Vol. XVI, N$^{\underline{o}}$ 1, Junio (2008)
Matemáticas: 51–62

Matemáticas: Enseñanza Universitaria

Plane viscous flows in a porous medium

Balswaroop Bhatt
University of the West Indies

Angela Shirley
University of the West Indies

Received Aug. 23, 2006 Accepted Mar. 12, 2007

Abstract

The methods employed by Martin [1971] and Chandna et al. [1982] to steady plane flows have been applied to the plane viscous flows in a porous medium using the Darcy - Brinkman - Lapwood equation. Various flows and corresponding geometries have been investigated.

Keywords: Plane flows, viscous flows, porous medium, Darcy - Brinkman - Lapood Equation.

MSC(2000): Primary 76S05, Secondary 76F10.

1 Introduction

We can divide the study of plane viscous flows in to two categories. Basically both follow Martin's [1] approach. The first set of problems are studied using curvilinear co-ordinates (ϕ, ψ), where ψ = constant, are taken to be the streamlines and ϕ = constant, are taken to be the isobars or the orthogonal trajectories of the streamlines, as used by Martin [1] whereas the second set of problems are studied using (u, v) - the velocity components as the independent variables, as used by Chandna et al. [2].

One can refer to Govindaraju [3], Nath and Chandna [4], Chandna and Kaloni [5] and Kaloni and Siddiqui [6] for the first set of problems whereas Barron and Chandna [7], Siddiqui et al. [8] and Bhatt [9] for the second set of problems. Recently Labropulu and Chandna [10, 11] have found some more exact solutions using the similar technique.

In the present paper we have extended the analysis Martin [1] and Chandna et al. [2] to plane viscous flows in a porous medium. Firstly we study the plane viscous flows in a porous medium by writing the equations of motion and continuity in terms of curvilinear coordinates (ϕ, ψ) and then establish interesting results corresponding to

$$\phi = \phi(\xi), \quad \psi = \psi(\eta), \tag{1}$$

$$\phi = \phi(\eta), \quad \psi = \psi(\xi), \tag{2}$$

for particular choice of coordinate lines. Secondly we write the equations of motion in terms of a vorticity function $\omega(x, y) = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$. We transform these equations into an equivalent system in any region in which $0 < \left|\frac{\partial(u,v)}{\partial(x,y)}\right| < \infty$, so that x, y and ω are dependent variables of u, v (i.e. we use the region of a hodograph plane as our domain). From the transformed equation of continuity, we define $L(u,v)$ - the Legendre transform related to the stream function $\psi(x,y)$ as

$$L(u,v) = vx - uy + \psi(x,y). \tag{3}$$

As an application we discuss some forms of $L(u,v)$ and $L^{\star}(q,\theta)$ in polar coordinates of (u,v) and find the flows and corresponding geometries in the physical plane.

2 The Equations of Motion

We consider steady plane flows of an incompressible viscous fluid in porous media. The flows are governed by Darcy - Brinkman - Lapwood equation, namely

$$\rho\left[\frac{\partial V}{\partial t} + V.\nabla V\right] = -\nabla p - \frac{B\mu}{k}v + \tilde{\mu}\nabla^2 v \ , \qquad v = \varepsilon V \ \ , \tag{4}$$

where

ρ - density of fluid,
ε - the porosity,
k - permeability,
$\tilde{\mu}$ - effective viscosity,
μ - dynamic viscosity and
B - binary number, $B = 0$ in the fluid and $B = 1$ in the porous media.

We have taken $\mu = \tilde{\mu}$. For steady two dimensional motion the equations of continuity and motion are:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \ , \tag{5}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\varepsilon^2\frac{\partial p}{\partial x} + \frac{\varepsilon^2}{R_e}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) - \frac{\varepsilon^2 B}{D_a R_e}u \tag{6}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\varepsilon^2\frac{\partial p}{\partial y} + \frac{\varepsilon^2}{R_e}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) - \frac{\varepsilon^2 B}{D_a R_e}v \tag{7}$$

where we have non-dimensionalized the velocity by U (characteristic velocity), pressure by ρU^2, length by L (the characteristic length), $R_e = \dfrac{\rho L U}{\mu}$ is the Reynolds number, $D_a = \dfrac{k}{L^2}$ (k is the permeability of the porous medium) is the Darcy number.

