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On the flow of an Oldroyd-B liquid through a straight circular tube performing longitudinal and torsional oscillations of different frequencies

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Abstract

The flow of an Oldroyd-B liquid in a straight, circular, tube oscillating longitudinally and torsionally at different frequencies is examined. The flow does not start from rest but is assumed to be fully established. We obtained analytical solutions for the velocity components which were assumed to have the frequencies of the velocities of the corresponding boundary components. We also obtained analytical expressions for the shear-stresses and drag on the cylinder. The velocity components and work done are displayed graphically using particular values of the flow parameters.

Keywords: Fluid flow, Oldroyd-B liquid, oscillating tube, longitudinal, torsional, different frequencies.

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1 Introduction

Studies involving the flow of liquids through a long straight cylindrical tube have aroused much interest since the development of the field of non-Newtonion fluids. Important applications of these kinds of flows are in the oil industry where, (i) knowledge of the behaviour of oil (non-Newtonian fluid) flowing through pipelines is essential, as well as (ii) knowledge of the behaviour of a non-Newtonian fluid in which a straight cylindrical rod moves, e.g. oscillates. One important application for this latter problem occurs in the oil industry in Ocean engineering, where for example, information on the drag forces on the rods is useful.

Previous studies involving flows in tubes and oscillating rods in liquids include Casarella and Laura [1], Rajagopal[2], Ramkissoon and Majumdar[3], Ramkissoon, Easwaran and Majumdar [4], Rahaman[5, 6]. In [1] the authors examined the motion of a viscous liquid due to a circular cylindrical rod which is oscillating longitudinally and torsionally in that liquid. Their interest was in determining the drag on the rod and they obtained an analytical solution. Ref.[2] investigated a problem similar to that of [1] with a non-Newtonian liquid of the second grade, an exact solution was also obtained. In ref.[3] a viscous liquid flowing in a circular tube was treated and analytical solutions for the shear-stresses and drag on the cylinder were obtained. Ref.[4] dealt with a solid rod oscillating longitudinally and torsionally in a Polar liquid, again an exact solution for the velocity field was obtained. Refs.[5],[6] examined the internal and external flows of an Upper-Convected Maxwell(UCM)

liquid in a tube again performing longitudinal and torsional oscillations respectively.
In the present work we examine the induced flow of an incompressible Oldroyd-B liquid in a straight circular cylinder due to the cylinder performing longitudinal and torsional oscillations of different frequencies when there is no applied pressure-gradient. As in the above works, we consider the flow to be fully developed. We were able to obtain analytical expressions for the velocity and drag fields and we displayed these for particular values of the fluid parameters. We feel our work compliments other existing works.
In Section 2 we describe the defining equations for the problem. Section 3 is devoted to the solutions for the axial and torsional components of the velocity field. In Section 4 we give the corresponding shear-stresses and tangential drag on the cylinder together with the work done by the drag force per half-cycle of the longitudinal and torsional motions of the tube. In Section 5 we give some numerical results while Section 6 contains some concluding remarks.

2 The defining equations

Following Oldroyd[7], we write the rheological equations of state for an Oldroyd-B liquid as

$$T = -pI + S \tag{1}$$

$$S + \lambda_1 \breve{S} = 2\mu D + 2\lambda_2 \breve{D} \tag{2}$$

where T is the total stress, I is the unit tensor, S is the added stress, p the isotopic pressure, $D = \frac{1}{2}(q_{i,j} + q_{j,i})$ is the deformation rate tensor, λ_1 the relaxation time, μ the viscosity coefficient and λ_2 the retardation time. The symbol $\breve{}$ denotes the upper-convected derivative defined by

$$\breve{C}^{ij} = \frac{\partial C^{ij}}{\partial t} + q^m C^{ij}_{,m} - C^{im} q^j_{,m} - C^{mj} q^i_{,m}$$

where q is the velocity field.
In the absence of external forces the dynamic equation is

