Vol. XIV Nº 1 Junio (2006)
Matemáticas: 1–15

Matemáticas:
Enseñanza Universitaria

A remark on the stability of solitary waves for a 1-D Benney-Luke equation

Eduardo Ibargüen Mondragón

Received Oct. 14 de 2005 Accepted Mar.18, 2006

Abstract
The aim of this work is to analyze the stability of solitary waves of the one-dimensional Benney-Luke equation. We prove that the null solution is asymptotically stable due to the presence of a Hamiltonian structure and to the existence of invariant quantities with regard to time. In the case of a non–null solitary wave, we find the Hamiltonian structure but the verification that some quantities are conserved with respect to time has turned out to be a difficult numerical calculation. We show that the criteriom of stability and orbital instability of M. Grillakis, J. Shatah and W. Strauss is not applicable and we present numerical evidence for unstable solitary waves.

Keywords: Solitary wave, Stability theory, Hamiltonian structure.

AMSC(2000): Primary: 74J35, Secondary 74H55, 74H15.

1 Introduction

In this paper we shall study the stability of solitary waves of the Benney-Luke equation

$$\Phi_{tt} - \Phi_{xx} + a\Phi_{xxxx} - b\Phi_{xxtt} + \Phi_t\Phi_{xx} + 2\Phi_x\Phi_{xt} = 0, \qquad (1)$$

where a and b are positive numbers such that $a - b = \sigma - 1/3$ (σ is named Bond's number).

M. Grillakis, J. Shatah and W. Strauss in [5] established a result that characterizes the stability and orbital instability of solitary waves for problems framed as abstract Hamiltonian systems of the form

$$u_t = J\delta\mathcal{H}(u),$$

defined on a space of Hilbert X, where J is a skew-symmetric linear operator in X.

This class of problems has special solutions called solitary waves of the form $u(x,t) = u_c(x-ct)$, where $c > 0$ is denominated speed-wave. Due to the translation invariance of the motion equation, we can show the existence of a quantity N that is conserved with respect to time (see [4]). This quantity is fundamental in the study of the stability of solitary waves since these are characterized as stationary points of the functional of energy $\mathcal{F} = \mathcal{H} + cN$. The criterion of Grillakis, Shatah and Strauss in [5] for stability and orbital instability of a solitary wave u_c requieres the second variation of $\mathcal{F}$ in u_c, $\delta^2\mathcal{F}(u_c)$, to be positive definite around u_c, except possibly in two directions. Once all the hypotheses of the criteria have been verified we conclude that the solitary wave u_c is orbitally stable if and only if the function d defined as $d(c) = \mathcal{F}(u_c)$ is strictly convex.

In an interesting work Peter Smereka [13] considers the study of the Boussinesq type equation

$$u_{tt} = u_{xx} + \frac{1}{n+1}\left(u^{n+1}\right)_{xx} + u_{xxtt}. \tag{2}$$

In this case the corresponding energy functional $\mathcal{F}_1$ is such that $\delta^2\mathcal{F}_1(\omega_c)$ is negative in an infinite number of directions around ω_c, where ω_c is the solitary wave. This fact makes it impossible to use the result of Grillakis et al. to obtain nonlinear stability of the solitary waves. Before noticing this difficulty, Smereka makes a linear analysis of stability. First he proves the stability of the null solution of equation (2) using the Hamiltonian structure and a quantity N_1 that is conserved with respect to time. Following a similar analysis to establish stability as in the previous case, Smereka tries to prove the stability of solitary waves considering the linearization of the Boussinesq equation and determining its Hamiltonian structure. However, he can not prove that the corresponding quantity N_1 is conserved with respect to time. Finally Smereka presents heuristic arguments for the linear stability of solitary waves of equation (2), showing numerical evidence of unstable solitary waves.

On the other hand,using the results of Grillakis, Shatah and Strauss in [5], Quintero in [8] showed that solitary waves of the Benney-Luke equation (1) are orbitally stable when $0 < c < \sqrt{a/b} < 1$ and orbitally unstable when $0 < c < \sqrt{a/b} < 1$.

In this paper we will consider the case $c > 0$ and $c^2 > \max(1, a/b)$ studied by Quintero in [8]. We make an analysis similar to the one carried out by Smereka in [13].

The paper is organized in the following way. In the second section we deduce the Hamiltonian structure of the Benney-Luke equation (1), we find its solitary waves solutions and we show the non applicability of the criterion of Grillakis, Shatah and Strauss. In the third section we make the linear analysis of stability. In the fourth section we present the numerical work.