We introduce the vorticity function

$$\omega = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right). \tag{8}$$

Eliminating p between equations (6) and (7) we obtain:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0\ , \qquad u\frac{\partial \omega}{\partial x}+v\frac{\partial \omega}{\partial y}=\eta_1\nabla^2\omega-\eta_1\eta_2\omega\ , \tag{9}$$

where $\eta_1=\dfrac{\varepsilon^2}{R_e}$ and $\eta_2=\dfrac{B}{D_a}$.

3 Hodograph transformation (in terms of ψ,ϕ)

Equation of continuity implies the existence of the stream function $\psi(x,y)$ such that

$$u=\frac{\partial \psi}{\partial y}\ ,v=-\frac{\partial \psi}{\partial x} \tag{10}$$

We introduce a curvilinear coordinate system (ϕ,ψ) in place of x,y where $\phi(x,y)=$ constant, be the arbitrary family of curves which generates with the streamlines $\psi(x,y)=$ constant, a curvilinear net such that :

$$x=x(\phi,\psi)\quad,\quad y=y(\phi,\psi) \tag{11}$$

which defines a curvilinear net in the (x,y) plane with the squared element of arc length given by the well known equation

$$ds^2=Ed\phi^2+2Fd\phi d\psi+Gd\psi^2 \tag{12}$$

where

$$E=\left(\frac{\partial x}{\partial \phi}\right)^2+\left(\frac{\partial y}{\partial \phi}\right)^2,\quad F=\frac{\partial x}{\partial \phi}\frac{\partial x}{\partial \psi}+\frac{\partial y}{\partial \phi}\frac{\partial y}{\partial \psi},\quad G=\left(\frac{\partial x}{\partial \psi}\right)^2+\left(\frac{\partial y}{\partial \psi}\right)^2. \tag{13}$$

Equation (11) can be solved to determine ϕ,ψ as functions of x and y so that

$$\frac{\partial x}{\partial \phi}=J\frac{\partial \psi}{\partial y},\quad \frac{\partial x}{\partial \psi}=-J\frac{\partial \phi}{\partial y},\quad \frac{\partial y}{\partial \phi}=-J\frac{\partial \psi}{\partial x},\quad \frac{\partial y}{\partial \psi}=J\frac{\partial \phi}{\partial x}, \tag{14}$$

where the Jacobian J is such that

$$J=\frac{\partial(x,y)}{\partial(\phi,\psi)}=\pm\sqrt{EG-F^2}=\pm W\ \text{(say)}.$$

We assume $0<|J|<\infty$. Denoting by γ, the local angle of inclination of the tangent to the coordinate line $\psi=$ constant, directed in the sense of increasing ϕ, we have from the differential geometry (using Martin [1]) the following:

$$\frac{\partial x}{\partial \phi}=E\cos\gamma\ ,\quad \frac{\partial y}{\partial \phi}=E\sin\gamma,$$

$$\frac{\partial x}{\partial \psi}=\frac{F}{\sqrt{E}}\cos\gamma-\frac{J}{\sqrt{E}}\sin\gamma\ \ ,\ \ \frac{\partial y}{\partial \psi}=\frac{F}{\sqrt{E}}\sin\gamma+\frac{J}{\sqrt{E}}\cos\gamma\ ,$$

$$\frac{\partial\gamma}{\partial\phi} = \frac{J}{E}{\Gamma_{11}}^2, \quad \frac{\partial\gamma}{\partial\psi} = \frac{J}{E}{\Gamma_{12}}^2, \tag{15}$$

$$K = \frac{1}{W}\left[\frac{\partial}{\partial\psi}\left(\frac{W}{E}{\Gamma_{11}}^2\right) - \frac{\partial}{\partial\phi}\left(\frac{W}{E}{\Gamma_{12}}^2\right)\right] = 0 \ ,$$

where

$${\Gamma_{11}}^2 = \frac{-F(\partial E/\partial\phi) + 2E(\partial F/\partial\phi) - E(\partial E/\partial\psi)}{2W^2} \ ,$$

$${\Gamma_{12}}^2 = \frac{E(\partial G/\partial\phi) - F(\partial E/\partial\psi)}{2W^2} \ ,$$

and K is the Gaussian curvature. The vorticity function ω is such that (Martin [1]):

$$\omega = \frac{1}{J}\left[\frac{\partial}{\partial\phi}\left(\frac{F}{J}\right) - \frac{\partial}{\partial\psi}\left(\frac{E}{J}\right)\right]$$

$$\nabla^2\omega = \frac{1}{J}\left[\frac{\partial}{\partial\phi}\left(\frac{G(\partial\omega/\partial\phi) - F(\partial\omega/\partial\psi)}{J}\right) + \frac{\partial}{\partial\psi}\left(\frac{E(\partial\omega/\partial\psi) - F(\partial\omega/\partial\phi)}{J}\right)\right]. \tag{16}$$