$$\nabla \cdot S - \nabla p = \rho \frac{dq}{dt} \tag{3}$$

and the equation of continuity is

$$\nabla \cdot q = 0. \tag{4}$$

Since we shall be examining flow within a long circular tube, we shall use cylindrical polar coordinates (r, θ, z) with the z-axis as the axis of the cylinder and assume the velocity field, q, is given by

$$q = (0, v(r,t), w(r,t)). \tag{5}$$

It is easily seen that the continuity equation (4) is automatically satisfied by eqn.(5).
We shall take the cylinder to be performing oscillations described by the vector q_b, where

$$q_b = q\Big)_{r=a} = q_0 \cos(\Omega_1 t)\cos(\beta)\hat{\theta} + q_0 \cos(\Omega_2 t)\sin(\beta)\hat{z} \tag{6}$$

where $\hat{\theta}$ and $\hat{z}$ are unit vectors in the $\theta-$ and $z-$increasing directions respectively. The parameters $q_0, \Omega_1, \Omega_2, \beta$ are such that q_0 determines the magnitude of the oscillations, Ω_1, Ω_2 the respective frequencies and β is the angle which the boundary velocity, q_b given in eqn.(2.6), makes with the $\hat{\theta}$ direction. When $\beta = 0$ the cylinder has only torsional oscillations whereas for $\beta = \frac{\pi}{2}$, it has only longitudinal oscillations.
Now, substituting eqn.(5) into eqns.(2) and (3), we get after some algebra, the following equations for $S_{r\theta},\ S_{rz}$

$$S_{r\theta} + \lambda_1 \frac{\partial S_{r\theta}}{\partial t} = \mu(\frac{\partial v}{\partial r} - \frac{v}{r}) + \lambda_2 \frac{\partial}{\partial t}(\frac{\partial v}{\partial r} - \frac{v}{r}) \tag{7}$$

$$S_{rz} + \lambda_1 \frac{\partial S_{rz}}{\partial t} = \mu \frac{\partial w}{\partial r} + \lambda_2 \frac{\partial}{\partial t}(\frac{\partial w}{\partial r}) \tag{8}$$

with

$$\frac{\partial p}{\partial r} = \frac{\rho v^2}{r} \tag{9}$$

$$\rho \frac{\partial v}{\partial t} = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \frac{\partial S_{r\theta}}{\partial r} + \frac{2S_{r\theta}}{r} \tag{10}$$

$$\rho \frac{\partial w}{\partial t} = -\frac{\partial p}{\partial z} + \frac{\partial S_{rz}}{\partial r} + \frac{1}{r} S_{rz}. \tag{11}$$

Eliminating $S_{r\theta}$ and S_{rz} from these equations we end up with the following equations for $v(r,t)$ and $w(r,t)$. The equation for $v(r,t)$ is

$$\rho\lambda_1 \frac{\partial^2 v}{\partial t^2} + \rho \frac{\partial v}{\partial t} - (\mu + \lambda_2 \frac{\partial}{\partial t})(\frac{2}{r}(\frac{\partial v}{\partial r} - \frac{v}{r}) + \frac{\partial}{\partial r}(\frac{\partial v}{\partial r} - \frac{v}{r})) = -\frac{1}{r}(\frac{\partial p}{\partial \theta} + \lambda_1 \frac{\partial}{\partial t}(\frac{\partial p}{\partial \theta})) \tag{12}$$

and the equation for $w(r,t)$ is

$$\rho\lambda_1 \frac{\partial^2 w}{\partial t^2} + \rho \frac{\partial w}{\partial t} - (\mu + \lambda_2 \frac{\partial}{\partial t})(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{\partial}{\partial r}(\frac{\partial w}{\partial r})) = -(\frac{\partial p}{\partial z} + \lambda_1 \frac{\partial}{\partial t}(\frac{\partial p}{\partial z})). \tag{13}$$

We shall assume that $p = p(r,t)$ hence terms involving $\frac{\partial p}{\partial \theta}$, $\frac{\partial p}{\partial z}$ will be put to zero.
We immediately observe that if we put $\lambda_2 = 0$ we recover the equations for the UCM liquid as in [5] and if we put $\lambda_1 = \lambda_2 = 0$ we get the equations for the Classical liquid, Batchelor[8].

