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Sturm-Liouville boundary conditions for a second order ODE

Pablo Amster Pedro P. Cárdenas

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Abstract

We study the semilinear second order ODE $u'' + g(t,u) = 0$ under the following Sturm-Liouville boundary condition $au(0) + bu'(0) = u_0$ and $cu(T) + du'(T) = u_T$. We obtain solutions by topological methods. Moreover, we show that a solution may be constructed recursively, under appropriate conditions.

Keywords: Sturm-Liouville boundary conditions - Topological methods

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1 Introduction

We study the semilinear second order problem

$$\begin{cases} u'' + g(t,u) = 0 \\ au(0) + bu'(0) = u_0 \\ cu(T) + du'(T) = u_T \end{cases} \tag{1}$$

with $g : [0,T]\times\mathbb{R} \to \mathbb{R}$ continuous, and $ad-bc \neq 0$. Problems of this kind have been considered since the fifties by, among others, Ehrmann [4] and Struwe [7] using shooting arguments, and by Bahri-Berestycki [1], Rabinowitz [6], using critical point theory. In the nineties, Capietto, Henrard, Mawhin and Zanolin [2], [3] applied the Leray-Schauder continuation method for a nonlinearity of the type $g = g_1(u) + p(t,u,u')$, where g_1 is superlinear and p satisfies a linear growth condition.

Throughout the paper, we shall assume that all the eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}}$ of the problem

$$-u'' = \lambda u, \qquad au(0) + bu'(0) = cu(T) + du'(T) = 0$$

are non-negative. Writing $u = \gamma e^{rt} + \delta e^{-rt}$ as a possible eigenfunction (corresponding to an eigenvalue $\lambda = -r^2$), it is easy to verify that the previous non-negativity assumption is equivalent to the following condition:

$$(a+br)(c-dr) \neq (a-br)(c+dr)e^{2rT} \qquad \text{for } r > 0. \tag{2}$$

If furthermore $ad - bc + acT \neq 0$, then $\lambda_1 > 0$, and the problem is called *non-resonant*. On the other hand, if $ad - bc + acT = 0$, then $\lambda_1 = 0$. This situation corresponds to the *resonant* case, for which a simple computation shows that the corresponding (normalized) eigenfunction φ_1 is given by

$$\varphi_1(t) = \left(\frac{a^2T^3}{3} - abT^2 + b^2T\right)^{-1/2}(b - at). \tag{3}$$

We shall prove the existence of solutions of (1) by topological methods. More precisely, for the non-resonant case we obtain in section 2.1 an existence result under a linear growth condition on g using Schauder's fixed point theorem. On the other hand, we shall prove the existence of at least one solution when g is subquadratic and satisfies the one-sided growth condition

$$\frac{g(t,u) - g(t,v)}{u - v} \leq \gamma < \lambda_1. \tag{4}$$

We recall that the first eigenvalue can be computed by the Rayleigh quotient:

$$\lambda_1 = \inf_{u \in E - \{0\}} \frac{-\int_0^T u''(t)u(t)dt}{\int_0^T u^2(t)dt} \tag{5}$$

with $E = \{u \in H^2(0,T) : au(0) + bu'(0) = cu(T) + du'(T) = 0\}$.

In section 2.2 we shall embed problem (1) in a family $(1)_\sigma$ of problems with a parameter $\sigma \in [0,1]$. Thus, starting at a solution u_σ for some $\sigma < 1$ we shall define recursively a sequence which converges to a solution of $(1)_{\sigma+\varepsilon}$ for some appropriate step ε. In particular, when ε does not depend on u_σ, we obtain recursively solutions for $0 = \sigma_0 < \sigma_1 < \ldots < \sigma_N = 1$, which gives a solution of the original problem. Finally, in section 3 we obtain solutions for the resonant case under the so-called *Landesman-Lazer* type conditions.

Remark 1.1. *For simplicity, we consider only the case $g = g(t,u)$, although the methods presented in this paper can be extended to the non-variational case $g = g(t,u,u')$.*

2 The non-resonant case

In this section we study the non-resonant case, in which condition

$$ad - bc + acT \neq 0 \tag{6}$$

holds. In section 2.1 we establish two existence results by topological methods, and in section 2.2 we define an iterative scheme that converges to a solution of (1).