2 Hamiltonian Structure

In order to discuss the Hamiltonian structure, it is convenient first to perform in equation (1) the following change of variables $q = \Phi_x$, $r = \Phi_t$. Thus, equation (1) becomes the system

$$\begin{aligned} q_t &= r_x \\ r_t &= Aq_x - rq_x - 2r_xq + br_{xxt}, \end{aligned}$$

where $A = I - a\partial_x^2$. In particular if we define the operator $B = I - b\partial_x^2$, we have that

$$\left[r + B^{-1}\left(\frac{1}{2}q^2\right)\right]_t = \partial_x B^{-1}\left(Aq - rq\right)$$

Equation (1) is the Euler-Lagrange equation for the action functional

$$S = \int_{t_0}^{t_1} \mathcal{L}(\Phi, \Phi_t)dt,$$

where the Lagrangian is given by

$$\mathcal{L}(\Phi, r) = \frac{1}{2}\int_{\mathbb{R}} (rBr - \Phi_x A(\Phi_x) + r(\Phi_x)^2)dx.$$

Now we introduce the conjugate momentum variable

$$p = r + B^{-1}\left(\frac{1}{2}q^2\right).$$

In terms of these variables, a Hamiltonian structure for equation (1) is given by

$$\begin{aligned}\mathcal{H}\begin{pmatrix} q \\ p \end{pmatrix} &= \textstyle\int_{\mathbb{R}} rBpdx - L(q,r) \\ &= \tfrac{1}{2}\textstyle\int_{\mathbb{R}} \left\{\left(p - B^{-1}\left(\tfrac{1}{2}q^2\right)\right) B\left(p - B^{-1}\left(\tfrac{1}{2}q^2\right)\right) + qAq\right\} dx.\end{aligned}$$

A straightforward calculation shows that the first variation of the Hamiltonian is given by

$$\delta\mathcal{H}\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} -\left[p - B^{-1}\left(\frac{1}{2}q^2\right)\right]q + Aq \\ \\ B\left[p - B^{-1}\left(\frac{1}{2}q^2\right)\right] \end{pmatrix}. \tag{3}$$

Note that

$$\partial_x B^{-1}\left[B\left(p - B^{-1}\left(\tfrac{1}{2}q^2\right)\right)\right] = r_x = q_t.$$

On the other hand

$$\partial_x B^{-1}\left(-\left[p - B^{-1}\left(\tfrac{1}{2}q^2\right)\right]q + Aq\right) = \partial_x B^{-1}(-qr + Aq) = p_t.$$

This implies that the Benney-Luke equation (1) is equivalent to the system in canonical Hamiltonian form

$$\begin{pmatrix} q \\ p \end{pmatrix}_t = \begin{pmatrix} 0 & \partial_x B^{-1} \\ \partial_x B^{-1} & 0 \end{pmatrix} \delta\mathcal{H}\begin{pmatrix} q \\ p \end{pmatrix}. \tag{4}$$

Due to the translation invariance of the equation of motion, Noether's Theorem assures the existence of the following conserved quantity

$$N(q,p) = \int_{\mathbb{R}} B(p)qdx.$$

This quantity is crucial in the study of solitary waves with speed $c > 0$ because they can be characterized as the stationary points of the functional $\mathcal{F} = \mathcal{H} + cN$. Note that $\delta\mathcal{F} = 0$ is equivalent to the following system

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -[p - B^{-1}(\frac{1}{2}q^2)]q + Aq + cBp \\ \\ B\left[p - B^{-1}\left(\frac{1}{2}q^2\right)\right] + cBq \end{pmatrix}.$$

From the second equation, we get

$$p = -cq + B^{-1}\left(\tfrac{1}{2}q^2\right).$$

Using the previous equation into the first equation, we get

$$-cBp = -\left[p - B^{-1}\left(\tfrac{1}{2}q^2\right)\right]q + Aq = cq^2 + Aq.$$

Therefore

$$-c\left(-cBq + \tfrac{1}{2}q^2\right) = cq^2 + Aq.$$

But this equation is the same as

$$c^2Bq - Aq - \frac{3}{2}cq^2 = 0$$

or equivalently

$$(bc^2 - a)q_{xx} - (c^2 - 1)q + \frac{3}{2}cq^2 = 0. \tag{5}$$

Since we assume that $c > 0$ and $c^2 > \max(1, a/b)$, then we can define $\lambda^2 = (bc^2 - a)\beta^2$, $\lambda^2 - 1 = c^2 - 1$ and $3c\alpha = 1$. Thus the function η defined by $q(x) = \alpha\eta(z)$, where $z = \beta x$, satisfies the equation

$$\lambda^2\eta'' + (1 - \lambda^2)\eta + \frac{1}{2}\eta^2 = 0,$$

which for $\lambda > 1$ has the solution

$$\eta(z) = 3(\lambda^2 - 1)\,\mathrm{sech}^2\left(\frac{1}{2}\left[\frac{\sqrt{\lambda^2 - 1}}{\lambda}\right]z\right).$$

Therefore, equation (5) has for $c > 0$ and $c^2 > \max(1, a/b)$ the following solution

$$q_c(x) = \frac{c^2 - 1}{c}\,\mathrm{sech}^2\left(\frac{1}{2}\left[\sqrt{\frac{c^2 - 1}{bc^2 - a}}\right]x\right). \tag{6}$$

Since $p = -cq + B^{-1}(q^2/2)$, the stationary point of $\mathcal{F}$ is

$$\phi_c = \begin{pmatrix} q_c \\ p_c \end{pmatrix} = \begin{pmatrix} q_c \\ -cq_c + B^{-1}(q_c^2/2) \end{pmatrix}.$$