3 Solutions of the flow equations

We now solve eqns.(12), (13) subject to the boundary conditions of Section2. These boundary conditions are

$$v_b(a,t) = q_0 \cos(\beta)\cos(\Omega_1 t)$$
$$w_b(a,t) = q_0 \sin(\beta)\cos(\Omega_2 t).$$

We shall look for solutions of the form

$$v(r,t) = \Re\Big[f(r)exp(i\Omega_1 t)\Big]$$

and

$$w(r,t) = \Re\Big[g(r)exp(i\Omega_2 t)\Big]$$

where $\Re[z]$ denotes the real part of z and $f(r), g(r)$ satisfy the boundary conditions $f(a) = q_0\cos(\beta),\ g(a) = q_0\sin(\beta)$.
Substituting for $v(r,t)$ into eqn.(12), with $p = p(r,t)$, we get the following equation for $f(r)$:

$$f'' + \frac{f'}{r} + (-\frac{1}{r^2} + K_1^2)f = 0 \tag{14}$$

where

$$K_1 = \sqrt{\frac{1}{K}(\lambda_1\Omega_1^2 - i\Omega_1)} \tag{15}$$

and $K = \frac{\mu + i\Omega_1\lambda_2}{\rho}$.
The solution of eqn.(14) is

$$f(r) = AJ_1(K_1 r) + BY_1(K_1 r)$$

where J_1 and Y_1 are Bessel functions of the first and second kind respectively, each of order one, and A and B are constants. Now $v(r,t)$, and hence $f(r)$, must be finite at $r = 0$ hence we must put $B = 0$, and so

$$v(r,t) = \Re\Big[AJ_1(K_1 r)exp(i\Omega_1 t)\Big].$$

Setting $K_1 = i\hat{K}_1$ and using $J_1(i\hat{K}_1 r) = iI_1(\hat{K}_1 r)$ we get, on incorporating the boundary condition,

$$v(r,t) = \Re\Big[\frac{I_1(\hat{K}_1 r)}{I_1(\hat{K}_1 a)}exp(i\Omega_1 t)\Big]q_0\cos(\beta) \tag{16}$$

where $\hat{K}_1 = \sqrt{\frac{\rho(i\Omega_1 - \Omega_1^2\lambda_1)}{\mu + i\Omega_1\lambda_2}}$ and I_1 is the modified Bessel function of the first kind of order one.
Similarly using $w(r,t) = \Re\Big[g(r)exp(i\Omega_2 t)\Big]$ we see that $g(r)$ satisfies

$$g'' + \frac{g'}{r} + K_2^2 g = 0 \tag{17}$$

where K_2 is given by eqn.(15), with Ω_1 replaced by Ω_2.
The solution for $w(r,t)$ including the boundary condition is thus

$$w(r,t) = \Re\Big[\frac{I_0(\hat{K}_2 r)}{I_0(\hat{K}_2 a)} exp(i\Omega_2 t)\Big] q_0 \sin(\beta) \tag{18}$$

where $\hat{K}_2 = \sqrt{\frac{\rho(i\Omega_2 - \Omega_2^2\lambda_1)}{\mu + i\Omega_2\lambda_2}}$, I_0 is the modified Bessel function of the first kind of order zero and we have used $J_0(K_2 r) = J_0(i\hat{K}_2 r) = I_0(\hat{K}_2 r)$.
We mention again that the forms of the solutions we have sought imply that the liquid, as mentioned in Section1, at time $t = 0$, was in motion and the velocity component fields had the forms

$$v(r,t)\mid_{t=0} = \Re\Big[\frac{I_1(\hat{K}_1 r)}{I_1(\hat{K}_1 a)}\Big] q_0 \cos(\beta)$$

and

$$w(r,t)\mid_{t=0} = \Re\Big[\frac{I_0(\hat{K}_2 r)}{I_0(\hat{K}_2 a)}\Big] q_0 \sin(\beta).$$

We note that the torsional oscillations affect only the θ-component of velocity, v, and the longitudinal oscillations affect only the z- component of velocity, w.