Using the equations (9), (10) and (14), the vorticity equation can be written as

$$\frac{\partial\omega}{\partial\phi} = \eta_1 J \ \nabla^2 \ \omega - \eta_1\eta_2 J\omega. \tag{17}$$

Applications

(a) Straight streamlines:

We study now plane flows with straight streamlines and assume the streamlines are not parallel but envelope a curve C. We assume that the tangent lines to C and their orthogonal trajectories (involutes) determine an orthogonal curvilinear net as in Chandna and Kaloni [5]. Letting χ denote the arc length of C, η the angle subtend by the tangent line to C with x axis, ξ the parameter constant along each involute, we have

$$ds^2 = d\xi^2 + \{\xi - \chi(\eta)\}^2 d\eta^2. \tag{18}$$

The curves $\xi =$ constant are the involutes of C and the curves $\eta =$ constant its tangent lines. We proceed to determine flows for which

$$\phi = \phi(\xi) \ , \ \psi = \psi(\eta), \tag{19}$$

so that

$$ds^2 = E{\phi'}^2 d\xi^2 + 2F\phi'\psi' d\xi d\eta + G{\psi'}^2 d\eta^2. \tag{20}$$

Comparing equations (20) with (18) we have

$$E = \frac{1}{\phi'^2(\xi)}, \quad F = 0, \quad G = \left[\frac{\xi - \chi(\eta)}{\psi'(\eta)}\right]^2, \quad J = \frac{\xi - \chi}{\phi'\psi'} \quad \text{and}$$

$$\omega = -\left[\frac{\{\xi - \chi(\eta)\}\psi'' + \psi'\chi'}{\{\xi - \chi(\eta)\}^3}\right]. \tag{21}$$

Therefore we write

$$\omega_\xi = \frac{2\psi''}{(\xi - \chi)^3} + \frac{3\psi'\chi'}{(\xi - \chi)^4}\ ,\ \omega_{\xi\xi} = -\frac{6\psi''}{(\xi - \chi)^4} - \frac{12\psi'\chi'}{(\xi - \chi)^5}\ ,$$

$$\omega_\eta = -\frac{\psi'''}{(\xi - \chi)^2} - \frac{2\psi''\chi'}{(\xi - \chi)^3} - \frac{(\psi''\chi' + \psi'\chi'')}{(\xi - \chi)^3} - \frac{3\psi'\chi'^2}{(\xi - \chi)^4}\ ,$$

$$\omega_{\eta\eta} = -\frac{\psi^{iv}}{(\xi - \chi)^2} - \frac{4\psi'''\chi'}{(\xi - \chi)^3} - \frac{2\psi''\chi''}{(\xi - \chi)^3} - \frac{6\psi''\chi'^2}{(\xi - \chi)^4} - \frac{(\psi'''\chi' + 2\psi''\chi'' + \psi'\chi''')}{(\xi - \chi)^3}$$
$$- \frac{3(\psi''\chi' + \psi'\chi'')\chi'}{(\xi - \chi)^4} - \frac{3(\psi''\chi'^2 + 2\psi'\chi'\chi'')}{(\xi - \chi)^4} - \frac{12\psi'\chi'^3}{(\xi - \chi)^5}\ . \tag{22}$$

Using equations (21) and (22) in (16) and (17) we obtain

$$\frac{\psi'}{\eta_1}\left[2\psi''(\xi - \chi)^3 + 3\psi'\chi'(\xi - \chi)^2\right] + (4\psi'' + \psi^{iv})(\xi - \chi)^3 +$$
$$(\xi - \chi)^2[9\psi'\chi' + 6\chi'\psi''' + 4\psi''\chi'' + \psi'\chi'''] + (\xi - \chi)[15\psi''\chi'^2 + 10\psi'\chi'\chi'']+$$
$$15\psi'\chi'^3 - \eta_2[(\xi - \chi)^5\psi'' + \psi'\chi'(\xi - \chi)^4] = 0. \tag{23}$$