As we mentioned in the introduction, Grillakis, Shatah and Strauss established a general result which characterizes orbital stability for a class of systems with a special Hamiltonian structure, as in the case considered here. We will see that the result of Grillakis et al fails in this case. In fact, a direct computation of $\delta^2\mathcal{F}$ around the solitary waves ϕ_c shows that

$$(R,S)\delta^2\mathcal{F}(\phi_c)\begin{pmatrix} R \\ S \end{pmatrix} = \int_{\mathbb{R}} (R,S)\begin{pmatrix} cq + B^{-1}(q_c(\cdot))q_c + A & -q_c + cB \\ -q_c + cB & B \end{pmatrix}\begin{pmatrix} R \\ S \end{pmatrix} dx.$$

This is more easily studied with the transformation

$$\begin{pmatrix} R \\ S \end{pmatrix} = \begin{pmatrix} I & 0 \\ B^{-1}(q_c(\cdot)) - c & I \end{pmatrix}\begin{pmatrix} R \\ W \end{pmatrix},$$

which simplifies $\delta^2\mathcal{F}$ to

$$D\begin{pmatrix} R \\ W \end{pmatrix}^T \delta^2\mathcal{F}(\phi_c)\, D\begin{pmatrix} R \\ W \end{pmatrix} = \int_{\mathbb{R}} (R,W)\widetilde{\mathcal{A}}\begin{pmatrix} R \\ W \end{pmatrix} dx,$$

where

$$\widetilde{\mathcal{A}} = \begin{pmatrix} -L & 0 \\ 0 & B \end{pmatrix}, \quad \text{with} \quad L = -(bc^2 - a)\partial_{xx} + (c^2 - 1 - 3cq_c).$$

Now, we can easily verify that zero is an eigenvalue of L with eigenfunction $(q_c)_x$. On the other hand, Ibarguen [6] shows that there is a single eigenfunction of L with a negative eigenvalue. Also, the essential spectrum of L is on the positive real axis bounded away from zero by $c^2 - 1$ (recall $c^2 > 1$). Therefore we see that L is strictly positive except for two directions which are associated with the two degrees of freedom of the solitary waves, namely phase and speed. Since B is a positive definite operator we see that $\delta^2\mathcal{F}$ is negative in an infinite number of directions. Thus we can not use the criterion of Grillakis et al to establish orbital stability of solitary waves for Benney-Luke equation.

3 Linear Stability

In order to recognize the mechanism of stability we start by analysiing the stability of the linear part of the Benney-Luke equation. We consider the equation

$$\Phi_{tt} - \Phi_{xx} + a\Phi_{xxxx} - b\Phi_{xxtt} = 0,$$

which is equivalent to

$$(1 - b\partial_{xx})\Phi_{tt} = (1 - a\partial_{xx})\Phi_{xx}. \tag{7}$$

We will see that for this equation there exists a quadratic form E (energy) which is conserved in time, but E is indefinite. This means that E can not be used to obtain stability at the linear level. However, we will see that the Hamiltonian structure is enough to establish stability.

We consider equation (7) in a frame of reference moving in the positive direction with speed c, that is

$$t = \tau \quad , \quad z = x - ct. \tag{8}$$

And we consider $\Psi(\tau, z) = \Phi(t, x)$. By differentiating we get that

$$\begin{aligned}&\Phi_x = \Psi_z \, , \, \Phi_{xx} = \Psi_{zz} \\ &\Phi_t = \Psi_\tau - c\Psi_z \, , \, \Phi_{tt} = \Psi_{\tau\tau} - 2c\Psi_{z\tau} + c^2\Psi_{zz}.\end{aligned}$$

Using this we see the equation (7) is reduced to

$$(1 - b\partial_{zz})(\Psi_{\tau\tau} - 2c\Psi_{z\tau} + c^2\Psi_{zz}) = (1 - a\partial_{zz})\Psi_{zz}. \tag{9}$$

This first attempt to establish stability of the trivial solution $\Psi = 0$ is to try the energy method. Using Fourier transform we found that equation (9) gives that $\widehat{\Psi}$ satisfies

$$\widehat{\Psi}_{\tau\tau} - 2cki\widehat{\Psi}_\tau - c^2k^2\widehat{\Psi} + bk^2\widehat{\Psi}_{\tau\tau} - 2bck^3i\widehat{\Psi}_\tau - bc^2k^4\widehat{\Psi} = -k^2(1 + ak^2)\widehat{\Psi}$$

or equivalently

$$(1 + bk^2)\widehat{\Psi}_{\tau\tau} - 2cki(1 + bk^2)\widehat{\Psi}_\tau + \left(1 + ak^2 - c^2(1 + bk^2)\right) k^2\widehat{\Psi} = 0.$$

Consequently

$$\widehat{\Psi}_{\tau\tau} - 2cki\widehat{\Psi}_\tau + \left(\frac{1 + ak^2}{1 + bk^2} - c^2\right) k^2\widehat{\Psi} = 0. \tag{10}$$