4 Shear-stress, tangential drag and work done

Having obtained the velocity components $v(r,t), w(r,t)$ we shall now determine the shear-stress components $S_{r\theta}, S_{rz}$ at $r = a$. These are needed in order to find the drag, $\mathbf{S}\cdot\mathbf{n}$, along the cylinder, where $\mathbf{n}$, is the normal to the surface and $\mathbf{S}$ is the stress, which per unit length is given by, $\mathcal{D}$, where

$$\mathcal{D} = -2\pi a(S_{r\theta}\hat{\theta} + S_{rz}\hat{z})\mid_{r=a}. \tag{19}$$

We have seen that the shear-stress components satisfy equations (7) and (8). Using the expressions for $v(r,t), w(r,t)$ in eqns.(16), (18) and properties of the Bessel functions $J_n(x),\ J_n'(x), I_n(x), I_n'(x)$ we get, on solving eqns.(7), (8), for $S_{r\theta},\ S_{rz}$,

$$S_{r\theta}\mid_{r=a} = q_0\cos(\beta)\Re\Big[\frac{(\mu+i\Omega_1\lambda_2)}{(1+i\Omega_1\lambda_1)}\frac{(\hat{K}_1 I_0(\hat{K}_1 a)-\frac{2}{a}I_1(\hat{K}_1 a))}{I_1(\hat{K}_1 a)}exp(i\Omega_1 t)\Big] \quad (20)$$

and

$$S_{rz}\mid_{r=a} = q_0\sin(\beta)\Re\Big[\hat{K}_2\frac{I_1(\hat{K}_2 a)}{I_0(\hat{K}_2 a)}\frac{(\mu+i\Omega_2\lambda_2)}{(1+i\Omega_2\lambda_1)}exp(i\Omega_2 t)\Big]. \quad (21)$$

Using eqns.(19)–(21) we have

$$\mathcal{D} = -2\pi a\Re\left[q_0\cos(\beta)\left(\frac{\mu+i\Omega_1\lambda_2}{1+i\Omega_1\lambda_1}\right)\left(\frac{\hat{K}_1 I_0(\hat{K}_1 a)-\frac{2}{a}I_1(\hat{K}_1 a)}{I_1(\hat{K}_1 a)}\right)\exp(i\Omega_1 t)\hat{\theta}\right.$$
$$\left.+q_0\sin(\beta)\left(\frac{\mu+i\Omega_2\lambda_2}{1+i\Omega_2\lambda_1}\right)\left(\frac{\hat{K}_2 I_1(\hat{K}_2 a)}{I_0(\hat{K}_2 a)}\right)\exp(i\,\Omega_2\,t)\hat{z}\right]. \quad (22)$$

The work done, W_j, by the drag force, $\mathcal{D}$, on the fluid per half-cycle of torsional and longitudinal motion is given by

$$W_j = -\int_0^{\frac{\pi}{\Omega_j}}\mathcal{D}\cdot q_b dt, \quad (23)$$

where $j=1,2$, refer to the torsional and longitudinal motions respectively. Substituting eqns.(6) and (22) into eqn.(23) gives

$$W_j = \pi a q_0^2\Re\left[\cos^2(\beta)\hat{I}(\Omega_j,\Omega_1)\left(\frac{\mu+i\Omega_1\lambda_2}{1+i\Omega_1\lambda_1}\right)\left(\frac{\hat{K}_1 I_0(\hat{K}_1 a)-\frac{2}{a}I_1(\hat{K}_1 a)}{I_1(\hat{K}_1 a)}\right)+\right.$$
$$\left.+\sin^2(\beta)\hat{I}(\Omega_j,\Omega_2)\left(\frac{\mu+i\Omega_2\lambda_2}{1+i\Omega_2\lambda_1}\right)\hat{K}_2\left(\frac{I_1(\hat{K}_2 a)}{I_0(\hat{K}_2 a)}\right)\right], \quad (24)$$

