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Hessians, warped products and eigenvalues

Nelson Castañeda

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Abstract

We use the Hessian - Weitzenböck formula to simplify the exposition of several well known theorems. We present a unified treatment of the theorems of Lichnerowicz - Obata, Reilly and Escobar regarding the first eigenvalue of the Laplacian on Manifolds.

Keywords: Hessians, warped products, eigenvalue bounds, Riemannian manifold

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1 Introduction

In this article we use the Hessian - Weitzenböck formula introduced in the author's Ph.D. thesis [4] to simplify the proofs of several well known theorems.

In section 2 we introduce the notation and prove the Hessian - Weitzenböck formula. We then use it to provide a simple proof of the Hessian comparison theorem based on the observation that the maximal and the minimal eigenvalues of the Hessian of a distance function t restricted to the orthogonal complement of $\partial_t = \nabla t$ satisfy a differential equation that involves certain radial curvature. A comparison theorem for ordinary differential inequalities required in the proof of the Hessian comparison theorem is provided in section 7.

In section 3 we discuss warped products and prove Theorem 3.1 that establishes the equivalence of three conditions of rotational symmetry about a point p : of the metric, of the Hessian of the distance from p, and of the radial curvatures.

In section 4 we provide Theorem 4.3 that plays a key role in the study of manifolds admitting concircular fields. From it we derive Obata's theorem that characterizes the sphere as the only complete manifold with nonconstant solutions to the equation $\mathcal{H}u + ug = 0$. We synthesize some results of Ishihara and Tashiro in Theorem 4.9. We prove that a manifold that admits $n-1$ linearly independent concircular vector fields is a manifold of constant sectional curvature. This generalizes a theorem of Tandai.

Section 5 deals with manifolds with boundary and section 6 contains a unified treatment of the theorems of Lichnerowicz - Obata, Reilly, and Escobar regarding the first nonzero eigenvalue of the Laplacian under a positive lower bound on the Ricci curvature and suitable conditions of convexity of the boundary.

2 Notation and preliminaries

(M, g) denotes a Riemannian manifold. All the manifolds in this article are assumed to be of class C^∞.

If X is a vector field, $X_p f$ will denote the directional derivative of f in the direction of the tangent vector X_p. All the vector fields in this article are assume to be of class C^∞ unless specified otherwise.

The differential df of a smooth function f is the one-form defined by $df(X) = Xf$.

The gradient of a function f will be denoted by ∇f. It is the vector field defined by the condition $g(\nabla f, X) = df(X) = Xf$ for all smooth vector fields X.

The covariant derivative $D_X Y$ of the vector field Y in the direction of the vector field X is the one defined by the Levi-Civita connection with the main properties that

$$Xg(Y, Z) = g(D_X Y, Z) + g(Y, D_X Z)$$

and

$$[X, Y] = D_X Y - D_Y X,$$

where $[X, Y]$ is the vector field that acts on functions by the formula

$$[X, Y]f = X(Yf) - Y(Xf).$$

$D_X Y$ can be computed from

$$\begin{aligned} 2g(D_X Y, Z) &= Xg(Y, Z) + Yg(X, Z) - Zg(X, Y) \\ &+ g(X, [Z, Y]) + g(Y, [Z, X]) + g(Z, [X, Y]). \end{aligned} \tag{1}$$

A local orthonormal frame $\{e_1, \ldots, e_n\}$ around a point p is a system of vector fields defined on a open neighborhood of p that forms an orthonormal basis for the tangent space at each point of the neighborhood.

If $\{e_1, \ldots, e_n\}$ is a local orthonormal frame and f is a smooth function then $\nabla f = \sum_{i=1}^n e_i(f) e_i$.

The Hessian of a function f, denoted by $\mathcal{H}f$, is the symmetric $2-$form associated to the covariant derivative of the gradient of f according to the formula

$$\mathcal{H}f(X, Y) = g(D_X \nabla f, Y) = g(X, D_Y \nabla f).$$

The norm of the Hessian of f is the scalar $|\mathcal{H}f|$ defined by $|\mathcal{H}f|^2 = \sum_{i=1}^n \lambda_i^2$, where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $\mathcal{H}f$.

The norm $|X|$ of a vector is given by $|X| = \sqrt{g(X, X)}$. An important observation is that $\nabla |\nabla f|^2 = 2D_{\nabla f} \nabla f$. Indeed, $g(X, \nabla |\nabla f|^2) = X(|\nabla f|^2) = 2g(D_X \nabla f, \nabla f) = 2g(D_{\nabla f} \nabla f, X)$, where the last identity follows from the symmetry of the Hessian.

Given a vector field X and a point $p \in M$ the mapping

$$(DX)_p : Y_p \in T_pM \to D_{Y_p}X \in T_pM$$

is a linear transformation of the tangent space at p that we call the covariant derivative of X at the point p. The divergence of a vector field X is a scalar field div$X$ that at each point p equals the trace of the covariant derivative of X at p. In other words, if $\{e_1, \ldots, e_n\}$ is a local orthonormal field, then

$$\mathrm{div}X = \sum_{i=1}^{n} g(D_{e_i}X, e_i).$$

The Laplacian of a function f is denoted by Δf and is defined to be the divergence of the gradient ∇f or, equivalently, the trace of $\mathcal{H}f$. If $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $\mathcal{H}f$ then $\Delta f = \sum_{i=1}^{n} \lambda_i$. It follows that $|\mathcal{H}f|^2 \geq \frac{(\Delta f)^2}{n}$ with equality if and only if $\lambda_1 = \cdots = \lambda_n$.

For any vector fields X, Y we denote by $R(X,Y)$ the operator $D_X D_Y - D_Y D_X - D_{[X,Y]}$. The Riemann curvature tensor is defined by $R(X,Y,Z,W) = g(R(Z,W)Y,X)$. Being a tensor, $R(X,Y,Z,W)$ at a point p depends only on the specific vectors X_p, Y_p, Z_p, and W_p. If X_p and Y_p are orthogonal and of unit length then $R(X_p, Y_p, X_p, Y_p)$ is called the sectional curvature along the given vectors and is usually denoted by $K(X,Y)$. If X_p is a unit vector, the Ricci curvature of the manifold at the point p and in the direction of X is the number $Ric(X) = \sum_{i=1}^{n-1} R(X, e_i, X, e_i)$, where $e_1, \ldots e_{n-1}$ is any orthonormal basis of the orthogonal complement of X_p in T_pM. We will write $Ric_M \geq c$ to indicate that $Ric(X) \geq c|X|^2$ for every vector in TM.

