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A more direct proof of Gerschgorin's theorem

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Abstract

The aim of this paper is to show a different presentation of the classical proof of the first part of Gerschgorin's theorem. Besides for the second part of this theorem, there is a more straightforward and understandable proof than the one given on most of the classical books.

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1 Introduction

The Gerschgorin circle theorem is a theorem which may be used to bound the size of the eigenvalues of a square matrix. It was first published by Belorussian mathematician Semyon Aranovich Gerschgorin in 1931.

Informally, the theorem says that if the off-diagonal entries of a square matrix over the complex numbers have small norms then its eigenvalues are similar in norm to the diagonal entries of the matrix.

This theorem is a very useful tool in numerical analysis, particularly in perturbation theory. In orden to understand its importance the reader may look up the classical book [2].

The prerequisites for this note are only the basic properties of double summations, triangle inequalities and a little bit of topology of the real and complex numbers.

Let $A = (a_{ij}) \in \mathbb{C}_{n\times n}$ be a square matrix of order n with complex entries and let $D^i = \{z \in \mathbb{C} \mid |z - a_{ii}| \leq \sum_{j=1,j\neq i}^{n} |a_{ij}|\} = B(a_{ii}, \sum_{j=1,j\neq i}^{n} |a_{ij}|)$; $i = 1, 2, \ldots, n$.

Theorem 1.1 (Gerschgorin)**.**

1. *Every eigenvalue of A lies in some D^i.*
2. *If M is the union of m disks D_i such that M is disjoint from all other disks of this type, then M contains precisely m eigenvalues of A (counting multiplicities).*

In this note we show a different presentation of the classical proof of Theorem 1,cf [1], [2],...,[8]. For the second part of the Theorem 1, we start our proof with the same setup as in [1], [3], [8], but we develop it in a clearer way.

Proof. Let λ be an eigenvalue of A and let $B = A - \lambda I$, where I is the identity. That is $B = (b_{ij})$, with $b_{ii} = a_{ii} - \lambda$ and $b_{ij} = a_{ij}$, for $i \neq j$. Since $\det(B) = \det(A - \lambda I) = 0$, then the rows of B are linearly dependent. We can assume that $B_{r+1} = \alpha_1 B_1 + \alpha_2 B_2 + \cdots + \alpha_r B_r$, where B_i is the i-the row of B, $r+1 \leq n$, and $\alpha_i \in \mathbb{C}$ for $i = 1, \ldots, r$. So

$$B_{r+1} = \left(\sum_{k=1}^{r} \alpha_k b_{k,1}, \sum_{k=1}^{r} \alpha_k b_{k,2}, \ldots, \sum_{k=1}^{r} \alpha_k b_{k,n} \right).$$

Now if for some k between 1 and r we have that $|b_{k,k}| \leq \sum_{i=1, i\neq k}^{r+1} |b_{k,i}|$ then λ lies in D^k and we are done. Next, we consider the remaining case. Suppose that $|b_{k,k}| > \sum_{i=1,i\neq k}^{r+1} |b_{k,i}|$, for every $k \in \{1, 2, \ldots, r\}$, then $|\alpha_k b_{k,k}| \geq \sum_{i=1,i\neq k}^{r+1} |\alpha_k b_{k,i}|$, thus $|\alpha_k b_{k,r+1}| \leq -\sum_{i=1,i\neq k}^{r} |\alpha_k b_{k,i}| + |\alpha_k b_{k,k}|$. This inequality being strict for $\alpha_k \neq 0$, for some $k \in \{1, 2, \ldots, r\}$.
Therefore,

$$\begin{aligned}
|b_{r+1,r+1}| &= |\textstyle\sum_{k=1}^{r} \alpha_k b_{k,r+1}| \\
&\leq \textstyle\sum_{k=1}^{r} |\alpha_k b_{k,r+1}| \\
&< \textstyle\sum_{k=1}^{r} (|\alpha_k b_{k,k}| - \sum_{i=1,i\neq k}^{r} |\alpha_k b_{k,i}|) \\
&= \textstyle\sum_{k=1}^{r} |\alpha_k b_{k,k}| - \sum_{k=1}^{r} \left(\sum_{i=1,i\neq k}^{r} |\alpha_k b_{k,i}| \right) \\
&= \textstyle\sum_{i=1}^{r} |\alpha_i b_{i,i}| - \sum_{i=1}^{r} \left(\sum_{k=1,k\neq i}^{r} |\alpha_k b_{k,i}| \right) \\
&= \textstyle\sum_{i=1}^{r} \left(|\alpha_i b_{i,i}| - \sum_{k=1,k\neq i}^{r} |\alpha_k b_{k,i}| \right) \\
&\leq \textstyle\sum_{i=1}^{r} (|\alpha_i b_{i,i}| - |\sum_{k=1,k\neq i}^{r} \alpha_k b_{k,i}|) \\
&\leq \textstyle\sum_{i=1}^{r} \left| \alpha_i b_{i,i} + \sum_{k=1,k\neq i}^{r} \alpha_k b_{k,i} \right| \\
&= \textstyle\sum_{i=1}^{r} |\sum_{k=1}^{r} \alpha_k b_{k,i}| \\
&= \textstyle\sum_{i=1}^{r} |b_{r+1,i}| \\
&\leq \textstyle\sum_{i=1,i\neq r+1}^{n} |b_{r+1,i}|,
\end{aligned}$$

since $|a+b| \geq |a| - |b|$. Hence, $\lambda \in D^{r+1}$ and the assertion follows. □

For the proof of the second part of Theorem 1.1 we need two lemmas, the first of them is stated next without proof cf. [1].