where

$$\hat{I}(\Omega_j,\Omega) = \int_0^{\frac{\pi}{\Omega_j}}\cos(\Omega t)exp(i\Omega t)dt$$
$$= \frac{\Omega_j\cos(\frac{\pi\Omega}{\Omega_j})\sin(\frac{\pi\Omega}{\Omega_j})+\pi\Omega-i\Omega_j\cos^2(\frac{\pi\Omega}{\Omega_j})+i\Omega_j}{\Omega\Omega_j}.$$

5 Numerical results and concluding remarks

We compared, numerically, the velocity components, $v(r,t), w(r,t)$, as well as the work done by the drag force, for different oscillation frequencies Ω_1, Ω_2. We examined several combinations of frequencies, however, since we do not apply our results to any special problem, we chose to display our results using two particular frequency sets, I: $\Omega_1 = 3.6,\ \Omega_2 = 1$ and II: $\Omega_1 = 0.5,\ \Omega_2 = 2.0$, together with $\lambda_1 = 0.3,\ \lambda_2 = 0.06,\ \mu = 0.05,\ \nu = 0.1$. For set I we chose the value 3.6 since the equal frequency case, $\Omega_1 = \Omega_2 = 3.6$, was considered by Rahaman[5]. The values of the parameters $\lambda_1, \lambda_2, \mu, \nu$ were also chosen to be similar to their values in [5] and to obey the conditions stated in the work of Toms and Strawbridge[9].

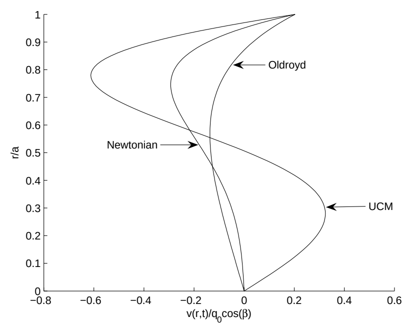

Figure 1: θ-components of velocity for Oldroyd, UCM, Newtonian liquids

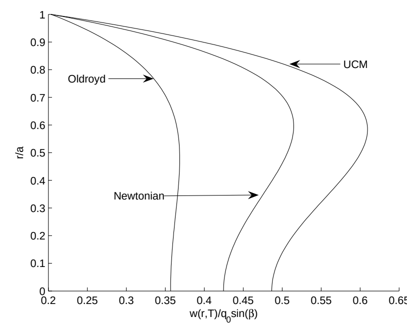

Figure 2: z-components of velocity for Oldroyd,UCM, Newtonian liquids

To display our results we plotted $\frac{v(r,t)}{q_0\cos(\beta)}, \frac{w(r,t)}{q_0\sin(\beta)}$ where $t = \frac{\pi}{\frac{(\Omega_1+\Omega_2)}{2}}, \frac{n\pi}{2\Omega_i}, n = 0,1,2,3$. The value $t = \frac{\pi}{\frac{\Omega_1+\Omega_2}{2}}$ was chosen because

1. it is somewhat consistent with the corresponding value of t used in the UCM and Newtonian case for equal frequencies, i.e. $t = \frac{\pi}{\Omega}$, as in [5],

2. for particular values of $\Omega_1,\ \Omega_2$ we do not have to distinguish between the case $\Omega_1 > \Omega_2$ and $\Omega_1 < \Omega_2$, the only difference will be in the arguments of the modified Bessel functions which occur in $v(r,t), w(r,t)$.

