq y \leq 1. \end{cases}$$

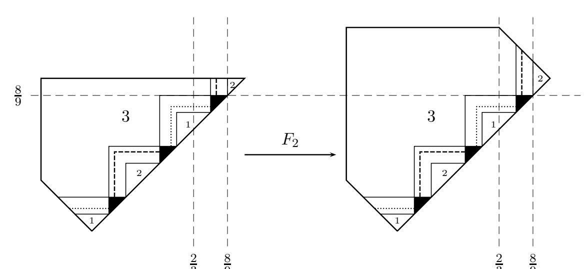
F_1 is represented by the following picture:



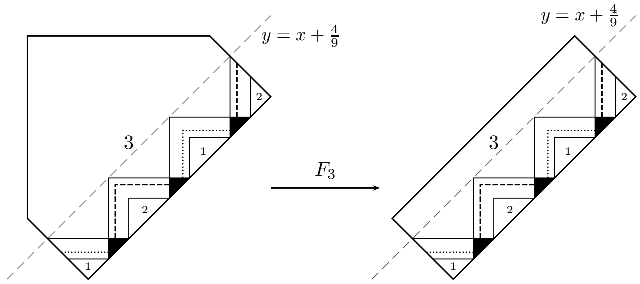
Note that F_1 is a homeomorphism. Let $R = \{(x, y) \in \mathbb{R}^2 : -1/3 \leq x,\ y \leq 4/3 \text{ and } |x| \leq y \leq -x + 2\}$ and define $F_2 : L \to R$ by:

$$F_2((x,y)) = \begin{cases} (x,y), & \text{if } y \leq 8/9; \\ (x,2y-x), & \text{if } 8/9 \leq x \leq 1 \text{ and } 8/9 \leq y; \\ (x,(10-9x)y+8(x-1)), & \text{if } 2/3 \leq x \leq 8/9 \text{ and } 8/9 \leq y; \\ (x,4y-8/3), & \text{if } -1/3 \leq x \leq 2/3 \text{ and } 8/9 \leq y. \end{cases}$$

F_2 is represented by the following picture:



It is possible verify that F_2 is also a homeomorphism. Next, let $S = \{(x, y) \in \mathbb{R}^2 : |x| \leq y \leq -|x - 2/3| + 4/3\}$ and let $F_3 : R \to S$ be a homeomorphism such that $F_3((x, y)) = (x, y)$ for each $(x, y) \in R$, where $y \leq x + 4/9$. Thus, F_3 may be represented by:



We write $F = F_3 \circ F_2 \circ F_1$. Clearly, F is a homeomorphism from T onto S. Let N_1, N_2 and N be subsets of S such that $S = N_1 \cup N_2 \cup N$, where:

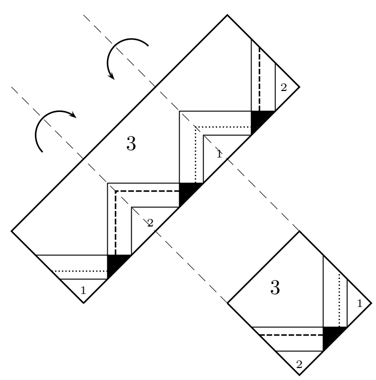
$$N_1 = \{(x,y) \in S : y \leq -x + 2/3\}, \quad N_2 = \{(x,y) \in S : -x + 4/3 \leq y\}$$

$$\text{and } N = \{(x,y) \in S : -x + 2/3 \leq y \leq -x + 4/3\}.$$

Define the open mapping $O : S \to N$ by:

$$O((x,y)) = \begin{cases} (-y + 2/3, -x + 2/3), & \text{if } (x,y) \in N_1; \\ (x,y), & \text{if } (x,y) \in N; \\ (-y + 4/3, -x + 4/3), & \text{if } (x,y) \in N_2. \end{cases}$$

The mapping O may be represented in the following way:



It is very important to emphasize that if $O((x_1, y_1)) = O((x_2, y_2))$, then $C(f)(F^{-1}((x_1, y_1))) = C(f)(F^{-1}((x_2, y_2)))$.

Let $M : N \to T$ be the mapping defined by:

$$M((x,y)) = C(f)((O \circ F)^{-1}((x,y))).$$

Notice that M is a mapping such that $C(f) = M \circ O \circ F$, by [3, Theorem 3.2, p.123]. Furthermore, observe that $O(F(C(f)^{-1}(x,y)))$ is connected, for each $(x,y) \in T$. Since $M^{-1}((x,y)) = O(F(C(f)^{-1}(x,y)))$, M is monotone. Therefore, $C(f)$ is an MO-mapping and our proof is complete. □

We finish this section with a more particular question than [2, (12.2), p.251].

Let $f : [0,1] \to [0,1]$ be a mapping such that $C(f)$ is an MO-mapping. Then does it follow that f is an MO-mapping?