Given a point p in a Riemannian manifold M, and a tangent vector $X \in T_pM$ we will denote by γ_X the geodesic emanating from p with initial velocity $\gamma'(0) = X$. If (M, g) is a complete Riemannian manifold, the mapping $Exp_p : X \in T_pM \to \gamma_X(1) \in M$ is well defined and differentiable on the whole space T_pM [[14]; Proposition 2.3]. The differential of the mapping Exp_p at the origin $0 \in T_pM$, viewed as a map from T_pM to T_pM, is the identity map. It follows from the inverse function theorem that Exp_p maps an open neighborhood $\mathcal{O}$ of $0 \in T_pM$ diffeomorphically onto an open neighborhood $\mathcal{U}$ of p in M. In this case, if the neighborhood $\mathcal{O}$ is star-shaped from 0, it is called a normal neighborhood of 0 in T_pM and the corresponding neighborhood $\mathcal{U}$ of p is called a *normal neighborhood of* p in M. Clearly for every $q \in \mathcal{U}$ there is a unique minimizing geodesic in M that connects q to p and all the points on this geodesic are also contained in $\mathcal{U}$. If $e_1, \ldots, e_n$ is any orthonormal basis of T_pM, and $q \in \mathcal{U}$, then there exists a unique $n-$tuple $(x^1, \ldots, x^n)$ of real numbers such that $Exp_p^{-1}(q) = \displaystyle\sum_{i=1}^{n} x^i e_i$. The numbers $x^1, \ldots, x^n$ are called the *normal coordinates of* q.

The *geodesic ball of radius* a *and center* p in a manifold M is the set $B_p(a)$ of all the points $q \in M$ that are at a distance less than a away from p. The *injectivity radius of a manifold* M *at a point* p is the supremum of the set of real numbers a such that the geodesic ball $B_p(a)$ is contained in a normal neighborhood of p. It will be denoted by $inj_p(M)$.

Lemma 2.1. *Let* r *denote the distance function from a fixed point* p *in a Riemannian manifold* (M, g). *Then, in a neighborhood of* p, $\mathcal{H}(\frac{1}{2}r^2) = g + o(1)$, *where* $o(1)$ *is a symmetric two-tensor whose eigenvalues approach zero as* r *approaches zero.*

Proof. Let $x^1, \dots, x^n$ be normal coordinates around p. Then $r^2 = \sum_{i=1}^n (x^i)^2$, $\nabla(\frac{1}{2}r^2) = \sum_{i=1}^n x^i \nabla x^i$, and therefore, $\mathcal{H}(\frac{1}{2}r^2) = \sum_{i=1}^n dx^i \otimes dx^i + \sum_{i=1}^n x^i \mathcal{H} x^i$ $= g + (\sum_{i=1}^n dx^i \otimes dx^i - g) + \sum_{i=1}^n x^i \mathcal{H} x^i = g + o(1)$. □

We point out that if r is a distance function then $D_{\nabla r} \nabla r = 0$ which implies that ∇r is an eigenvector of the Hessian of r with eigenvalue zero. Therefore, the eigenvalues of $\mathcal{H} r$ restricted to the orthogonal complement of ∇r are also eigenvalues of the Hessian of r.

Corollary 2.2. *If* λ *is any of the eigenvalues of the Hessian of* r *restricted to the orthogonal complement of* ∇r *then* $\lambda = \frac{1}{r} + o(\frac{1}{r})$, *where* $o(\frac{1}{r})$ *is a function such that* $\lim_{r \to 0} r \cdot o(\frac{1}{r}) = 0$.

A local orthonormal frame $\{X_1, \dots, X_n\}$ is said to be *normal at* p if $(DX_i)_p = 0$ for $1 \le i \le n$. Around any point p of a manifold M one can always find a local orthonormal frame that is normal at p.

Lemma 2.3. *Let* u *be a smooth function and* X *a vector field defined on a neighborhood of a point* $p \in M$ *and such that* $D_{\nabla u} X(p) = 0$. *Then at* p

$$\mathcal{H}|\nabla u|^2(X_p, X_p) = 2\Big[|D_{X_p} \nabla u|^2 + R(\nabla u, X_p, \nabla u, X_p) + \nabla u(\mathcal{H}u(X, X)) \Big]. \quad (2)$$

Proof. First we observe that the condition $(D_{\nabla u} X)_p$ implies that at p

$$g(D_{\nabla u} D_X \nabla u, X_p) = \nabla u(D_X \nabla u, X) = \nabla u(\mathcal{H}u(X, X))$$

and

$$g(D_{[X,\nabla u]} \nabla u, X_p) = g(D_{X_p} \nabla u, [X, \nabla u]) = g(D_{X_p} \nabla u, D_{X_p} \nabla u) = |D_{X_p} \nabla u|^2.$$

Therefore,

$$\mathcal{H}|\nabla u|^2(X_p, X_p) = g(D_{X_p} \nabla |\nabla u|^2, X_p) = 2g(D_{X_p} D_{\nabla u} \nabla u, X_p) =$$

$$2\left[R(X_p,\nabla u,X_p,\nabla u)+g(D_{\nabla u}D_X\nabla u,X_p)+g(D_{[X,\nabla u]}\nabla u,X_p)\right]=$$
$$2\left[|D_{X_p}\nabla u|^2+R(X_p,\nabla u,X_p,\nabla u)+\nabla u(\mathcal{H}u(X,X))\right].$$

□

We will refer to formula (2) as the Hessian-Weitzenbock formula. This formula was introduced in [4] in connection to a simple proof of the Hessian comparison theorem.

Lemma 2.4. *([4]; Lemma 1) Let (M,g) be a complete Riemannian manifold and p a point in M. Let $r(x)$ denote the distance from x to p. Assume that q is not in the cut locus of p. Along the minimizing geodesic connecting q to p we consider the function λ equal to the maximal or to the minimal eigenvalue of $\mathcal{H}r$ restricted to the orthogonal complement of ∂_r. Then*

$$\lambda^2(x)+K(\partial_r,X_x)+(\partial_r)_x\lambda=0, \tag{3}$$

where X_x is a unit length eigenvector corresponding to the eigenvalue $\lambda(x)$. In other words, $D_{X_x}\partial_r=\lambda(x)X_x$.