The graphs of $\frac{v(r,T)}{q_0\cos(\beta)}, \frac{w(r,T)}{q_0\sin(\beta)}$ for the Oldroyd, UCM and Newtonian liquids are displayed in Fig.1 and Fig.2 respectively for the above values of $\Omega_i \in I$ at $T = \frac{\pi}{\frac{\Omega_1+\Omega_2}{2}}$. Fig.3 and Fig.4 depict the velocity profiles for $\frac{v(r,t)}{q_0\cos(\beta)}, \frac{w(r,t)}{q_0\sin(\beta)}$ respectively, for the Oldroyd liquid alone for the values of $\Omega_i \in II$ at the times, $t = 0, \frac{\pi}{2\Omega_i}, \frac{\pi}{\Omega_i}, \frac{3\pi}{2\Omega_i}$. The work done by the drag force on the various liquids mentioned, per half cycle of the torsional and longitudinal motions, using $\Omega_i \in I$, are displayed in Fig.5 and Fig.6 respectively.

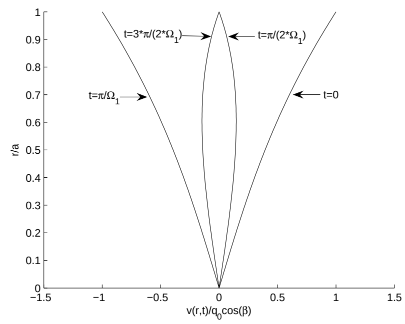

Figure 3: θ-component of velocity for Oldroyd liquid at different times

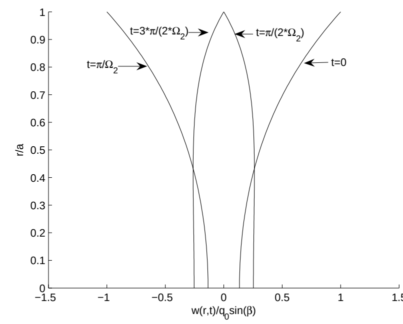

Figure 4: z-component of velocity for Oldroyd liquid at different times

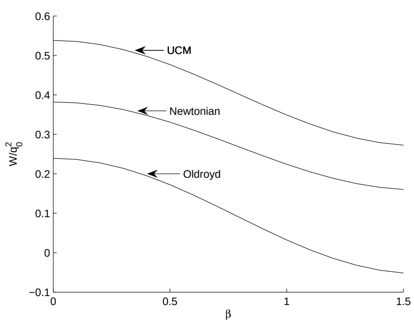

Figure 5: Work done by drag force on the liquids per half cycle of the torsional motion with $\Omega_i \in I$

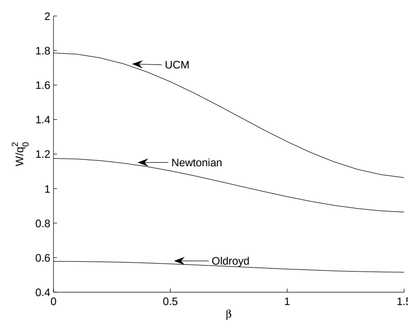

Figure 6: Work done by drag force on the liquids per half cycle of the longitudinal motion with $\Omega_i \in I$

We observe in Fig.1 that the deviation of the $\hat{\theta}$-component of velocity at $t = T$, is greatest for UCM liquids and least for Oldroyd liquids. A similar situation occurs for the $\hat{z}$-component as illustrated in Fig.2. The main difference observed is that of flow reversal occurring with the $\hat{\theta}$ component at this particular time.

In Fig.3 and Fig.4 we see in both motions that there is flow reversal and for the times observed, the respective profiles are symmetric about axes of zero velocity. In particular, symmetry is observed at times $t = 0, \frac{\pi}{\Omega_i}$ and $t = \frac{\pi}{2\Omega_i}, \frac{3\pi}{2\Omega_i}$.

In Fig.5 we see that the work done per half-cycle of torsional motion, ($j = 1$ in eqn.(24)), is greatest for UCM liquids and least for Oldroyd liquids, and the same is observed in Fig.6 for the work done per half-cycle of longitudinal motion, ($j = 2$ in eqn.(24)).

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