Now suppose that $\widehat{\Psi} = u + iv$. Then equation (10) is written as

$$u_{\tau\tau} + iv_{\tau\tau} - 2cki(u_\tau + iv_\tau) + \left(\frac{1 + ak^2}{1 + bk^2} - c^2\right) k^2(u + iv) = 0,$$

which, in terms of the real and the imaginary part, becomes the system

$$\begin{aligned} u_{\tau\tau} + 2ckv_\tau + k^2\left(\frac{1 + ak^2}{1 + bk^2} - c^2\right) u &= 0 \\ v_{\tau\tau} - 2cku_\tau + k^2\left(\frac{1 + ak^2}{1 + bk^2} - c^2\right) v &= 0. \end{aligned}$$

Take $\tau = k\sqrt{c^2 - \frac{1+ak^2}{1+bk^2}}\, s$ and define $u(\tau) = U(s)$ y $v(\tau) = V(s)$. Then

$$u_{\tau\tau} = k^2\left(c^2 - \frac{1 + ak^2}{1 + bk^2}\right) U_{ss}, \qquad v_{\tau\tau} = k^2\left(c^2 - \frac{1 + ak^2}{1 + bk^2}\right) V_{ss}.$$

Thus we conclude that U and V satisfy the system

$$\begin{aligned} U_{ss} + 2\alpha V_s - U = 0, \\ V_{ss} - 2\alpha U_s - V = 0, \end{aligned} \tag{11}$$

where $\alpha = c\left(c^2 - \frac{1+ak^2}{1+bk^2}\right)^{-1/2} > 1$, since $a \leq b$ and $c^2 > 1$. Note that by defining $W = U_s$ end $Z = V_s$, we see that system (11) becomes

$$\begin{pmatrix} U \\ V \\ W \\ Z \end{pmatrix}_s = \begin{pmatrix} W \\ Z \\ U - 2\alpha Z \\ V + 2\alpha W \end{pmatrix}. \tag{12}$$

As we know, the analysis of stability of the zero solution depends on the eigenvalues of the constant coefficient system associated with the linearization of (12). In this case, the Jacobian matrix is given by

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & -2\alpha \\ 0 & 1 & 2\alpha & 0 \end{pmatrix}, \tag{13}$$

whose characteristic equation is

$$\lambda^4 + 2(2\alpha^2 - 1)\lambda^2 + 1 = 0,$$

This equation has the following roots

$$\lambda_+^2 = 1 - 2\alpha^2 + 2\sqrt{\alpha^2(\alpha^2-1)},\ \lambda_-^2 = 1 - 2\alpha^2 - 2\sqrt{\alpha^2(\alpha^2-1)}.$$

The first observation is that we have that $\lambda_-^2 < 0$, since $\alpha > 1$. Moreover, $\lambda_+^2 < 0$. This follows by noticing that

$$\left(\alpha^2 - \frac{1}{2}\right)^2 = \alpha^4 - \alpha^2 + \frac{1}{4} > \alpha^2(\alpha^2 - 1), \text{ or } \alpha^2 - \frac{1}{2} > \sqrt{\alpha^2(\alpha^2-1)},$$

which implies that $\lambda_+^2 = 1 - 2\left(\alpha^2 + \sqrt{\alpha^2(\alpha^2-1)}\right) < 0$.

In other words, the eigenvalues of (13) have zero real part, and so with respect to the Euclidean norm, the criterion is enough to establish stability of the trivial solution, but not asymptotic stability. Note that we have the existence of the conserved energy E for the system (11) defined as the quadratic form

$$E\begin{pmatrix} U \\ V \\ W \\ Z \end{pmatrix} = \frac{1}{2}\left(W^2+Z^2\right)-\frac{1}{2}\left(U^2+V^2\right) \tag{14}$$

$$= \frac{1}{2}\begin{pmatrix} U \\ V \\ W \\ Z \end{pmatrix}^T \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} U \\ V \\ W \\ Z \end{pmatrix},$$

We observe that this quadratic form is indefinite. So, it can not be used to establish stability of the trivial solution of the system (11). This means that the energy method also fails to obtain stability of the trivial solution of the system (11). The last approach is to see if this problem has a Hamiltonian structure which allows us to prove the desired stability result of the trivial solution of the system (11). We start by considering the new set of variables U, V, P_1 and P_2 where P_i corresponds to a type of the momentum conjugate variable. Note that the system (11) has the following Hamiltonian structure

$$\frac{dA}{ds} = J\delta\mathcal{H}_1(A), \tag{15}$$

where $A = (U, V, P_1, P_2)^T$ and J is the antisymmetric operator

$$J = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

and $\mathcal{H}_1$ is of the form

$$\mathcal{H}_1(A) = \frac{1}{2}\left(P_1^2+P_2^2\right)+\frac{1}{2}\left(\alpha^2-1\right)\left(U^2+V^2\right)+\alpha N_1,$$

being $N_1(A) = VP_1 - UP_2$.