Proof. We extend the vector X_x to a vector field X defined on a neighborhood of x and parallel along the geodesic connecting x to p. We apply the Hessian-Weitzenböck formula (2) to the function $u=r$ and the vector field X to obtain

$$0=\lambda^2(x)+K(\partial_r,X_x)+(\partial_r)_x\,g(D_X\partial_r,X).$$

If λ is the maximal eigenvalue then along the geodesic connecting x to p we have $g(D_X\partial_r,X)\le\lambda$ with equality at the point x. Therefore, $(\partial_r)_x\,g(D_X\partial_r,X)=(\partial_r)_x\lambda$ and this implies (3). Similar argument works for the minimal eigenvalue. □

Theorem 2.5. *(Hessian Comparison Theorem) Let (M^m,g) and (N^n,h) be two Riemannian manifolds, γ,α be geodesics parametrized by arc-length in M and N respectively, with $\gamma(0)=p$ and $\alpha(0)=q$, and such that they don't hit the cut locus of their respective initial points. Assume further that for every unit vector X orthogonal to $\gamma'(t)$ and every vector Y orthogonal to $\alpha'(t)$ we have $K^M(\gamma'(t),X)\ge K^N(\alpha'(t),Y)$. Then $\mathcal{H}^Mt(X,X)\le\mathcal{H}^Nt(Y,Y)$ for all unit vectors X orthogonal to $\gamma'(t)$ and Y orthogonal to $\alpha'(t)$. Here t denotes the distance function from p in M and also the distance function from q in N.*

Proof. Let λ be the maximal eigenvalue of $\mathcal{H}^Mt$ restricted to the orthogonal complement of $\gamma'(t)$, and μ be the minimal eigenvalue $\mathcal{H}^Nt$ restricted to the orthogonal complement of $\alpha'(t)$. According to Lemma 2.4 these functions satisfy the ordinary differential equations

$$\lambda^2(t)+K^M(\gamma'(t),X)+\frac{d\lambda(t)}{dt}=0$$

D_XY	TM	TN
TM	$D^M_X Y$	$\frac{Xf}{f}Y$
TN	$\frac{Yf}{f}X$	$D^N_X Y - k(X,Y)\frac{\nabla f}{f}$

Table 1: Main covariant derivatives

$$\mu^2(t) + K^N(\gamma'(t), Y) + \frac{d\mu(t)}{dt} = 0$$

for appropriate unit vector fields X, Y. Besides, by Corollary 2.2, they satisfy the initial conditions $\lambda(t), \mu(t) = \frac{1}{t} + o(\frac{1}{t})$ as $t \to 0^+$. It follows from a comparison theorem for ODE's that $\lambda(t) \leq \mu(t)$ for all t. (See Lemma 7.4 and the remark 3. after it.) □

The Weitzenbock formula

$$\Delta|\nabla u|^2 = 2\left[|\mathcal{H}u|^2 + Ric(\nabla u, \nabla u) + \langle \nabla u, \nabla \Delta u\rangle\right] \tag{4}$$

follows immediately from the Hessian-Weitzenböck formula when the latter is applied to all the vectors of a local orthornormal frame normal at p.

3 Warped Products

Let M and N be two differentiable manifolds. Then for any points $p \in M$ and $q \in N$ the tangent space $T_{(p,q)}M \times N$ is identified with the direct sum $T_pM \oplus T_qN$ in a canonic way. If X is a vector field on $M \times N$ then the notation X_M, X_N will refer to the components of the vector field X relative to the mentioned identification.

We say that a vector field X is independent of M if for every point $(p,q) \in M \times N$ the vector $X_{(p,q)}$ equals $\frac{d}{dt}_{|t=0}\gamma(t)$ for a curve $\gamma(t) = (p, \alpha(t))$, where α is a curve on N with $\alpha(0) = q$. Vector fields independent of N are defined in a similar way. When X is independent of N we will say that it belongs to TM, and when it is independent of M that it belongs to TN.

Let (M^m, g), and (N^n, h) be two Riemannian manifolds, and f a positive function defined on M. The warped product $M \times_f N$ is the Riemannain manifold $(M \times N, k)$, with k defined by $k(X,Y) = g(X_M, Y_M) + f^2 h(X_N, Y_N)$. Using formula (1) one can compute the covariant derivatives D_XY when the vector fields involved are in TM or in TN. They are summarized in Table 1.

Using this table and the definition of sectional curvature one can compute the sectional curvatures along pairs of orthonormal vectors that are independent from M or from N. They are summarized in Table 2, where $K(X,Y)$ is

$K(X,Y)$	TM	TN
TM	$K^M(X,Y)$	$-\left\lvert\frac{Xf}{f}\right\rvert^2 - k(D^M_X \frac{\nabla f}{f}, X)$
TN	$\dots$	$-\left\lvert\frac{\nabla f}{f}\right\rvert^2 + \frac{1}{f^2}K^N(fX, fY)$

Table 2: Main sectional curvatures

used to denote the sectional curvature $R(X, Y, X, Y)$, and K^M, K^N to denote the corresponding sectional curvatures on the manifolds M and N.

We will say that a manifold (M^n, g) is *rotationally symmetric about* p, if there exists an interval I and a positive function f defined on I such that $(M \setminus \{p\}, g)$ is isometric to the warped product $I \times_f S^{n-1}$, where S^{n-1} is the sphere of constant sectional curvature 1. The map f must satisfy the conditions $f(t) \to 0, f'(t) \to 1$ as $t \to 0$, for the metric g to be smooth at p.

Theorem 3.1. *Let (M^n, g) be a Riemannian manifold, and $B_p(a)$ the geodesic ball of radius a centered at p. Assume further that a is not greater than the injectivity radius of M at p. We denote by $t(q)$ the distance function from q to p. Then the following three statements are equivalent*

1. *There exists a function $\varphi(t)$ defined on $(0, a)$ such that $R(\partial_t, X, \partial_t, X) = \varphi(t)$ for all unit vectors $X \in TB_p(a)$, that are orthogonal to ∂_t.*

2. *There exists a function $\psi(t)$ defined on $(0, a)$ such that $\mathcal{H}t(X, X) = \psi(t)$ for all unit vectors $X \in TB_p(a)$ that are orthogonal to ∂_t.*

3. *The manifold $(B_p(a), g)$ is rotationally symmetric about p. More precisely, the manifold $(B_p(a) \setminus \{p\}, g)$ is isometric to the warped product $I \times_f S^{n-1}$, where $I = (0, a)$, f is a positive function on I, and S^{n-1} is the sphere of constant sectional curvature* 1.

The functions φ, ψ, and f are related by $\varphi = -\frac{f''}{f}$, and $\psi = \frac{f'}{f}$.

Proof. It is obvious that (3) implies both (1) and (2). Moreover, from Table 1 we see that if X is a unit vector orthogonal to ∂_t, then $D_X \nabla t = \frac{\partial_t f}{f} X$. This implies that $\psi = \frac{f'}{f}$.

If (1) holds, then according to Lemma 2.4, along the geodesics emanating from p the maximal and the minimal eigenvalues of $\mathcal{H}t$ restricted to the orthogonal complement of ∂_t satisfy the same ordinary differential equation $\lambda' = -\lambda^2 - \varphi(t)$, with the same asymptotic initial conditions as $t \to 0^+$. Lemma 7.4 and the remarks after it imply that the above mentioned maximal and minimal eigenvalues coincide, and therefore that (2) holds.