For $\xi = \chi(\eta)$ the equation (23) is satisfied provided $\chi' = 0$, which means that C has zero radius of curvature. Therefore we have the following theorem:
Theorem 1: *In a steady plane flows in porous media the streamlines are straight lines, then these are concurrent or parallel.*
(b) Streamlines are involute of a curve:

Here we consider the involute of curve C as the streamlines and the tangents to curve C as the orthogonal trajectories. As in (a) we have

$$E = \left[\frac{\xi - \chi(\eta)}{\phi'}\right]^2\ ,\ G = \frac{1}{\psi'^2}\ ,\ J = \frac{\xi - \chi}{\phi'\psi'} \text{ and } \omega = -\psi'' - \frac{\psi'}{\xi - \chi}.$$

Therefore we write

$$\omega_\xi = -\psi''' - \frac{\psi''}{\xi - \chi} + \frac{\psi'}{(\xi - \chi)^2}\ ,$$
$$\omega_\eta = -\frac{\psi'\chi'}{(\xi - \chi)^2}\ ,$$
$$\omega_{\xi\xi} = -\psi^{iv} - \frac{\psi'''}{\xi - \chi} + \frac{2\psi''}{(\xi - \chi)^2} - \frac{2\psi'}{(\xi - \chi)^3} \quad \text{and}$$

$$\omega_{\eta\eta} = -\frac{\psi'\chi''}{(\xi-\chi)^2} - \frac{2\psi'\chi'^2}{(\xi-\chi)^3}.$$

Then the vorticity equation becomes

$$(\xi-\chi)^5\psi^{iv} + (\xi-\chi)^4\psi''' - (\xi-\chi)^3\psi'' + (\xi-\chi)^2\left[\psi' - \frac{\psi'^2\chi'}{\eta_1}\right]$$
$$(\xi-\chi)\psi'\chi'' + 3\psi'\chi'^2 - \eta_2[(\xi-\chi)^5\psi'' + (\xi-\chi)^4\psi'] = 0. \quad (24)$$

Since equation (24) holds identically, it should hold along the curve $\xi = \chi(\eta)$, therefore

$$3\psi'\chi'^2 = 0.$$

Since ψ' can not vanish identically, we have $\chi' = 0$. Thus we have the following theorem:

Theorem2: *The streamlines in two dimensional plane flows in porous media can be involutes of a curve C only if C reduces to a point and the streamlines are circles concentric at this point.*

4 Hodograph transformation (in terms of u, v)

We take the functions $u = u(x, y)$,$v = v(x, y)$ to be such that in the region of the flow, the Jacobian

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} \neq 0, |J| < \infty.$$

We may consider x and y as functions of u and v. By means of $x = x(u, v)$ and $y = y(u, v)$, we have the following relations.

$$\frac{\partial u}{\partial x} = J\frac{\partial y}{\partial v}, \quad \frac{\partial u}{\partial y} = -J\frac{\partial x}{\partial v}, \quad \frac{\partial v}{\partial x} = -J\frac{\partial y}{\partial u}, \quad \frac{\partial v}{\partial y} = J\frac{\partial x}{\partial u} \quad (25)$$

and

$$\frac{\partial f}{\partial x} = \frac{\partial(f,y)}{\partial(x,y)} = J\frac{\partial(f,y)}{\partial(u,v)}, \quad \frac{\partial f}{\partial y} = \frac{\partial(f,x)}{\partial(x,y)} = J\frac{\partial(x,f)}{\partial(u,v)}, \quad (26)$$

where $f = f(x, y)$ is any continuously differential function and

$$J = J(x,y) = \frac{\partial(u,v)}{\partial(x,y)} = \left[\frac{\partial(x,y)}{\partial(u,v)}\right]^{-1} = j(u,v). \quad (27)$$

With the help of equations (25)-(27) and the transformation equation for the vorticity function defined by

$$\omega(x,y) = \omega(x(u,v)\ ,\ y(u,v)) = \bar{\omega}(u,v)\ ,$$

the system (8) - (9) becomes:

$$\frac{\partial x}{\partial u} + \frac{\partial y}{\partial v} = 0, \tag{28}$$

$$j\left(\frac{\partial x}{\partial v} - \frac{\partial y}{\partial u}\right) = \bar{\omega} \tag{29}$$

and

$$\eta_1 \left[\frac{\partial(jQ_2, y)}{\partial(u,v)} + \frac{\partial(x, jQ_1)}{\partial(u,v)}\right] - \eta_1\eta_2\frac{\bar{\omega}}{j} = uQ_2 + vQ_1. \tag{30}$$

where

$$Q_1 = \frac{\partial(x,\bar{\omega})}{\partial(u,v)} \text{ and } Q_2 = \frac{\partial(\bar{\omega}, y)}{\partial(u,v)}. \tag{31}$$