In fact, from (15)

$$\begin{pmatrix} U \\ V \\ P_1 \\ P_2 \end{pmatrix}_s = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} \frac{\partial\mathcal{H}_2}{\partial U} \\ \frac{\partial\mathcal{H}_2}{\partial V} \\ \frac{\partial\mathcal{H}_2}{\partial P_1} \\ \frac{\partial\mathcal{H}_2}{\partial P_2} \end{pmatrix} = \begin{pmatrix} -\frac{\partial\mathcal{H}_2}{\partial P_1} \\ -\frac{\partial\mathcal{H}_2}{\partial P_2} \\ \frac{\partial\mathcal{H}_2}{\partial U} \\ \frac{\partial\mathcal{H}_2}{\partial V} \end{pmatrix},$$

then

$$U_s = -\left(P_1+\alpha V\right), \tag{16}$$

$$V_s = -\left(P_2-\alpha U\right), \tag{17}$$

$$P_{1s} = \left(\alpha^2-1\right)U-\alpha P_2, \tag{18}$$

$$P_{2s} = \left(\alpha^2-1\right)V+\alpha P_1. \tag{19}$$

Solving for P_1 in (16) we have that

$$P_1 = -U_s - \alpha V, \tag{20}$$

and differentiating (20)

$$(P_1)_s = -U_{ss} - \alpha V_s. \tag{21}$$

Now matching equations (20) and (21) we conclude that

$$\left(\alpha^2 - 1\right) U - \alpha P_2 = -U_{ss} - \alpha V_s,$$

and substituting P_2 into equation (17) we find that

$$U_{ss} + 2\alpha V_s - U = 0.$$

In a similar way

$$V_{ss} - 2\alpha U_s - V = 0.$$

Using equations (16)-(19) we verify that N_1 as $\mathcal{H}_1$ are conserved quantities on the solutions of systems (11). As a consequence, we have that $\mathcal{K}_\alpha$ defined by

$$\mathcal{K}_\alpha = \mathcal{H}_1 - \alpha N_1 = \frac{1}{2}\left(P_1^2 + P_2^2\right) + \frac{1}{2}\left(\alpha^2 - 1\right)\left(U^2 + V^2\right) \tag{22}$$

is also a conserved quantity on any solutions of the system (11). In other words $\mathcal{K}_\alpha\left(A(s)\right) = \mathcal{K}_\alpha\left(A(s_0)\right)$ for all $s \in \mathbb{R}$.

The most important feature is that for $\alpha > 1$, $\mathcal{K}_\alpha$ defines a norm for $\mathbb{R}^4$ for which the trivial solution of system (11) is asymptotically stable. In other words, we have found that the mechanism of stability at the linear level is based on the existence of a Hamiltonian structure and the existence of a positive conserved quantity $\mathcal{K}_\alpha$ (norm) which depends on the quantity N_1 that is conserved for solutions of system (11).

We want to establish some sort of criterion of stability of the solitary wave for the Benney-Luke equation (1) for speed wave $c > 1$ at least from the numerical point of view. We proceed as in the linear case by linearization of equation (1) around the solitary wave solution. Once this is done, we determine the existence of a Hamiltonian structure and analogous conserved quantities $\tilde{\mathcal{H}}$, $\tilde{N}$ and $\tilde{\mathcal{K}}$. The final analysis will depend on finding a meaning for $\tilde{N}$ being a conserved quantity.

A straightforward computation of $\delta^2\mathcal{H}$ around the solitary wave shows that

$$\delta^2\mathcal{H}\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} cq + B^{-1}(q(\cdot))q + A & -q \\ -q & B \end{pmatrix}.$$

Then the linearization of (4) around any solution is given by

$$\begin{pmatrix} Q \\ P \end{pmatrix}_t = \begin{pmatrix} 0 & \partial_x B^{-1} \\ \partial_x B^{-1} & 0 \end{pmatrix}\begin{pmatrix} cq_c + B^{-1}(q_c(\cdot))q_c + A & -q_c \\ -q_c & B \end{pmatrix}\begin{pmatrix} Q \\ P \end{pmatrix} \tag{23}$$

In terms of the variables (τ, z) defined by (8) system (23) takes the form

$$\begin{aligned}\begin{pmatrix} Q \\ P \end{pmatrix}_{\tau} &= \begin{pmatrix} 0 & \partial_z B^{-1} \\ \partial_z B^{-1} & 0 \end{pmatrix} \begin{pmatrix} cq_c Q + B^{-1}(qQ)q_c + AQ - q_c P \\ -q_c Q + BP \end{pmatrix} \\ &+ c\partial_z \begin{pmatrix} Q \\ P \end{pmatrix}\end{aligned}$$

It is not hard to see that the Hamiltonian is given by

$$\tilde{\mathcal{H}}\begin{pmatrix} Q \\ P \end{pmatrix} = \int_{\mathbb{R}} \frac{1}{2}\left[cq_c Q^2 + q_c Q B^{-1}(q_c Q) + QAQ + PBP\right] dz + \tilde{N}\begin{pmatrix} Q \\ P \end{pmatrix},$$

where

$$\tilde{N}\begin{pmatrix} Q \\ P \end{pmatrix} = \int_{\mathbb{R}} (cBQ - q_c Q)\, P dz.$$