We assume now (2) and will prove (3). Let $I = (0, a)$, and consider the hypersurface S given by $t = b$, where b is a positive constant less than a. For every point $q \in S$ there is a unique unit vector $\xi = \xi(q) \in T_pM$ such that $q = Exp_p(b\xi)$. We use the diffeomorphism $(t, q) \in I \times S \longrightarrow Exp_p(t\xi(q)) \in B_p(a) \setminus \{p\}$ to identify $B_p(a) \setminus \{p\}$ with $I \times S$. Let X_q, Y_q be vectors tangent to S at some point q. We extend these vectors to obtain two vector fields X, Y independent of I. Then, along the normal geodesic γ connecting p to q in $B_p(a)$ the function $w(t) := g(X_{\gamma(t)}, Y_{\gamma(t)})$ satisfies the equation

$$\begin{aligned} w'(t) &= \nabla t\,(g(X,Y)) = g(D_{\nabla t}X, Y) + g(X, D_{\nabla t}Y) \\ &= g(D_X\nabla t, Y) + g(X, D_Y\nabla t) = 2\mathcal{H}t(X,Y) = 2\psi g(X,Y) = 2\psi w(t) \end{aligned}$$

with initial condition $w(b) = g(X_q, Y_q)$. Integrating this equation we obtain $w(t) = w(b)\exp\left(-\int_t^b 2\psi(s)\,ds\right)$. This shows that $(B_p(a) \setminus \{p\})$ is isometric to a warped product $I \times_F S$, where $F(t) = \exp\left(-\int_t^b \psi(s)\,ds\right)$. The function $F(t)$ approaches zero as t approaches zero because $\psi(t)$ behaves like $\frac{1}{t}$ near zero. The manifold S is diffeomorphic to the sphere in the Euclidean space because $B_p(a)$ is contained in a normal neighborhood of p. We only need to prove now that S with the induced metric from g is isometric to a round sphere, i.e., that it is a manifold of constant sectional curvature. Assume now that X_q, Y_q are orthonormal and tangent to S at q, and that X and Y are extensions of X_q, Y_q that are independent of I. Along the geodesic connecting q to p the vector fields $\widetilde{X} = \frac{1}{F}X, \widetilde{Y} = \frac{1}{F}Y$ are orthonormal. From Table 2 we have

$$K(\widetilde{X}, \widetilde{Y}) = \frac{1}{F^2}\left[K(X_q, Y_q) - |F'(t)|^2\right]. \tag{5}$$

Letting t approach zero, we conclude that $K(X_q, Y_q) = \lim_{t \to 0^+} |F'(t)|^2$, which shows that the sectional curvatures are constant along S. In addition to this the principal curvatures of S are constant and therefore, by the Gauss equation, S with the metric induced by g is a manifold of constant sectional curvature. □

The arguments in the proof of Theorem 3.1 can be applied with very minor changes to obtain the following theorem

Theorem 3.2. *Let (M^n, g) be a Riemannian manifold, S an oriented hypersurface in M with constant principal curvatures. We denote by $t(q)$ the oriented distance from q to S, and let $B_S(-b, a) = \{x \in M : -b < t(x) < a\}$. Assume further that for every point $q \in B_S(-b, a)$ there is only one minimizing geodesic from q to S. Then the following three statements are equivalent (1) There exists a function $\varphi(t)$ defined on $(-b, a)$ such that $R(\partial_t, X, \partial_t, X) = \varphi(t)$ for all unit vectors $X \in TB_S(-b, a)$, that are orthogonal to ∂_t. (2) There exists a function $\psi(t)$ defined on $(-b, a)$ such that $\mathcal{H}t(X, X) = \psi(t)$ for all* *unit vectors $X \in TB_S(-b,a)$ that are orthogonal to ∂_t. (3) The manifold $(B_S(-b,a),g)$ is isometric to the warped product $I \times_f S$ where I is the interval $(-b,a)$ and $f(t) = \exp\left(\int_0^t \psi(s)\,ds\right)$.*

4 Constant Energy and Distinguished Vector Fields

In this section we introduce the notion of manifolds with constant energy, prove Theorem 4.3 from which Obata's theorem follows readily and discuss certain classes of vector fields whose covariant derivatives have special properties.

Proposition 4.1. *Let u be a smooth function defined on (M,g). Let $\Omega = \{x \in M : \nabla u(x) \neq 0\}$. The integral curves of the unit vector field $T = \frac{\nabla u}{|\nabla u|}$ are geodesics in Ω if and only if ∇u is an eigenvector of $D\nabla u$ at every point in Ω.*

Proof. We observe that

$$D_T T = \frac{1}{|\nabla u|}\left[\nabla u\left(\frac{1}{|\nabla u|}\right)\nabla u + \frac{1}{|\nabla u|}D_{\nabla u}\nabla u\right]$$

from where it is clear that ∇u is an eigenvector of $D\nabla u$ if and only if $D_T T$ is proportional to ∇u and hence proportional to T. Since T is of unit length this is possible if and only if $D_T T = 0$. □

We say that a smooth function defined on an open subset Ω of (M,g) has **a constant energy in** Ω if there exists a real valued function Φ such that $|\nabla u|^2 + \Phi(u) \equiv C$ for some constant C.

Corollary 4.2. *If a function u has a constant energy on Ω and has no critical points there then the integral curves of $T = \frac{\nabla u}{|\nabla u|}$ are geodesics.*

Proof. Since $|\nabla u|^2 + \Phi(u) \equiv C$ in Ω then $D_{\nabla u}\nabla u = \frac{1}{2}\nabla|\nabla u|^2 = -\frac{1}{2}\Phi'(u)\nabla u$ which shows that ∇u is an eigenvector of $D\nabla u$ and consequently the integral curves of T are geodesics in Ω. □

Theorem 4.3. *Let (M,g) be a Riemannian manifold, and u a smooth function on M that satisfies the equation $\mathcal{H}u = \varphi(u)g$ for some differentiable real-valued function φ. Then (1) The integral curves of ∇u are geodesics in the set of regular points of u. (2) $|\nabla u|$ is constant along level surfaces of u. (3) If X is a unit vector orthogonal to $T = \frac{\nabla u}{|\nabla u|}$ then $R(X,T,X,T) = -\varphi'(u)$. (4) If p is an isolated point of local maximum or local minimum of u then $(B_p(a),g)$ is rotationally symmetric about p for every $a < inj_b(M)$.*

Proof. Let $\Phi(t)$ be any antiderivative of $-2\varphi(t)$. Then

$$\nabla\left[|\nabla u|^2+\Phi(u)\right]=2D_{\nabla u}\nabla u-2\varphi(u)\nabla u=0$$

which shows that the function u is a function of a constant energy $|\nabla u|^2+\Phi(u)\equiv C$. Thus (1) follows from Corollary 4.2 and (2) from the constancy of energy.

(3) follows directly from the Hessian - Weitzenböck formula when the terms involved in it are expressed through $\varphi(u)$ as follows: $\mathcal{H}|\nabla u|^2(X,X)=-\Phi'(u)\mathcal{H}u(X,X)=2\varphi^2(u)$, $|D_X\nabla u|^2=\varphi^2(u)$, and $\nabla u\mathcal{H}u(X,X)=\nabla u\varphi(u)=\varphi'(u)|\nabla u|^2$.