5 Equation for Legendre transform function

The equation of continuity implies the existence of a stream function $\psi(x, y)$ such that

$$d\psi = -v\ dx + u\ dy \ \text{ or } \ \frac{\partial \psi}{\partial x} = -v \ , \ \frac{\partial \psi}{\partial y} = u. \tag{32}$$

Likewise (28) implies the existence of a function $L(u, v)$ called the Legendre transform function of stream function $\psi(x, y)$, such that

$$dL = -y\ du + x\ dv \ \text{ or } \ \frac{\partial L}{\partial u} = -y, \frac{\partial L}{\partial v} = x, \tag{33}$$

and the two functions $\psi(x, y)$, $L(u, v)$ are related by

$$L(u, v) = v\ x - u\ y + \psi(x, y). \tag{34}$$

Introducing $L(u, v)$ in (28) - (31) , we see that (28) is identically satisfied and the remaining equations are :

$$j\left(\frac{\partial^2 L}{\partial u^2} + \frac{\partial^2 L}{\partial v^2}\right) = \bar{\omega}, \tag{35}$$

and

$$\eta_1 \left[\frac{\partial\left(\frac{\partial L}{\partial u}, jQ_2\right)}{\partial(u,v)} + \frac{\partial\left(\frac{\partial L}{\partial v}, jQ_1\right)}{\partial(u,v)}\right] - \eta_1\eta_2\frac{\bar{\omega}}{j} = uQ_2 + vQ_1, \tag{36}$$

where

$$Q_1(u,v) = \frac{\partial\left(\frac{\partial L}{\partial v}, \bar{\omega}\right)}{\partial(u,v)}, \quad Q_2(u,v) = \frac{\partial\left(\frac{\partial L}{\partial u}, \bar{\omega}\right)}{\partial(u,v)} \ , \tag{37}$$

and

$$j = \left[\frac{\partial^2 L}{\partial u^2}\frac{\partial^2 L}{\partial v^2} - \left(\frac{\partial^2 L}{\partial u \partial v}\right)^2\right]^{-1}. \tag{38}$$

Summing up we have the following theorem:

Theorem 3: *If $L(u,v)$ is the Legendre transform of a stream function of the equations of motion* (4)-(7) *governing the plane steady flow of a viscous incompressible fluid then $L(u,v)$ must satisfy* (36).

We now define the polar coordinates (q,θ) in (u,v) plane given by

$$q = \sqrt{(u^2+v^2)}\ , \quad \theta = \tan^{-1}\left(\frac{v}{u}\right)\ , \quad \text{or } u = q\cos\theta\ , \quad v = q\sin\theta \tag{39}$$

so that we have

$$\frac{\partial}{\partial u} = \cos\theta\frac{\partial}{\partial q} - \frac{\sin\theta}{q}\frac{\partial}{\partial\theta}\ , \ \frac{\partial}{\partial v} = \sin\theta\frac{\partial}{\partial q} + \frac{\cos\theta}{q}\frac{\partial}{\partial\theta}. \tag{40}$$

Define $L^\star(q,\theta), \omega^\star(q,\theta), j^\star(q,\theta)$ to be the Legendre transform, vorticity function, Jacobian function in (q,θ) of (u,v) coordinates and using

$$\frac{\partial(F,G)}{\partial(u,v)} = \frac{\partial(F^\star,G^\star)}{\partial(q,\theta)}\frac{\partial(q,\theta)}{\partial(u,v)} = \frac{1}{q}\frac{\partial(F^\star,G^\star)}{\partial(q,\theta)} \tag{41}$$

where $F(u,v) = F^\star(q,\theta)$ and $G(u,v) = G^\star(q,\theta)$ are continuously differential functions, we get the following corollary from theorem 1:

Corollary: *If $L^\star(q,\theta)$ is the Legendre transform function of a stream function of the equations of motion* (4) - (7) *governing the plane steady viscous flow then $L^\star(q,\theta)$ must satisfy*

$$\eta_1\left[\frac{\partial\left(\sin\theta\frac{\partial L^\star}{\partial q} + \frac{\cos\theta}{q}\frac{\partial L^\star}{\partial\theta}, j^\star {Q_2}^\star\right)}{\partial(q,\theta)} + \frac{\partial\left(\cos\theta\frac{\partial L^\star}{\partial q} - \frac{\sin\theta}{q}\frac{\partial L^\star}{\partial\theta}, j^\star {Q_2}^\star\right)}{\partial(q,\theta)}\right] - \eta_1\eta_2\frac{\omega^\star q}{j^\star} = q^2(\sin\theta {Q_1}^\star + \cos\theta {Q_2}^\star) \tag{42}$$

where ${Q_1}^\star, {Q_2}^\star, j^\star$ and $\omega^\star$ are same as obtained by Chandna et al. (2). Once $L^\star(q,\theta)$ of (41) is obtained, we employ

$$x = \sin\theta\frac{\partial L^\star}{\partial q} + \frac{\cos\theta}{q}\frac{\partial L^\star}{\partial\theta}\ , \ y = \frac{\sin\theta}{q}\frac{\partial L^\star}{\partial\theta} - \cos\theta\frac{\partial L^\star}{\partial q} \tag{43}$$

and (39) to get $u(x,y), v(x,y)$ in the physical plane.

Applications

Application 1. Let

$$L(u,v)=F(u)+G(v)\ , \tag{44}$$

such that first and second derivatives of $F(u)$ and $G(v)$ are not zero. Then (29) to (31) give

$$\bar{\omega}=\frac{1}{F''(u)}+\frac{1}{G''(v)}\ ,\ j=\frac{1}{F''(u)G''(v)}\ ,\ Q_1=\frac{F'''(u)G''(v)}{F''^2(u)} \tag{45}$$

$$Q_2=\frac{F''(u)G'''(v)}{G''^2(v)}.$$

Also (36) with the help of (43) and (44) give

$$\eta_1\left[\frac{1}{G''}\left(\frac{G'''}{G''^2}\right)'+\frac{1}{F''}\left(\frac{F'''}{F''^2}\right)'\right]+\eta_1\eta_2\frac{(G''+F'')}{F''G''}+v\frac{F'''}{F''^3}-u\frac{G'''}{G''^3}=0. \tag{46}$$

If (44) defines the Legendre transformation such that $F'''(u)=0$ or $G'''(v)=0$, then (46) is satisfied only when $\eta_2=0$ or when $G''(v)+F''(u)=0$. The case $\eta_2=0$ is considered in Chandna et al. [2]. $F'''(u)=G'''(v)=0$ requires that $F''(u)=K_1$ and $G''(v)=K_2$ for arbitrary constants K_1 and K_2. Then $G''(v)+F''(u)=0$ implies $K_1=-K_2$. In this case we can take

$$L(u,v)=C_1u^2+C_2u+C_3+D_1v^2+D_2v+D_3$$

for arbitrary constants C_1,C_2,C_3,D_1,D_2,D_3 with $D_1=-C_1$.

Then (33) gives

$$u=-\frac{1}{2C_1}\left(y+C_2\right)\ ,\ v=-\frac{1}{2C_1}\left(x-D_2\right). \tag{47}$$

Now (32) implies

$$\psi=\frac{(x-D_2)^2}{4C_1}-\frac{(y+C_2)^2}{4C_1}=\text{ constant or} \tag{48}$$

$$(x-D_2)^2-(y+C_2)^2=\text{constant} \tag{49}$$

We then calculate pressure and vorticity.

$$p=\frac{1}{2C_1}\left[\frac{B}{D_aR_e}(xy+C_2x-D_2y)-\frac{1}{2C_1\varepsilon^2}\left(\frac{1}{2}(x^2+y^2)-D_2x+C_2y\right)\right]+N_1, \tag{50}$$

$$\omega=0 \tag{51}$$

The above result can be summed up in the next theorem:
Theorem 4: *If $L(u,v) = F(u) + G(v)$ is the Legendre transform of a stream function of the equations of motion* (4)-(7) *such that $F''(u) = 0 = G''(v)$ (which is satisfied only if $\eta_2 = 0$ or $F''(u) + G''(v) = 0$) gives u and v which are given by* (47) *where as pressure and vorticity are given by* (50) *and* (51) *respectively and the streamlines are the curves given by* (49).
Application 2. Let

$$L(u,v) = u^m v^n \tag{52}$$

be the Legendre transform function such that $m \neq 0$, $n \neq 0$ and $m + n \neq 1$. Substituting (52) in (35)-(38) yields