Now we consider the operator $\tilde{\mathcal{K}}$ defined by

$$\begin{aligned}\tilde{\mathcal{K}}\begin{pmatrix} Q \\ P \end{pmatrix} &= \tilde{\mathcal{H}}\begin{pmatrix} Q \\ P \end{pmatrix} - \tilde{N}\begin{pmatrix} Q \\ P \end{pmatrix} \\ &= \int_{\mathbb{R}} \frac{1}{2}\left[cq_c Q^2 + q_c Q B^{-1}(q_c Q) + QAQ + PBP\right] dz.\end{aligned}$$

We observe that as in the linear case, $\tilde{\mathcal{K}}$ is positive. If we differentiate with respect to τ, we conclude that

$$\begin{aligned}\frac{d\tilde{\mathcal{H}}}{d\tau} &= 0. \\ \frac{d\tilde{N}}{d\tau} &= \int_{\mathbb{R}} \left\{(cBQ_\tau - q_c Q_\tau)\, P + (cBQ - q_c Q)\, P_\tau\right\} dz \\ &= \int_{\mathbb{R}} \left\{\tfrac{1}{2}(q_c)_z P^2 + \tfrac{1}{2}c^2 (q_z)_z Q^2 + c(q_c)_z Q B^{-1}(q_c Q)\right\} dz. \\ &\quad + \int_{\mathbb{R}} \left\{\partial_z (q_c Q)\, B^{-1}\left[q_c B^{-1}(q_c Q) + AQ\right]\right\} dz.\end{aligned}$$

In case that we are dealing with the trivial solution $q_c = 0$ (which is also a travelling wave solution), we have

$$\frac{d\tilde{N}}{d\tau} = 0.$$

But this implies that $\tilde{\mathcal{K}}$ is also conserved with respect to the time variable τ, and

$$\begin{aligned}\tilde{\mathcal{K}}\begin{pmatrix} Q \\ P \end{pmatrix} &= \frac{1}{2}\int_{\mathbb{R}} (QAQ + PBP)\, dz \\ &= \frac{1}{2}\int_{\mathbb{R}} \left(Q^2 + a\left(Q'\right)^2 + P^2 + b\left(P'\right)^2\right) dz,\end{aligned}$$

We observe that $\tilde{\mathcal{K}}$ defines a norm in $H^2(\mathbb{R})$ on which the trivial solution is stable, as in the linear case discussed above.

Now in case we are dealing with a trivial norm of the solitary wave we find that

$$\begin{aligned} \frac{d\tilde{N}}{d\tau} &= \int_{\mathbb{R}} \left\{ \tfrac{1}{2}(q_c)_z P^2 + \tfrac{1}{2}c^2 (q_z)_z Q^2 + c(q_c)_z Q B^{-1}(q_c Q) \right\} dz. \\ &\quad + \int_{\mathbb{R}} \left\{ \partial_z (q_c Q) B^{-1} \left[q_c B^{-1}(q_c Q) + AQ \right] \right\} dz. \end{aligned}$$

which is not necessarily zero, as it occurred in the work of Smereka. The perturbations P and Q are unknown functions that depend τ as z, for this reason it is quite difficult to estimate $d\tilde{N}/d\tau$. In the next section we will show numeric evidence of instability of the solitary waves for certain range of the wave speed.

4 Numerical Experiments

Note that $\phi_c = (q_c, p_c)^T$ is also a stationary point of the functional $\mathcal{T} = -\mathcal{F} = -(\mathcal{H} + cN)$. In this case again $\delta^2 \mathcal{T}(\phi_c)$ is negative in an infinite number of directions; this is due to the fact that the operator $-B$ is negative definite . Even in this situation, there is some work suggesting that the sign of $dN(\phi_c)/dc$ can be used to establish some sort of stability or instability at the linear level (see [13], [10]).

In the case of the Benney-Luke equation (1), the quantity N evaluated in the solitary wave is given by

$$\begin{aligned} N(\phi_c) &= \int_{\mathbb{R}} B(p_c) q_c dx \\ &= -\frac{8}{15}\left(c^2-1\right)^{3/2}\left(bc^2-a\right)^{1/2}\left(3c^2+2\right)c^{-3} \\ &\qquad -\frac{8}{15}\left(c^2-1\right)^{5/2}\left(bc^2-a\right)^{-1/2} bc^{-1}. \end{aligned}$$

Let us notice that

$$\frac{d}{dc}N(\phi_c) = -\frac{8}{15}\left(c^2-1\right)^{1/2}\left(bc^2-a\right)^{-3/2} c^{-4} Y(a,b,c),$$

where the function Y comes defined by

$$\begin{aligned} Y(a,b,c) &= 3\left(bc^2-a\right)^2\left(3c^2+2\right) \\ &\quad + bc^4\left(c^2-1\right)(b-a) + bc^2\left(c^2-1\right)\left(bc^2-a\right)\left(2c^2+3\right) \\ &\quad + 4bc^4\left(c^2-1\right)\left(bc^2-a\right) + 6c^2\left(c^2-1\right)\left(bc^2-a\right)^2 \\ &= 3\left(bc^2-a\right)^2\left(3c^2+2\right) + b^2c^4\left(c^2-1\right)\left(4c^2+1-5a/b\right) \\ &\quad + bc^2\left(c^2-1\right)\left(bc^2-a\right)\left(2c^2+3\right) + 6c^2\left(c^2-1\right)\left(bc^2-a\right)^2 \end{aligned}$$

A direct observation shows that if $c^2 > b/a \geq 1$ or $c^2 > 5a/4b - 1/4$, then $Y(a,b,c) > 0$.