Lemma 1.2. *Let $A \in \mathbb{C}_{n\times n}$ and let $\lambda_1, \lambda_2, \ldots, \lambda_m$ be its different eigenvalues. Then, for any sufficiently small $\varepsilon > 0$, there exists a $\delta > 0$ such that if $B \in \mathbb{C}_{n\times n}$ with $\|A - B\|^* < \delta$, then the matrices A and B have the same number of eigenvalues (counting multiplicities) in every ball $B_\varepsilon(\lambda_i)$.*

For simplicity of notation in the second part of Theorem 1.1 we will suppose that M is the union of the first m disks. We write $I = [0, 1]$. For

$x \in I$, let $A_x = \left(a'_{ij}\right)$ be the matrix defined by $a'_{ii} = a_{ii}$ and $a'_{ji} = xa_{ij}$ for $i, j = 1, 2, \ldots, n$ with $i \neq j$; $D^i(x) = B(a_{ii}, x\sum_{j=1, j\neq i}^{n} |a_{ij}|)$; $M(x) = \bigcup_{i=1}^{m} D^i(x), N(x) = \bigcup_{i=m+1}^{n} D^i(x)$. We define

$$\|A\|^* = \max_{x\neq 0} \frac{\|Ax\|}{\|x\|} = \max_{\|x\|=1} \|Ax\|,$$

it is well known that $\|\cdot\|^*$ is a matrix norm.
Let $f(x) = A_x$, $x \in I$. It is easy to prove that f is a continuous function with the matrix norm $\|\cdot\|^*$.

So, under the hypothesis of the second part we have that $M(x) \cap N(x) = \emptyset$ for $x \in I$, since $M \cap N = \emptyset$, $M(x) \subseteq M(1) = M$ and $N(x) \subseteq N(1) = N$.

Lemma 1.3. *Let $t \in I$ and $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the eigenvalues of A_t. Suppose A_t has exactly its first r eigenvalues in $M(t)$. Then, there exists a $\delta > 0$ such that A_x has exactly r eigenvalues in $M(x)$, for $x \in (t - \delta, t + \delta) \cap I$.*

Proof. Let $\varepsilon > 0$ be sufficiently small and such that Lemma 1.2 holds with $\delta' > 0$. Now since f is continuous, there exists a $\delta > 0$ such that $\|A_t - A_x\|^* < \delta'$, for $x \in (t - \delta, t + \delta) \cap I$. Hence, if $x \in (t - \delta, t + \delta) \cap I$ then A_x and A_t have the same number of eigenvalues in $B_\varepsilon(\alpha_i)$, for $i = 1, 2, \ldots, n$. We can choose ε even smaller in such a way that $(\bigcup_{i=1}^{r} B_\varepsilon(\alpha_i)) \cap N = \emptyset$ and $\left(\bigcup_{i=r+1}^{n} B_\varepsilon(\alpha_i)\right) \cap M = \emptyset$.
Now, for $x \in (t - \delta, t + \delta) \cap [0, 1]$ we have that A_x and A_t have r eigenvalues in $\bigcup_{i=1}^{r} B_\varepsilon(\alpha_i)$, and the $n - r$ eigenvalues in $\bigcup_{i=r+1}^{n} B_\varepsilon(\alpha_i)$.
Let $s(x)$ be the number of eigenvalues of A_x that lie in $M(x)$. Our goal is to prove $s(x) = r$ for $x \in (t - \delta, t + \delta) \cap I$. If $s(x) < r$, then A_x would have $n - s(x)$ eigenvalues in $N(x)$. Certainly $M(x) \cap N(x) = \emptyset$ and, because of the first part of the Theorem, every eigenvalue lies in $M(x) \cup N(x)$. Besides A_x has r eigenvalues in $\bigcup_{i=1}^{r} B_\varepsilon(\alpha_i)$ and $N(x) \cap \bigcup_{i=1}^{r} B_\varepsilon(\alpha_i) = \emptyset$, thus A_x has $(n - s(x)) + r$ eigenvalues, which is an absurd given that $n - s(x) + r > n$. Similarly, if $s(x) > r$, A_x would have $(n - r) + s(x)$ eigenvalues, since A_x has $n - r$ eigenvalues in $\bigcup_{i=r+1}^{n} B_\varepsilon(\alpha_i)$ and $M(x) \cap \bigcup_{i=r+1}^{r} B_\varepsilon(\alpha_i) = \emptyset$. Again, we get a contradiction.
We conclude that $s(x) = r$ for every x in $(t - \delta, t + \delta) \cap [0, 1]$. □

Note that in the case $r = 0$ lemma 1.3 is still true, since A_t would have exactly its first n eigenvalues in $N(t)$ and clearly lemma 1.3 holds if we replace $M(t)$ by $N(t)$.

Proof of the second part of theorem 1.1. For $j = 0, 1, \ldots, n$ we define H_j as the set of $x \in I$ such that A_x has exactly j eigenvalues in $M(x)$ (counting multiplicities). Clearly, $I = \bigcup_{j=1}^{n} H_j$ and $H_j \cap H_i = \emptyset$ if $i \neq j$. Lemma 1.3 ensures that H_j is an open set of I for each j between 1 and n.

Since $0 \in H_m$ and I is a conected set, $H_i = \emptyset$ for $i \neq m$, thus, we conclude that $H_m = I$, which implies that $1 \in H_m$, that is, A has exactly m eigenvalues in M. □

In conclusion, theorem 1 not only state that the eigenvalues of the matrix A lies in $\bigcup_{j=1}^{n} D_j$, but also, assures that they are "regularly distributed" among the connected components of this set.

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