$$j = \frac{u^{2-2m}v^{2-2n}}{mn(1-m-n)}, \quad \bar{\omega} = \left[\frac{(m-1)}{n(1-m-n)}v^2 + \frac{(n-1)}{m(1-m-n)}u^2\right]u^{-m}v^{-n},$$

$$Q_1 = \frac{m(m-1)}{1-m-n}u^{-1} - \frac{n(n-1)(2n-2+m)}{m(1-m-n)}uv^{-2} \quad \text{and}$$
$$Q_2 = \frac{m(m-1)(2m-2+n)}{n(1-m-n)}vu^{-2} - \frac{n(n-1)}{1-m-n}v^{-1}\ . \tag{53}$$

Employing (52) and (53) in (36) shows that m and n must satisfy the equation

$$\begin{aligned}\eta_1 \Bigg[&\frac{2(m-1)(1-n)}{(1-m-n)^2}u^2v^2 + \frac{n(n-1)(2n+m-2)(3n+2m-3)}{m^2(1-m-n)^2}u^4 + \\ &\frac{m(m-1)(2m+n-2)(3m+2n-3)}{n^2(1-m-n)^2}v^4\Bigg] - \\ &\eta_1\eta_2[m(m-1)u^{2m}v^{2n+2} + n(n-1)v^{2m+2}u^{2n}] + \\ &\frac{2n(1-n)}{m}u^{m+3}v^{n+1} + \frac{2m(m-1)}{n}u^{m+1}v^{n+3} = 0\end{aligned} \tag{54}$$

The above equation is satisfied for $m = n = 1$, which gives

$$u = x, \quad v = -y, \quad \bar{\omega} = 0, \quad p = -\frac{1}{\varepsilon^2}\left(x^2+y^2\right) + \frac{B}{2D_aR_e}\left(x^2-y^2\right) + N_2 \tag{55}$$

Theorem 5: *If the Legendre transformation of a stream function for the equations of motion* (4) -(7) *has the form $L(u,v) = u^m n^n, m \neq 0, n \neq 0, m+n \neq 1$, then the velocity components, vorticity and pressure are given by* (55), *respectively and the stream lines are the curves $xy = M$, M is constant.*
Application 3: Let

$$L^\star(q,\theta) = F(q) \text{ such that } F'(q) \neq 0 \text{ and } F''(q) \neq 0. \tag{56}$$

Using (56) in (42) and (43), we evaluate $j^\star, \omega^\star, Q_1^\star, Q_2^\star, x, y$ to obtain

$$j^\star = \frac{q}{F'(q)F''(q)}, \ \omega^\star = \frac{q}{F'(q)} + \frac{1}{F''(q)}, \ x = F'(q)\sin\theta, \ y = -F'(q)\cos\theta,$$

$$Q_1{}^\star = -\frac{F'(q)}{q}\omega^\star(q)\cos\theta \ , \ Q_2{}^\star = -\frac{F'(q)}{q}\omega^\star(q)\sin\theta \ , \tag{57}$$

Eliminating $j^\star, \omega^\star, Q_1^\star, Q_2^\star$ and $L^\star$ from (42) by using (56) and (57) we obtain

$$\eta_1 \left\{ \frac{F'(q)}{F''(q)} \left[\frac{q}{F'(q)} + \frac{1}{F''(q)} \right]'' + \left(1 - \frac{F'(q)F''(q)}{F''^2(q)} \right) \left[\frac{q}{F'(q)} + \frac{1}{F''(q)} \right]' \right\} - \eta_1\eta_2 F'(q)F''(q) \left(\frac{q}{F'(q)} + \frac{1}{F''(q)} \right) = 0. \tag{58}$$

This will be satisfied when $\omega^\star = \dfrac{q}{F'(q)} + \dfrac{1}{F''(q)} = 0$. Which implies that

$$\frac{F''(q)}{F'(q)} = -\frac{1}{q} \ , \ \ln|F'(q)| = -\ln|q| + \ln S_1 \ , \ F'(q) = \frac{S_1}{q} \text{ and } F(q) = -\frac{S_1}{q^2} + S_2$$

where S_1 and S_2 are constants. Therefore $L^\star = -\dfrac{S_1}{q^2} + S_2$ gives

$$u = -\frac{S_1 y}{x^2 + y^2}, \ v = \frac{S_1 x}{x^2 + y^2}, \ \omega = 0, \tag{59}$$

$$p = \frac{S_1}{\varepsilon^2}\left[-\frac{S_1}{2(x^2+y^2)} + \frac{\varepsilon^2 B}{D_a R_e}\tan^{-1}\left(\frac{x}{y}\right)\right] + N_3. \tag{60}$$