Assume that $u(p)$ is a local maximum. Then, for a and δ sufficiently small the set $S=\{x\in B_p(a):u(x)=u(p)-\delta\}$ is an oriented closed hypersurface. The geodesics perpendicular to S must concur at the point p. This and (2) imply that the function u is a function of t, where t is the distance from p, and this will hold as long as $t\le inj_b(M)$. But then the radial curvatures $R(X,T,X,T)=-\varphi'(u(t))$ are also functions of t and hence (4) follows from Theorem 3.1.

□

Corollary 4.4. *(Obata's Theorem [9]) Let (M,g) be a complete Riemannian manifold and let u be a smooth non constant function on M such that $\mathcal{H}u=-ug$. Then M is isometric to the unit sphere (S^n,can) of constant sectional curvature* 1.

Proof. By Theorem 4.3 the integral curves of ∇u are geodesics and along any of these geodesics $\gamma(t)$, parametrized by arc-length, the function u satisfies the equation $u''+u=0$. It follows that u must have at least one isolated local maximum at some point p. The geodesics $\gamma(t)$ emanating from p are integral curves of ∇u and therefore, any two of them cannot meet at a point q unless $\nabla u(q)=0$. It follows that geodesics emanating from p minimize distance as long as $t<\pi$. By parts (3) and (4) of Theorem 4.3, $(B_p(\pi),g)$ is rotationally symmetric with radial curvatures identically equal to 1. Theorem 3.1 implies now that the eigenvalues of the Hessian $\mathcal{H}t$, of the distance function from p restricted to the orthogonal complement of ∇t satisfy the equation $\lambda'+\lambda^2=-1$ with $\lambda(t)=\frac{1}{t}+o(\frac{1}{t})$ as t approaches zero. Therefore, $\lambda(t)=\cot t$. It follows again from Theorem 3.1 that $(B_p(a)\setminus\{p\},g)$ is isometric to $(0,a)\times_f S^{n-1}$, where $f=\sin t$. It follows from here that (M,g) is isometric to (S^n,can). □

A vector field X is called *concurrent* if $D_YX=Y$ for every vector field Y. The following theorem follows from results in [7]. It was clearly stated and proven in [1] and in [3]. We provide a new proof.

Theorem 4.5. *Assume that a complete Riemannian manifold (M^n, g) admits a concurrent vector field X. Then M is isometric to the Euclidean space R^n and $X = r\nabla r$, where r is the distance function from a point $p \in M$.*

Proof. The function $u = \frac{1}{2}|X|^2$ satisfies $\nabla u \equiv X$ and $\mathcal{H}u = g$. By Theorem 4.3 the integral curves of ∇u are geodesics and along any of these geodesics $\gamma(t)$, parametrized by arc-length, the function $\varphi(t) = u(\gamma(t))$ satisfies the equation $\varphi'' = 1$. It follows that φ has a minimum at some real number τ. At $p = \gamma(\tau)$ we have $\nabla u(p) = \varphi'(\tau) = 0$. So p is a critical point for u. It is clear that u has an isolated minimum equal to zero at p. The geodesics emanating from p minimize distance up to infinity, hence the exponential map at p is a diffeomorphism. By Theorem 4.3 the radial curvatures are all equal to zero. By Theorem 3.1 $(M \setminus \{p\}, g)$ is isometric to $(0, \infty) \times_f S^{n-1}$, where $0 = -\frac{f''(t)}{f(t)}$ and $f(t) = t + o(t)$ as t approaches zero. It follows that $f(t) = t$ and therefore that (M, g) is isometric to the Euclidean space $\mathbf{R}^n$. If $(x_1, \ldots, x_n)$ are Cartesian coordinates around p then the function u satisfies $\frac{\partial u}{\partial x_i \partial x_j} = \delta_{ij}$ with initial conditions $\frac{\partial u}{\partial x_i} = 0, u(0) = 0$. It follows that $u = \frac{1}{2}r^2$ where r is the distance from p, and, consequently, $X = r\nabla r$. □

We will call a vector field X *concircular* if there exists a smooth function v such that for every vector field Y, $D_Y X = vY$. We call v the weight of X. A smooth function u is called a *concircular scalar field* if there exists a smooth function v such that $\mathcal{H}u = vg$. We will call v the weight of u. Concircular fields are associated to conformal transformation that preserve geodesic circles and were introduced by Yano in the forties.

Proposition 4.6. *Let X be a concircular vector field with weight v. Then the gradient of the function v is proportional to X whenever $X \neq 0$.*

Proof. Let $u = |X|^2$. Then $\nabla u = 2vX$, and

$$\mathcal{H}u(Y, Z) = g(D_Y \nabla u, Z) = g(D_Y(2vX), Z) = 2Y(v)g(X, Z) + 2vg(vY, Z).$$

Hence $Y(v)g(X, Z) = Z(v)g(X, Y)$ for all vectors Y, Z in virtue of the symmetry of the Hessian. This is possible at a point where $X \neq 0$ if and only if ∇v is proportional to X. □

Corollary 4.7. *Assume that a vector field X is defined on a neighborhood $\mathcal{U}$ of a point p, and that it is concircular with non vanishing weight v. Assume also that $X_p \neq 0$. Then there exists a real valued function F such that, perhaps in a smaller neighborhood of p, the vector field X is the gradient of the function $w = F(|X|^2)$. The function w is then a concircular scalar field with the same weight v.*

Proof. Since $\nabla|X|^2 = 2vX$ and ∇v is proportional to X nearby p, then $\nabla|X|^2$ is proportional to ∇v and therefore there is a function φ such that $v = \varphi(|X|^2)$ in some neighborhood of p. Let F be any antiderivative of $\frac{1}{2\varphi}$. Then $\nabla F(|X|^2) = F'(|X|^2)\nabla|X|^2 = 2vF'(|X|^2)X = 2\varphi(|X|^2)F'(|X|^2)X = X$. □

If u is a function such that $\mathcal{H}u = vg$ for some function v then ∇u is concircular with $D_Y\nabla u = vY$ for every vector field Y. Therefore we have

Corollary 4.8. *Let $\mathcal{H}u = vg$ and $\nabla u(p) \neq 0$. Then there exists a real valued function φ such that $v = \varphi(u)$ on a neighborhood of p.*

The following theorem is contained in the works of Ishihara and Tashiro [7], [10].