4 MO-mappings between simple closed curves

The goal of this section is to prove Theorem 4.2, where we show that if $f : S^1 \to S^1$ is an MO-mapping, then $C(f)$ is also an MO-mapping.

Let $w \in S^1$ and let $A \in C(S^1)$. We denote wA in the natural way; i.e., $wA = \{wz : z \in A\}$. Note that $wA \in C(S^1)$, for each $w \in S^1$ and $A \in C(S^1)$.

Proposition 4.1. *Let $f : S^1 \to S^1$ be a mapping. If f is open, then $C(f) : C(S^1) \to C(S^1)$ is an MO-mapping.*

Proof. It is known that if $f : S^1 \to S^1$ is open, then there exists a positive integer k such that f is topologically equivalent to g_k, where $g_k(z) = z^k$, for each $z \in S^1$ [11, 1.2, p.184]. Hence, we prove that $C(g_k)$ is an MO-mapping, for each $k \in \mathbb{N}$, and by Remark 2.4, $C(f)$ is an MO-mapping.

Let $k \in \mathbb{N}$. Let A and B be points in $C(S^1)$, we define $\sim$ a relation on $C(S^1)$ by:

$$A \sim B \text{ if and only if } A = e^{2\pi il/k}B, \text{ for some } 0 \leq l < k. \tag{1}$$

Notice that $\sim$ is an equivalence relation on $C(S^1)$. Moreover, the equivalence class of A, denoted by $[A]$, is the set $\{A, e^{2\pi i/k}A, ..., e^{2(k-1)\pi i/k}A\}$. Thus, it is not difficult to show that $\sim$ induces a continuous decomposition of $C(S^1)$, [7, (1.2.14), p.11]. Let $Z = C(S^1)/\sim$ and let $q : C(S^1) \to Z$ be the quotient mapping. Since Z is a continuous decomposition, Z is a continuum and q is an open mapping (see [10, Theorem 3.10, p.40] and [7, Corollary 1.2.24, p.16]).

Let $m : Z \to C(S^1)$ be the function defined by $m([A]) = g_k(A)$. Notice that if $A = e^{2\pi il/k}B$ for some $l \in \{0, 1, ..., k-1\}$, then $g_k(A) = A^k = (e^{2\pi il/k})^k B^k = B^k = g_k(B)$. Hence, m is well defined. Observe that $m \circ q = C(g_k)$. Therefore, m is continuous, by [12, Theorem 9.4, p.60]. Finally, we show that m is monotone. Let $D \in C(S^1)$. We consider two cases:

1. $D \neq S^1$. Note that $g_k^{-1}(D)$ has exactly k components and, since g_k is an open mapping, every component of $g_k^{-1}(D)$ is mapped onto D by g_k, [11, Theorem 7.5, p.148]. Assume $g_k^{-1}(D) = E_1 \cup E_2 \cup ... \cup E_k$, where E_i is a component, for each $i \in \{1, 2, ..., k\}$. Furthermore, it is not difficult to check that these k components of $g_k^{-1}(D)$, $E_1, E_2, ..., E_{k-1}$ and E_k, are the unique points of $C(S^1)$ in $C(g_k)^{-1}(D)$, and $[E_1] = \{E_1, E_2, ..., E_k\}$. Therefore, $m^{-1}(D) = \{[E_1]\}$ and $m^{-1}(D)$ is connected.

2. $D = S^1$. Let $E \in C(g_k)^{-1}(S^1)$. Then there exists an order arc α from E to S^1, [6, Theorem 14.6, p.112]. Notice that if $E' \in \alpha$, then $g_k(E) \subset g_k(E') \subset S^1$ [6, Definition 14.1, p.110]. Hence, $C(g_k)(E') = S^1$. Thus,

$\alpha \subset C(g_k)^{-1}(S^1)$. Since E was an arbitrary point of $C(g_k)^{-1}(S^1)$, we have that $C(g_k)^{-1}(S^1)$ is arcwise connected. Since $m \circ q = C(g_k)$, $m^{-1}(D) = q(C(g_k)^{-1}(S^1))$. Thus, $m^{-1}(D)$ is connected.

Therefore, m is monotone and our proof is complete. □

Theorem 4.2. *Let $f : S^1 \to S^1$ be a mapping. If f is an MO-mapping, then $C(f) : C(S^1) \to C(S^1)$ is an MO-mapping.*

Proof. Let $o : S^1 \to Z$ be an open mapping and let $m : Z \to S^1$ be a monotone mapping such that $f = m \circ o$. Since o is open, Z is either an arc or a simple closed curve [11, (1.2), p.184]. Since m is monotone and m is defined onto S^1, we have that Z is not an arc [10, Proposition 8.22, p.129]. Hence, Z is a simple closed curve.

We know that $C(o)$ is an MO-mapping, by Proposition 4.1. Therefore, $C(f)$ is an MO-mapping, by Proposition 3.1. □

It is important to emphasize that we do not know if the converse of Theorem 4.2 is true.

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