2.1 Solutions by fixed point methods

We shall define a fixed point operator in order to obtain solutions of (1) by topological methods, under the assumption $ad - bc + acT \neq 0$. In this case, for any $\theta \in L^2(0,T)$ there exists a unique solution of the problem

$$u'' = \theta, \qquad au(0) + bu'(0) = cu(T) + du'(T) = 0$$

given by the integral formula

$$u(t) = \int_0^T G(t,s)\theta(s)ds,$$

where G is the following Green function:

$$G(t,s) = \frac{(b - at)(c(T - s) + d)}{ad - bc + acT} + \max\{t - s, 0\}.$$

Thus, the solutions of (1) can be regarded as fixed points of the operator T given by

$$Tu(t) = \alpha t + \beta - \int_0^T G(t,s)g(s,u(s))ds, \tag{7}$$

where

$$\alpha = \frac{au_T - cu_0}{ad - bc + acT}, \qquad \beta = \frac{(cT + d)u_0 - bu_T}{ad - bc + acT}.$$

Thus we obtain:

Theorem 2.1. *Let (2) and (6) hold, and assume that* $|g(t,u)| \leq k|u| + l$, *with* $k < \lambda_1$. *Then problem (1) admits at least one solution.*

Proof. From the assumption on g, it follows that $T : L^2(0,T) \to L^2(0,T)$ is well defined. Furthermore, by Arzelá-Ascoli's Theorem we deduce that T is compact. Moreover, from the Rayleigh quotient (5) we get, for fixed $\tilde{u}$:

$$\|Tu - T\tilde{u}\|_{L^2} \leq \frac{1}{\lambda_1}\|(Tu - T\tilde{u})''\|_{L^2} = \frac{1}{\lambda_1}\|g(\cdot,u) - g(\cdot,\tilde{u})\|_{L^2} \leq \frac{k}{\lambda_1}\|u\|_{L^2} + s$$

for some constant $s \geq 0$. Thus, for R large enough we conclude that $T(B_R(0)) \subset B_R(0)$, and the proof follows from Schauder's Fixed Point theorem. □

Theorem 2.2. *Let (2) and (6) hold. Further, assume that g satisfies (4), and that* $|g(t,u)| \leq k|u|^r + l$ *for some constants* k, l *and some* $r < 2$. *Then problem (1) admits a unique solution.*

Proof. From the assumptions, if $u \in L^2(0,T)$ then $g(\cdot,u) \in L^p(0,T)$ for some $p > 1$, and the operator $T : L^2(0,T) \to L^2(0,T)$ given by (7) is well defined. Moreover, if $u = \sigma T u$ for some $\sigma \in [0,1]$, then

$$S_\sigma u := u'' + \sigma g(t,u) = 0, \qquad au(0) + bu'(0) = \sigma u_0, \quad cu(T) + du'(T) = \sigma u_T.$$

Let $\tilde{u} \in H^2(0,T)$ satisfy $a\tilde{u}(0) + b\tilde{u}'(0) = \sigma u_0, c\tilde{u}(T) + d\tilde{u}'(T) = \sigma u_T$. Then:

$$\|S_\sigma u - S_\sigma \tilde{u}\|_{L^2} \|u - \tilde{u}\|_{L^2} \geq -\int_0^T (S_\sigma u - S_\sigma \tilde{u})(u - \tilde{u})dt$$

$$\geq \lambda_1 \|u - \tilde{u}\|_{L^2}^2 - \int_0^T (g(t,u) - g(t,\tilde{u}))(u - \tilde{u})dt \geq (\lambda_1 - \gamma)\|u - \tilde{u}\|_{L^2}^2.$$

It follows that

$$\|u - \tilde{u}\|_{L^2} \leq \frac{1}{\lambda_1 - \gamma} \|S_\sigma u - S_\sigma \tilde{u}\|_{L^2} = \frac{1}{\lambda_1 - \gamma} \|S_\sigma \tilde{u}\|_{L^2}.$$

Thus, if we fix $z \in H^2(0,T)$ such that $az(0)+bz'(0) = u_0, cz(T)+dz'(T) = u_T$, then setting $\tilde{u} = \sigma z$ we obtain:

$$\|u - \sigma z\|_{L^2} \leq \frac{\sigma}{\lambda_1 - \gamma} \|z'' + g(\cdot, \sigma z)\|_{L^2} \leq C$$

for some constant C independent of σ. This implies that all solutions of the problem $u = \sigma T u$ satisfy $\|u\|_{L^2} \leq M$ for some constant M, and the existence of a fixed point of T follows from the Leray-Schauder theorem (see e.g. [5]).