Theorem 4.9. *(Ishihara-Tashiro) Let (M, g) be a connected compact Riemannian manifold, and $u : M \to \mathbf{R}$ a nonconstant smooth function such that $\mathcal{H}u = vg$ for some function v. Then the function u has exactly two isolated critical points p, q, and (M, g) is rotationally symmetric. More precisely, if S is any of the nonsingular level surfaces of u then $(M \setminus \{p, q\}, g)$ is isometric to the warped product $I \times_{\frac{|\nabla u|}{c}} S$, where I is some open bounded interval, c is the common value of the gradient of u at the points of S, and S with the induced metric from g is isometric to a sphere of constant sectional curvature. In particular, the manifold (M, g) is conformally equivalent to (S^n, can).*

Proof. Let S be any nonsingular level surface of u, $\widehat{S}$ one of its connected components, and Ω the connected component of the set of regular points of u that contains $\widehat{S}$. By Corollary 4.8 and Theorem 4.3 we know that $\mathcal{H}u = \varphi(u)g$, the integral curves of ∇u are geodesics that minimize the distance t to $\widehat{S}$ in Ω, and $\mathcal{H}t = \dfrac{\varphi(u)}{|\nabla u|}g = \dfrac{u''(t)}{u'(t)}g$. Besides, the principal curvatures of $\widehat{S}$ are constant. Therefore Theorem 3.2 applies and we conclude that (Ω, g) is isometric to $I \times_{\frac{u'(t)}{u'(0)}} \widehat{S}$, where I is some open interval $(-b, a)$. It follows that all the geodesics perpendicular to $\widehat{S}$ converge at a critical point p at distance a on one side, and to a critical point q at distance b on the opposite side of Ω relative to $\widehat{S}$. Since the directions of geodesics emanating from p or from q and hitting $\widehat{S}$ perpendicularly form an open and closed subset of directions and therefore they are all, we conclude that actually $\Omega = M \setminus \{p, q\}$ and $\widehat{S} = S$. Finally, Theorem 3.1 implies that S with the metric induced from g is isometric to a sphere of constant sectional curvature. □

We observe that Obata's theorem can be derive as a corollary to Ishihara-Tahshiro's theorem.

If in Theorem 4.9 the manifold is complete but not compact then the function u has at most one critical point. For a classification of complete manifolds admitting concircular scalar fields we refer the reader to the article [10].

Tandai [11] proved that a manifold admiting $(n-1)$ linearly independent solutions of $\mathcal{H}u + c^2ug = 0$ is a manifold of constant sectional curvature c^2. Here we provide a generalization of Tandai's theorem.

Theorem 4.10. *Let (M^n, g) be a Riemannian, not necessarily complete, manifold of dimension $n \geq 3$ that admits $n-1$ concircular vector fields $Y_1, \ldots Y_{n-1}$ such that $(Y_1)_q, \ldots, (Y_{n-1})_q$ are linearly independent vectors in T_qM for every $q \in M$. Then M is a manifold of constant sectional curvature.*

Proof. Let p be any point of M. Let $e_1, e_2, \ldots, e_n$ be an orthonormal basis of T_pM, where e_n is orthogonal to each of $(n-1)$ linearly independent concircular vector fields at p. We can choose $n-1$ linearly independent concircular vector fields $X_1, \ldots, X_{n-1}$ such that $(X_i)_p = e_i$ for $i = 1, \ldots, n-1$. According to Corollaries 4.7 and 4.8 each of these vector fields has an associated concircular scalar field w_i with $\mathcal{H}w_i = \varphi_i(w_i)g$, where the φ_i are certain real-valued functions. By Theorem 4.3 we have $K(e_i, e_j)(p) = -\varphi_i'(w_i(p)) = -\varphi_j'(w_j(p))$ for $1 \leq i < j \leq n-1$. Therefore, $\varphi'(w_1(p)) = \cdots = \varphi'(w_{n-1})$. This readily implies that all the sectional curvatures at p are equal to the same number. Schur's lemma implies that (M, g) is a manifold of constant sectional curvature. □

Gallot [6] provided examples of manifold of variable curvature and dimension n that admit $(n-2)$ linearly independent solutions to the equation $\mathcal{H}u + ug = 0$.

5 Manifolds with boundary

Let $(M, \partial M, g)$ be a Riemannian manifold with smooth boundary. If X is a vector field on M such that for each $p \in \partial M$ the vector X_p is tangent to the boundary we will say that X is a vector field tangent to the boundary. We recall that if X, Y are vector fields tangent to the boundary then $D_XY = D_X^{\partial M}Y + h(X,Y)\eta$ where, D is the covariant derivative on M, $D^{\partial M}$ is the covariant derivative on ∂M, η is the outer normal vector to ∂M, and h is a symmetric bilinear form field called the second fundamental form. The operator $-D\eta : X \in T_p(\partial M) \to D_{X_p}\eta \in T_p(\partial M)$ is symmetric with $g(-D_X\eta, Y) = h(X,Y)$. The mean curvature of the boundary at p is the number $H(p) = -\sum_{i=1}^{n-1} h(e_i, e_i)$, where $e_1, \ldots, e_{n-1}$ is any local orthonormal frame of ∂M around p. In other words, H is the trace of $D\eta$ when this is viewed as an operator acting on the tangent bundle of the boundary.

When the second fundamental form vanishes everywhere in ∂M it is said that this boundary is totally geodesic. In this case it is known that any geodesic emanating from a point p of the boundary with initial velocity tangent to the boundary will remain on the boundary forever. Therefore, all the geodesics coming from the interior hit the boundary transversally.

Lemma 5.1. *Let (M^n, g) be a compact manifold with smooth totally geodesic boundary ∂M. Let u be a nonconstant smooth function such that $\mathcal{H}u = -ug$, and $p \in M$ be a point of local maximum such that $\nabla u(p) = 0$. Then M is isometric to the unit upper hemisphere (S^n_+, can).*

Proof. If p is in the interior of the manifold one proceeds as in the proof of Corollary 4.4 to show that M can be isometrically embedded into the unit sphere (S^n, can). Since ∂M is then a totally geodesic hypersurface in S^n one concludes that M is isometric to (S^n_+, can). The case when p is a boundary point is similar. One needs to observe that the geodesic emanating from p towards the interior cannot hit the boundary as long as $|\nabla u| \neq 0$. This implies that they all meet at a point q that is at distance π away from p. From here, and from the expression of the metric computed as in Corollary 4.4 one obtains the desired result. □

Proposition 5.2. *(Reilly [12]) Let $(M^n, \partial M, g)$ be a compact Riemannian manifold with smooth boundary. Assume that the mean curvature H of the boundary is everywhere nonnegative. If there exists a nonconstant smooth function u such that $\mathcal{H}u = -ug$ on M, and $u \equiv 0$ on the boundary of M then ∂M is totally geodesic and M is isometric to the upper hemisphere (S^n_+, can).*

Proof. This follows from the same arguments used in the proof of Corollary 4.4. Alternatively, one can prove that the boundary is totally geodesic by observing that $\eta = \pm\frac{\nabla u}{|\nabla u|}$ from where it readily follows that $D_X \eta = 0$ for any X tangent to the boundary. Then Lemma 5.1 applies yielding the desired result. □

Finally we state Escobar's generalization of Obata's theorem. The proof is embedded in the proof of Theorem 6.4.