Theorem 6: *Let the Legendre transformation of a stream function for the equations of motion* (4) - (7) *has the form* $L^\star(q,\theta) = F(q)$ *such that* $F'(q) \neq 0$ *and* $F''(q) \neq 0$. *Then the velocity components, vorticity and pressure are given by* (59) *and* (60) *respectively.*

Application 4: Let

$$L^\star = q^2 G(\theta). \tag{61}$$

Then following Chandna et al. [2] , we have

$$j^\star = [4G^2 + 2GG'' - G'^2]^{-1} \ , \ \omega^\star = \frac{4G + G''}{8G^3 G'' - G'^2} \ ,$$

$$Q_1{}^\star = \frac{\omega^{\star\prime}}{q}\left(2G\sin\theta + G'\cos\theta\right) \text{ and } Q_2{}^\star = \frac{\omega^{\star\prime}}{q}\left(2G\cos\theta - G'\sin\theta\right) \tag{62}$$

where the prime denotes differentiation with respect to θ. Substituting (61) and (62) into (42) we obtain

$$\eta_1 \left[(4G^2 + G'^2) j^\star \omega^{\star\prime} \right]' - 2G\omega^{\star\prime} q^2 - \eta_1\eta_2 q\omega^\star = 0. \tag{63}$$

The above equation (63) is satisfied when $\omega^\star = 0$. Therefore $4G + G' = 0$, and the general solution of this equation is

$$G(\theta) = A_1\cos 2\theta + A_2\sin 2\theta \ , \tag{64}$$

where A_1 and A_2 are arbitrary constants. Equations (61) and (57) give

$$u = \frac{A_2x - A_1y}{2({A_1}^2 + {A_2}^2)},\ v = \frac{(-A_1x - A_2y)}{2({A_1}^2 + {A_2}^2)},\ \omega^\star = 0,\ p = N_5 - \frac{(x^2 + y^2)}{8\varepsilon^2({A_1}^2 + {A_2}^2)} \quad (65)$$

$$\psi = M_1xy + M_2(x^2 - y^2) + M_3\ , \quad (66)$$

where M_1, M_2 and M_3 are arbitrary constants.

Theorem 7: *Let the Legendre transformation of a stream function for the equations of motion* (4) - (7) *has the form* $L^\star(q,\theta) = q^2G(\theta)$. *Then the velocity components. vorticity and pressure are given by* (65)*, respectively and the stream lines are given by* (66).

Acknowledgements The authors are thankful to the referee for helpful suggestions.

References

[1] Martin, M.N. Arch. Rat. Mech. Anal. 41, p266 (1971).

[2] Chandna, O. P., Baron, R. M. and Smith A. C. SIAM (Soc. Ind. Appl. Math.) 42, p1323 (1982).

[3] Govindaraju, K.V. Arch. Rat. Mech. Anal. 45, p66 (1972).

[4] Nath, V.I. and Chandna, Qart. Appl. Math. 31, p351 (1973).

[5] Chandna, O.P. and Kaloni, P.N. SIAM (Soc. Ind. Appl. Math.) 31, p686 (1976).

[6] Kaloni, P.N. and Siddiqui, A.M. Int. J. Eng. Sci. 21, p1157 (1983).

[7] Barron, R.M. and Chandna, O.P. J. Eng. Math. 15, p211 (1981).

[8] Siddiqui, A.M. , Kaloni, P.N. and Chandna, O.P. J. Eng. Math. 19, p201 (1985).

[9] Bhatt, IL Nuvo Cimento, 92, p177 (1986).

[10] Labropulu, F. and Chandna, O.P. Int. J. Math. & Math. Sci. 20, p165 (1997).

[11] Labropulu, F. and Chandna, O.P. Int. J. Math. & Math. Sci. 23, p449 (2000).

Authors' address

Balswaroop Bhatt — University of the West Indies, Department of Mathematics and Computer Science, St. Augustine, Trinidad, W.I.,

e-mail: `Bal.Bhatt@sta.uwi.edu`

Angela Shirley — University of the West Indies, Department of Mathematics and Computer Science, St. Augustine, Trinidad, W. I.

e-mail: `angela.shirley@sta.uwi.edu`