Finally, if u and $\tilde{u}$ are solutions of (1), then $S_1 u = S_1 \tilde{u} = 0$. As before,

$$\|u - \tilde{u}\|_{L^2} \leq \frac{1}{\lambda_1 - \gamma} \|S_1 u - S_1 \tilde{u}\|_{L^2} = 0.$$

□

2.2 An iterative procedure for problem (1)

In what follows of this section we shall embed problem (1) in a family of problems

$$(1)_\sigma \begin{cases} u''(t) + \sigma g(t,u) = 0 \\ au(0) + bu'(0) = u_0 \\ cu(T) + du'(T) = u_T. \end{cases}$$

Starting at a solution u_σ for $\sigma < 1$ we shall define recursively a sequence that converges to a solution of $(1)_{\sigma+\varepsilon}$ for some step $\varepsilon \leq 1 - \sigma$.

As a basic assumption, we shall assume that g is C^2 with respect to u, and $\frac{\partial g}{\partial u} \leq \gamma < \lambda_1$. In particular, note that (4) holds.

Let u_σ be a solution of $(1)_\sigma$ and consider the sequence $\{u_n\} \subset H^2(0,T)$ given recursively by $u_1 = u_\sigma$, and u_{n+1} the unique solution of the linear problem:

$$\begin{cases} u''_{n+1} + (\sigma+\varepsilon)\Big(g(t,u_n) + \frac{\partial g}{\partial u}(t,u_n)(u_{n+1}-u_n)\Big) = 0 \\ au_{n+1}(0) + bu'_{n+1}(0) = u_0 \\ cu_{n+1}(T) + du'_{n+1}(T) = u_T. \end{cases} \tag{8}$$

From the Fredholm alternative for linear operators (and also as a particular case of Theorem 2.2) sequence $\{u_n\}$ is well defined. Moreover, if $u_n \to u$ in the L^2-norm, then it is easy to see that u is a solution of $(1)_{\sigma+\varepsilon}$.

Let $z_n = u_{n+1} - u_n$, then for $n \geq 2$ we have:

$$z''_n + (\sigma+\varepsilon)\frac{\partial g}{\partial u}(t,u_n)z_n = -(\sigma+\varepsilon)[g(t,u_n) - g(t,u_{n-1}) - \frac{\partial g}{\partial u}(t,u_{n-1})(u_n - u_{n-1})]$$

$$= -\frac{1}{2}(\sigma+\varepsilon)\frac{\partial^2 g}{\partial u^2}(t,\xi)z_{n-1}^2$$

for some mean value $\xi(t)$ between $u_n(t)$ and $u_{n-1}(t)$. Then, for some constant μ (independent of σ):

$$\|z_n\|_{H^1} \leq \mu \left\| z''_n + (\sigma+\varepsilon)\frac{\partial g}{\partial u}(\cdot,u_n)z_n \right\|_{L^2} \leq \frac{\mu}{2}\left\| \frac{\partial^2 g}{\partial u^2}(\cdot,\xi)z_{n-1}^2 \right\|_{L^2}$$
$$\leq C_n \|z_{n-1}\|_{H^1}^2$$

for some constant C_n. In particular, if $\frac{\partial^2 g}{\partial u^2}$ is bounded, we may consider $C_n = C := \frac{\mu\nu}{2}\|\frac{\partial^2 g}{\partial u^2}\|_{L^\infty}$ for every n, where ν is the constant of the imbedding $H^1(0,T) \hookrightarrow L^4(0,T)$. On the other hand,

$$z''_1 + (\sigma+\varepsilon)\frac{\partial g}{\partial u}(t,u_1)z_1 = -u''_1 - (\sigma+\varepsilon)g(t,u_1) = -\varepsilon g(t,u_1),$$

whence $\|z_1\|_{H^1} \leq \mu\varepsilon\|g(\cdot,u_1)\|_{L^2}$. Thus we obtain:

Theorem 2.3. *Assume that (2) and (6) hold, and let $u_1 = u_\sigma$ be a solution of $(1)_\sigma$ for some $\sigma \in [0,1)$. Furthermore, assume that $\frac{\partial g}{\partial u} \leq \gamma < \lambda_1$ for some constant γ, and that $\frac{\partial^2 g}{\partial u^2}$ is bounded. Then the iterative scheme defined by (8) converges to a solution of $(1)_{\sigma+\varepsilon}$, provided that $\mu\varepsilon C\|g(\cdot,u_\sigma)\|_{L^2} < 1$, with C and μ as before.*

Proof. From the previous computations, we deduce that

$$\|z_{n+1}\|_{H^1} \leq C^{2^n-1}\|z_1\|_{H^1}^{2^n} \leq \frac{1}{C}\left(\mu\varepsilon C\|g(\cdot,u_\sigma)\|_{L^2}\right)^{2^n}.$$

Then $\{u_n\}$ is a Cauchy sequence in $H^1(0,T)$, and the proof follows. □

Corollary 2.4. *Let the assumptions of the previous theorem hold. Further, assume that g is bounded. Then the step ε in the iterative scheme defined by (8) can be chosen independently of σ. In particular, there exists a sequence $0 = \sigma_0 < \sigma_1 < \ldots < \sigma_N = 1$, with u_{σ_j} solution of $(1)_{\sigma_j}$ constructed recursively from (8), and u_{σ_N} is a solution of (1).*

3 Resonant case: Landesman-Lazer type conditions

In this section we study problem (1) for $u_0 = u_T = 0$ under the assumption of resonance at the first eigenvalue $\lambda_1 = 0$; namely, we consider the case in which the condition

$$ad - bc + acT = 0 \tag{9}$$

holds. The proof of following lemma is straightforward:

Lemma 3.1. *Assume that (2) and (9) hold. Let $E \subset C^2([0,T])$ and $F \subset C([0,T])$ the closed subspaces defined by*

$$E = \{u \in C^2([0,T]) : au(0) + bu'(0) = cu(T) + du'(T) = 0, \int_0^T u(t)\varphi_1(t)dt = 0\}$$

and $F = \{\theta \in C([0,T]) : \int_0^T \theta(t)\varphi_1(t)dt = 0\}$. Then the continuous linear operator $L : E \to F$ given by $Lu = u''$ is bijective, and hence an isomorphism. In particular, there exists a constant γ such that $\|u\|_{C^2} \leq \gamma \|u''\|_C$ for every $u \in E$.

In order to introduce appropriate Landesman-Lazer conditions for our problem, we shall assume that the following limits exist:

$$\lim_{s\to\pm\infty} g(t, s\varphi_1(t)) := g^{\pm}(t). \tag{10}$$

Thus, the main result of this section reads:

Theorem 3.2. *Assume that (2) and (9) hold, and that the limits (10) exist. Then problem (1) for $u_0 = u_T = 0$ admits at least one solution, provided that one of the following conditions holds:*

$$\int_0^T g^+(t)\varphi_1(t)dt < 0 < \int_0^T g^-(t)\varphi_1(t)dt, \tag{11}$$

$$\int_0^T g^-(t)\varphi_1(t)dt < 0 < \int_0^T g^+(t)\varphi_1(t)dt. \tag{12}$$

Proof. Let us first observe that, for $\sigma > 0$, problem

$$\begin{cases} u'' + \sigma g(t,u) = 0 \\ au(0) + bu'(0) = cu(T) + du'(T) = 0 \end{cases} \tag{13}$$

is equivalent to the fixed point problem

$$u = \langle u - g(\cdot,u), \varphi_1\rangle\varphi_1 - \sigma K(g(\cdot,u) - \langle g(\cdot,u),\varphi_1\rangle\varphi_1), \tag{14}$$

where $K : F \to E$ is the inverse of the mapping L defined in Lemma 3.1, and $\langle\cdot,\cdot\rangle$ denotes the usual inner product of $L^2(0,T)$, namely $\langle\theta,\xi\rangle = \int_0^T \theta(t)\xi(t)dt$. Indeed, if u is a solution of (13) then $\langle u'', \varphi_1\rangle = \langle u, \varphi_1''\rangle = 0$, which implies $\langle g(\cdot,u),\varphi_1\rangle = 0$, and