Lemma 5.3. *(Escobar [5]) Let $(M^n, \partial M, g)$ be a compact Riemannian manifold with smooth boundary. Assume that the boundary is convex in the sense that $h(X, X) \leq 0$ for every vector X tangent to the boundary of M. If there exists a nonconstant smooth function u such that $\mathcal{H}u = -ug$ on M, and $\frac{\partial u}{\partial \eta} \equiv 0$ on the boundary of M then ∂M is totally geodesic and M is isometric to the upper hemisphere (S^n_+, can).*

6 From Lichnerowicz to Escobar

In this section we present a unified treatment of the theorems of Lichnerowicz - Obata, Reilly, and Escobar regarding the Laplacian on closed manifolds, and the Dirichlet and Neumann Laplacians on manifolds with boundary. These theorems characterize the sphere, in the first case, and the hemisphere, in the other two cases, as the unique manifolds with the smallest possible (nonzero) first eigenvalue given a lower bound on the Ricci curvature and suitable convexity conditions on the boundary. For simplicity we assume the lower bound of the Ricci curvature to be $(n-1)$, where n is the dimension of the manifold. The general statements follow immediately by scaling.

Theorem 6.1. *Let (M^n, g) be a complete connected Riemannian manifold with smooth or empty boundary ∂M and η the exterior normal vector to ∂M. Let $Ric_M \geq (n-1)$, let u be a smooth nonconstant function on $\overline{M}$, and let λ a positive number such that*

$$\Delta u + \lambda u = 0 \qquad \text{on } M, \tag{6}$$

$$\int_{\partial M} u \frac{\partial u}{\partial \eta} \leq 0, \tag{7}$$

and

$$\int_{\partial M} \eta(|\nabla u|^2) \leq 0. \tag{8}$$

Then $\lambda \geq n$ with equality if and only if the function u satisfies $\mathcal{H}u = -ug$.

Proof. Using the Weitzenböck formula

$$\Delta|\nabla u|^2 = 2\Big[|\mathcal{H}u|^2 + Ric(\nabla u, \nabla u) + g(\nabla \Delta u, \nabla u)\Big],$$

the equation (6), and the inequality $|\mathcal{H}u|^2 \geq \frac{(\Delta u)^2}{n}$ we obtain

$$\Delta|\nabla u|^2 \geq 2\Big[\frac{(\Delta u)^2}{n} + Ric(\nabla u, \nabla u) - \lambda|\nabla u|^2\Big]. \tag{9}$$

On the other hand, we observe that in virtue of (7)

$$\int_M (\Delta u)^2 = -\lambda \int_M u\Delta u = \lambda \int_M |\nabla u|^2 - \lambda \int_{\partial\Omega} u\frac{\partial u}{\partial \eta} \geq \lambda \int_\Omega |\nabla u|^2. \tag{10}$$

Integrating (9) over M taking into account (8), the lower bound of the Ricci curvature, and inequality (10), we obtain

$$0 \geq \Big[(n-1) + \lambda(\frac{1}{n} - 1)\Big] \int_M |\nabla u|^2 = (n-1)(1 - \frac{\lambda}{n}) \int_M |\nabla u|^2.$$

It follows that $\lambda \geq n$. The first part of the theorem is proven.

Assume now that $\lambda = n$. Then all the inequalities above are actually equalities. In particular $|\mathcal{H}u|^2 = \frac{(\Delta u)^2}{n}$ which is possible only if the Hessian of u is diagonal everywhere. In this case all the eigenvalues of $\mathcal{H}u$ must be equal to $-u$ because of equation (6). In other words, $\mathcal{H}u = -ug$. □

The theorem of Lichnerowicz and Obata is now an immediate corollary of Theorem 6.1 and Theorem 4.4.

Theorem 6.2. *(Lichnerowicz - Obata [8], [9]) Let M^n be a complete Riemannian manifold with $Ric_M \geq (n-1)k$. Then the first eigenvalue λ_1 of the Laplace operator is greater than or equal to nk with equality if and only if M is isometric to the Euclidean sphere of constant sectional curvature k.*

The bound on the eigenvalue is due to Lichnerowicz and the characterization of the extremal case is due to Obata.

Theorem 6.3. *(Reilly [12]) Let (M, g) be a Riemannian manifold with smooth boundary ∂M. Assume that $Ric_M \geq (n-1)$, and that the mean curvature H of the boundary is nonnegative. Then the first eigenvalue λ_1 of the Laplacian with Dirichlet boundary condition is greater than or equal to n with equality if and only if M is isometric to the unit upper hemisphere (S^n_+, can).*

Proof. Condition (7) is obviously met. We will prove that the condition (8) is also satisfied. First we observe that we may assume that u is positive in the interior of M. Let $e_1, \ldots, e_{n-1}$ be a local orthonormal frame tangent to ∂M. Then $\eta = -\frac{\nabla u}{|\nabla u|}$ and $\eta|\nabla u|^2 = -2\mathcal{H}u(\eta, \nabla u) = -2|\nabla u|\mathcal{H}u(\eta, \eta) = -2|\nabla u| \left(\Delta u - \sum_{i=1}^{n-1} h(e_i, e_i)\right) = -2|\nabla u| H \leq 0$. Theorem 6.1 applies and says that $\lambda_1 \geq n$ with equality if and only if $\mathcal{H}u = -ug$. In this case the manifold is isometric to (S^n_+, can) thanks to Proposition 5.2. □

Theorem 6.4. *(Escobar [5]) Let (M, g) be a Riemannian manifold with smooth boundary ∂M. Assume that $Ric_M \geq (n-1)$, and the boundary is strongly convex in the sense that $h(X, X) \leq 0$ for every $X \in T\partial M$. Then the first nonzero eigenvalue ν_1 of the Laplacian with Neumann boundary condition is greater than or equal to n with equality if and only if M is isometric to the unit upper hemisphere (S^n_+, can).*

Proof. Condition (7) is obviously met. On the other hand, the gradient of u is tangential to the ∂M because of the Neumann boundary condition. $\eta|\nabla u|^2 = 2g(D_{\nabla u}\nabla u, \eta) = 2h(\nabla u, \nabla u) \leq 0$, where the last inequality follows from the convexity of the boundary. This shows that (8) holds and Theorem 6.1 applies saying that $\nu_1 \geq n$ with equality if and only if $\mathcal{H}u = -ug$. Let p be the point of ∂M where u attains the absolute maximum relative to all the values of u in ∂M. It is clear that $\nabla u(p) = 0$ and that p is an isolated local maximum for u. As in the proof of Corollary 4.4 we have that all geodesics emanating from p are integral curves of ∇u. We observe that, because ∇u is tangential to the boundary, any geodesic starting at a point q of the boundary with initial velocity proportional to ∇u will remain on the boundary forever. It follows that the geodesics emanating from p with initial velocity toward the interior of the manifold cannot meet the boundary as long as $|\nabla u| \neq 0$. Therefore, they all will meet at a point q at a distance π away from p. It follows from the expression of the metric computed as in Corollary 4.4 that M is isometric to (S^n_+, can). □

7 Comparison Theorems For Differential Inequalities

In this section we prove two lemmas on comparison of solution of ordinary differential inequalities that are important in applications to comparison geometry. These lemmas and some applications of them are contained in an unpublished manuscript [2].