In the two previous cases we have that $\frac{dN(\phi_c)}{dc} < 0$. According to the work of Pego and Weinstein [10], this means that stability of the solitary waves of equation (1) is likely to occur, at least at the linear level. Following the same idea, we observe the existence of $c_0(a,b)$ such that for $1 < c^2 < c_0^2$, we have $\frac{dN(\phi_c)}{dc} > 0$, which would imply instability of linear type.

Now, we will consider some experiments where we establish the numerical evidence of instability for $a > b$. In the first place, fixing a and b, we can find numerically $c_0 > 1$ with $0 < c < c_0$ such that $\frac{dN(\phi_c)}{dc} > 0$. We will illustrate with some examples in which $a > b$ and $c > \sqrt{a/b}$.

Experiment 1: In the graphic *1a* we take $a = 0.7$ and $b = 0.1$, in this case $dN(\phi_c)/dc > 0$ if $0 < c < 2.8$. In the graphic *1b* we take $a = 0.5$ and $b = 0.3333$, in this case $dN(\phi_c)/dc > 0$ if $0 < c < 1.247$.

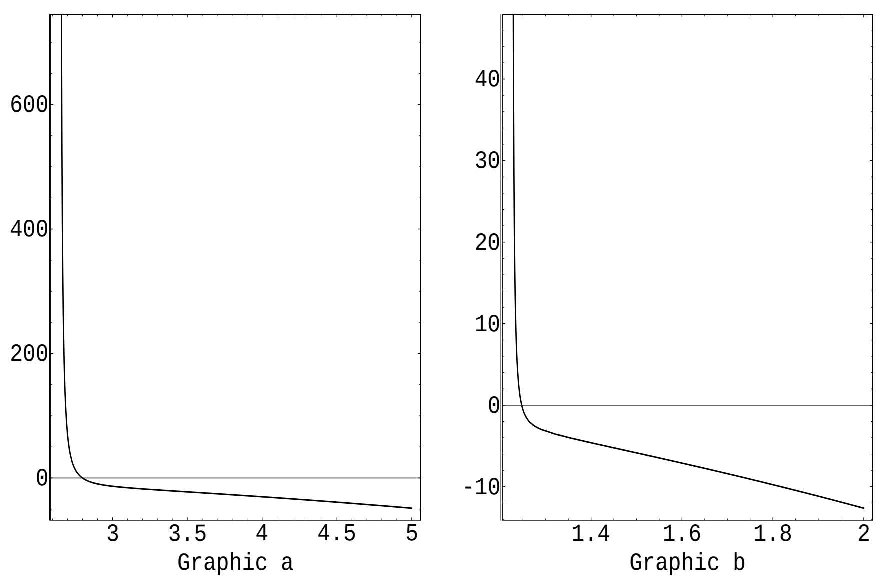

Fig. 1: Graphic of dN/dc vs wave speed c.

To complement our observation on the numerical evidence of instability, the following graphs of the evolution in time of a solitary wave of Benney-Luke equation (1) for the values of the experiments $1a$ and $1b$ are presented. For the numerical implementation we will use the results of Muï¿½z in [7] with the software MATLAB. To study the Benney-Luke equation (1) numerically

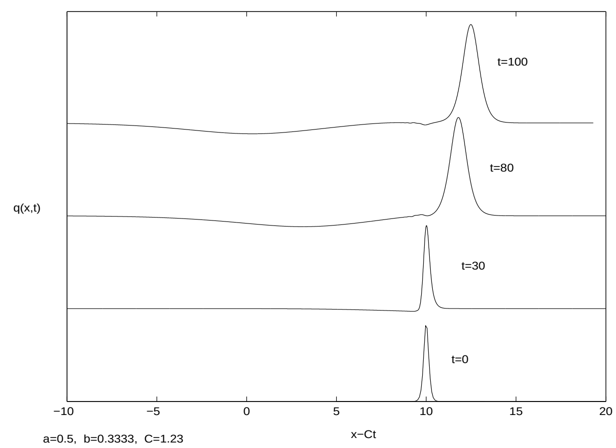

Figure 2: Evolution of an unstable solitary wave

we solve the system

$$\begin{aligned} q_t &= r_x \\ r_t &= Aq_x - rq_x - 2r_x q + br_{xxt}, \end{aligned}$$

subject to the initial conditions

$$\begin{aligned} q(x,0) &= (1+\epsilon)\frac{c^2-1}{c}\operatorname{sech}^2\left(\frac{1}{2}(1+\epsilon)\left[\sqrt{\frac{c^2-1}{bc^2-a}}\right]x\right) \\ r(x,0) &= -cq(x,0), \end{aligned}$$

where ϵ is the parameter that determines the interference of the solitary wave. Figures 2-4 show the graph of an unstable solitary wave for different values of a, b and c in the in the moving frame $x - ct$.