$$u - \langle u,\varphi_1\rangle\varphi_1 = -\sigma K(g(\cdot,u)).$$

Conversely, if u solves (14) then $u'' = -\sigma\left[g(t,u) - \langle g(\cdot,u),\varphi_1\rangle\varphi_1\right]$. Moreover, $\langle u,\varphi_1\rangle = \langle u - g(\cdot,u),\varphi_1\rangle$, and hence $\langle g(\cdot,u),\varphi_1\rangle = 0$. Thus, it suffices to prove that (14) is solvable for $\sigma = 1$. On the other hand, observe that if $\sigma = 0$ then (14) is equivalent to the equalities

$$u = k\varphi_1, \qquad \langle g(\cdot,u),\varphi_1\rangle = 0.$$

Let $T_\sigma : C([0,T]) \to C([0,T])$ be the compact operator defined by

$$T_\sigma u = \langle u - g(\cdot,u),\varphi_1\rangle\varphi_1 - \sigma K(g(\cdot,u) - \langle g(\cdot,u),\varphi_1\rangle\varphi_1),$$

and consider $F_\sigma(u) = u - T_\sigma u$. We claim that $F_1(u) = 0$ for some u, which corresponds to a solution of the original problem. Indeed, we shall prove that

1. $F_\sigma(u) \neq 0$ for $\|u\|_C$ large, and $\sigma \in [0,1]$.
2. $deg_{LS}(F_0, B_R, 0) = \pm 1$ for R large enough, where $B_R \subset C([0,T])$ is the ball of radius R centered at 0 and deg_{LS} denotes the Leray-Schauder degree.

We remark that once 1 and 2 are proved, the result follows from the homotopy invariance of the Leray-Schauder degree. In order to prove 1, assume first that $F_{\sigma_n}u_n = 0$, with $\|u_n\|_C \to \infty$ and $\sigma_n \in (0,1]$. Then $u_n'' + \sigma_n g(t,u_n) = 0$, and hence

$$0 = \langle u_n'',\varphi_1\rangle = -\sigma_n \int_0^T g(t,u_n)\varphi_1(t)dt.$$

On the other hand, we may write $u_n = v_n + \langle u_n,\varphi_1\rangle\varphi_1$, and from the previous lemma

$$\|v_n\|_C \leq \gamma\|v_n''\|_C = \gamma\|u_n''\|_C \leq \gamma\|g(\cdot,u_n)\|_C \leq M$$

for some constant M. We deduce that $c_n := \langle u_n, \varphi_1 \rangle \to \infty$. Taking a subsequence, assume for example that $c_n \to +\infty$, then by dominated convergence

$$0 = \int_0^T g(t, u_n)\varphi_1(t)dt = \int_0^T g(t, v_n + c_n\varphi_1)\varphi_1(t)dt \to \int_0^T g^+(t)\varphi(t)dt \neq 0,$$

a contradiction. On the other hand, if $F_0 u_n = 0$, with $\|u_n\|_C \to \infty$, then $u_n = c_n\varphi_1$ and $\int_0^T g(t, c_n\varphi_1(t))\varphi_1(t)dt = 0$. Applying dominated convergence as before, the claim follows.

Finally, we shall compute the Leray-Schauder degree $deg_{LS}(F_0, B_R, 0)$ for R large. As the range of T_0 is contained in $S := span\{\varphi_1\}$, it suffices to compute the Brouwer degree $deg_B(F_0|_S, B_R \cap S, 0)$. Furthermore, $F_0|_S$ can be identified with the mapping $\phi : \mathbb{R} \to \mathbb{R}$ given by $\phi(r) = \int_0^T g(t, r\varphi_1(t))\varphi_1(t)dt$. Again, by dominated convergence we have that

$$\lim_{r\to\pm\infty} \phi(r) = \int_0^T g^{\pm}(t)\varphi_1(t)dt.$$

Hence, $\phi(r).\phi(-r) < 0$ for $r \gg 0$, and it follows that $deg_B(F_0|_S, B_R \cap S, 0) = \pm 1$ for R large enough. □

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Dirección de los autores: Pablo Amster, Universidad de Buenos Aires, Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET), Argentina, pamster@dm.uba.ar. — Pedro P. Cárdenas, Departamento de Matemáticas - Universidad Tecnológica de Pereira, Colombia, ppablo@utp.edu.co.