Lemma 7.1. *Let u and v be two functions in $\mathbf{C}^1(0,T) \cap \mathbf{C}([0,T])$ for some $T > 0$. Assume further that $u(0) = v(0) = z$, and that for all $t \in (0,T)$*

$$u'(t) \geq F(u(t)) \tag{11}$$

and

$$v'(t) \leq F(v(t)), \tag{12}$$

where $F : \mathbf{R} \to \mathbf{R}$ is a continuous function that is locally Lipschitz at z in the sense that there exist two positive numbers $\delta = \delta(z)$, $L = L(z)$ such that

$$|F(x) - F(y)| \leq L|x - y|$$

whenever x and y belong to the open interval of radius δ centered at z. If $u(t) \neq v(t)$ for all $t \in (0,T)$, then

$$u(T) > v(T)$$

Proof. We consider the function $w(t) = u(t) - v(t)$ which equals zero at $t = 0$. We will prove that $w(t)' \geq 0$ for all $t \in (0,T)$ which implies that $u(T) > v(T)$ because $u(t) \neq v(t)$ for all $t \in (0,T)$.

We subtract inequality (12) from inequality (11) to obtain, for all $t \in (0,T)$

$$w'(t) + a(t)w \geq 0, \tag{13}$$

where $a(t) = -\dfrac{F(u(t)) - F(v(t))}{u(t) - v(t)}$ which is continuous on $(0,T)$. This function $a(t)$ is also bounded near zero because of the Lipschitz property of F at z. Indeed, if t is sufficiently small, then $u(t)$ and $v(t)$ will both lie inside the interval $(z - \delta, z + \delta)$ and therefore,

$$|a(t)| \leq \left|\frac{F(u(t)) - F(v(t))}{u(t) - v(t)}\right| \leq L\,. \tag{14}$$

For each interval $(0,\tau)$ on which $a(t)$ is defined and bounded we can define the function $f(t) = e^{\int_0^t a(s)\,ds}\, w(t)$ which belongs to $\mathbf{C}^1(0,\tau) \cap \mathbf{C}([0,\tau))$. Moreover,

$f(0) = w(0) = 0$, and $f'(t) \geq 0$ in virtue of (13). It follows that $w(t) \geq 0$, and that $w(\tau) > 0$. It also follows that $a(t)$ is defined and bounded on the whole $(0, T)$. Therefore, $w(T) > 0$. □

Remark 7.2. *If in Lemma 1 we require that $u(0) > v(0)$ then we just need the function F to be continuous to obtain the same conclusion that $u(T) > v(T)$.*

Corollary 7.3. *Let $F : \mathbf{R} \to \mathbf{R}$ be locally Lipschitz at each point, and u, v be two functions in $\mathbf{C}^1(0,T) \cap \mathbf{C}([0,T])$ for some $T > 0$. Assume further that $u(0) = v(0) = z$, and that for all $t \in (0,T)$, $u'(t) \geq F(u(t))$ and $v'(t) \leq F(v(t))$. Then $u(T) \geq v(T)$ with equality if and only if $u(t) \equiv v(t)$ on $[0,T]$.*

Proof. Consider the set $\{t \,|\, u(t) = v(t)\}$ and let τ be the maximum of the connected component of 0 in this set. If $\tau = T$ it means that $u(t) \equiv v(t)$ on $[0,T]$. If τ is less than T then we can apply Lemma 7.1 to the functions $u(t+\tau), v(t+\tau)$ to conclude that $u(T) > v(T)$. □

Lemma 7.4. *Let u and v be two functions in $\mathbf{C}^1(0,T) \cap \mathbf{C}(0,T]$ for some $T > 0$. Assume further that*

$$\lim_{t \to 0+} u(t) = \lim_{t \to 0+} v(t) = \infty,$$

and that for all $t \in (0,T)$

$$u'(t) \geq F(u(t))$$

and

$$v'(t) \leq F(v(t))\,,$$

where $F : \mathbf{R} \to \mathbf{R}$ is a continuous function that is locally Lipschitz and such that the function $G(s) = s^2 F(\frac{1}{s})$ can be extended to zero by continuity and becomes locally Lipschitz at zero. Then $u(t) \geq v(t)$ for all $t \in [0,T)$. Moreover, if $u(T) = v(T)$ then $u(t) \equiv v(t)$ for all $t \in [0,T]$.

Proof. For sufficiently small positive values of t both u and v are positive. For these values we define the functions

$$x(t) = \frac{1}{u(t)} \qquad \text{and} \qquad y(t) = \frac{1}{v(t)}$$

and set $x(0) = y(0) = 0$. A direct computation shows that $x'(t) \leq G(x(t))$ and $y'(t) \geq G(y(t))$, where $G(s) = -s^2\, F\big(\frac{1}{s}\big)$. Lemma 1 or its corollary applies and we conclude that $x(t) \leq y(t)$ for all sufficiently small positive values of t. Therefore, $u(t) \geq v(t)$ for small values of t. The Lemma follows now from the remark or Corollary 7.3. □

Remark 7.5. *All the proofs above apply to the cases when the differential inequalities are of the form* $u'(t) \geq F(u(t), t)$ *and* $v'(t) \leq F(v(t), t)$, *provided that* $F(x, t)$ *is a continuous function of two variables and that the Lipschitz conditions on* $F(z, t)$ *with respect to the first argument hold uniformly with respect to* t *whenever such conditions are required.*

Remark 7.6. *The Lipschitz condition required above can be replaced with some weaker conditions but cannot be removed completely. This is so because just continuity of* F *does not guarantee the uniqueness of a solution to the initial value problem* $u'(t) = F(u(t))$, $u(0) = 0$. *For instance* $u(t) \equiv 0$ *and* $v(t) = t^2$ *provide a counterexample to Lemma 1 with the Lipschitz condition removed. Here* $F(s) = 2\sqrt{s}$ *which is not locally Lipschitz at* 0.

Remark 7.7. *The function* $F(s) = -s^2 + K(t)$, *where* $K(t)$ *is any continuous function, satisfies the conditions of Lemma 7.4. This is the type of function that appears in the proof of the Hessian comparison theorem.*

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Dirección del autor: Nelson Castañeda, Department of Mathematical Sciences, Central Connecticut State University, New Britain, Connecticut 06050, castanedan@ccsu.edu