Experiment 2 (instability): In figure 2 we take $a = 0.5$, $b = 0.3333$, $c = 1.23$ and $\epsilon = 0.02$. In this case we observe that the initial solitary wave breaks down in a wave front that maintains its form but, its speed increases and it shows a deflection component. This deflection is due to the fact that the superficial tension is big because of being $a > b$ and the speed of the deflection being small.

Experiment 3 (instability): In figure 3 we take $a = 0.7$, $b = 0.1$, $c = 2.7$ and $\epsilon = 0.02$. The behavior of the solution is similar to that of figure 2. The fact that the wave speed c is bigger than in the previous case causes the decomposition of the solitary wave to occur in a shorter time.

Experiment 4 (instability): In figure 4 we take $a = 0.7$, $b = 0.1$, $c = 2.7$ and $\epsilon = -0.02$. In this case the behavior of the solitary wave changes completely. It loses its form and maintains the speed.

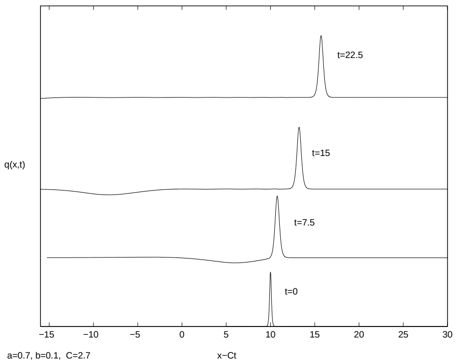

Figure 3: Evolution of an unstable solitary wave

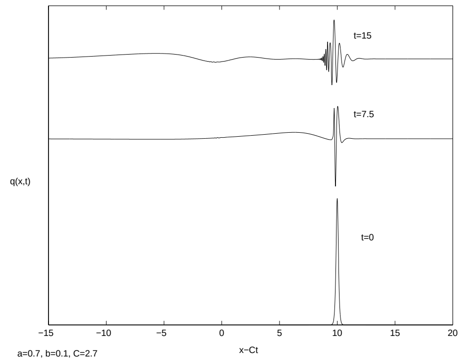

Figure 4: Evolution of an unstable solitary wav

Acknowledgments

The results of this paper are part of the research work done for my masters thesis [6] at Universidad del Valle. I am very gratefull to Prof. Josï¿½ Ral Quintero, my thesis director, and to professor Juan Carlos Muï¿½z, for the guidance, the help and the encouragment provided at every stage of my work.

References

[1] Angulo J. Existence and Stability of Solitary Wave Solutions to Nonlinear Dispersive Evolution Equation. Publicacões Matemáticas. IMPA. 2003.

[2] Birkhoff G., Rota G. Ordinary Differential Equations. Editorial Board. Blaisdell Publishing Company. 1962.

[3] Codington E., Levinson N.: Theory of Ordinary Differential Equations. Mc Graw-Hill.1995.

[4] Drazin P., Johnson S. Solitons: an introduction. Cambridge Texts in Applied Mathematics. 1993.

[5] Grillakis M., Shatah W. and Strauss W.A. Stability theory of solitary waves in the presence of Symetry I. Func. Anal 74(1987), pp 395-412.

[6] Ibargüen E. Sobre la estabilidad de las ondas solitarias para la ecuación de Benney-Luke uno-dimensional. Tesis de Maestría en Matemáticas, Universidad del Valle (2005).

[7] Muï¿½z J.C. Ecuaciones dipersivas para ondas acuaticas. Enseï¿½nza Univarsitaria (ERM), 13. pp 3-14. 2005.

[8] Quintero J. Nonlinear stability of a one-dimensional Boussinesq equation. Journal of Dynamics and Differential Equations, Vol 15, 2003.

[9] Pego R. and Quintero J. Two dimentional solitary waves for a Benney-Luke equation.Physica D 132 (1999), pp 476-496G.

[10] Pego R, Weintein M. Eigenvalues and instabilities of solitary waves. Phil. Trans. R. Soc. Lond. A (1992), 340 47-98.

[11] Renardy M., Rogers R. An Introduction to Partial Differential Equations. Springer-Verlag, 1992.

[12] Rynne B., Youngson M. Linear Functional Analysis. Springer-Verlag.2000.

[13] Smereka P. A remark on the solitary wave stability for a Boussinesq equation. Nonlinear Dispersive wave Systems (Orlando, Fl 1991). pp 255-263, World Scietific. Publishing, River, Edje, N.J 1992.

Dirección del autor: Eduardo Ibargüen Mondragón , Universidad de Nariï¿$\frac{1}{2}$, Departamento de Matemáticas y Estadística, edbargun@udenar.edu.co