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An explicit set of isolated points in $\mathbb{R}$ with uncountable closure

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Abstract
The aim of this paper is to show, in a explicit way, a set of isolated real points with uncontable closure. The advantage of our construction is the fact that we can get a clear idea of how is this set distributed among the real line, although such a set has a counter-intuitive property.

Keywords: isolated points, closure.

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1 Introduction

Although for most mathematicians the set of real numbers seems to be a very familiar object, there are lots of examples of subsets of $\mathbb{R}$ with a great number of counterintuitive properties. Unfortunately, most of these examples are just formal constructions for which we do not have any graphic or intuitive idea.

One is the case of a set of isolated points in $\mathbb{R}$ with uncountable closure [1]. A sketch of the solution of this problem known in the mathematical folklore is the following:

Let $\mathbb{Q} = \{q_n\}_{n\in\mathbb{N}}$ be the set of rational numbers with $0 < q_n < 1$ for every $n \in \mathbb{N}$. By induction we construct an open set $F = \bigcup_{i\in\mathbb{N}} V_i$,with positive measure, strictly less than one, where $V_i = (a_i, b_i)\,, 0 < a_i, b_i < 1$, $a_i \neq b_j$ and a_i, b_i are irrational numbers, furthermore, $V_i \cap V_j = \emptyset$, for $i \neq j$ and $\mathbb{Q} \subseteq F$. Now, for each $i \in \mathbb{N}$ we choose a number c_i in V_i and we form a set C. By construction $\mathcal{C}$ is a set of isolate points and it easy to prove that the closure $\mathcal{C}$ is exactly $I - \mathcal{F}$, which is uncountable since it has positive measure. This set has the required properties.

As we can see the former solution is not quite explicit and therefore we cannot get a good idea of how the points of the set are distributed along the real line.

The purpose of this short note is to construct a set with this property but in a very explicit way, that would enable us to visualize it.

2 Construction

I will denote the open interval $(0, 1)$, and let us consider real numbers with finite binary expansions. If $x \in I$ then $x = 0, x_1x_2\cdots x_n\cdots$, where $x_i = 0$ or $x_i = 1$, for all $i \in \mathbb{N}$. Define $J_x = \{i \in \mathbb{N} \mid x_i = 1\}$ and $m_x = \max J_x$, whenever J_x is a finite set. Let F be the set of $x \in I$ such that: *a*) J_x is a finite set. *b*) For all $i \in \mathbb{N}$, $x_{i+k} = 0$, for some $k \in \{0, 1, 2\}$. *c*) For $i \in J_x$, such that $i \neq m_x$ we have $i - 1 \in J_x$ or $i + 1 \in J_x$. *d*) $m_x - 1 \notin J_x$. Now let us notice the following facts:

1. F is a set of isolated points. Indeed, if $x \in F$, $x = 0, x_1x_2 \cdots x_n \cdots$ we define $y = y_1y_2 \cdots$, where $y_i = x_i$ if $i = 1, 2, \ldots, m_x - 1$; $y_{m_x} = 0$; $y_{m_x+k} = 1$ if $k = 1, 2, 3$; $y_i = 0$, if $i > m_x + 3$ and $z = 0, z_1z_2 \cdots z_n \cdots$, where $z_i = x_i$ if $i = 1, 2, \ldots, m_x$; $z_{m_x+1} = 1$ and $z_i = 0, \forall i > m_x + 1$.

Clearly we have $y < x < z$. If $w \in I$ is such that $y < w < x$, then $w_i = x_i$ if $i = 1, 2, \ldots, m_x - 1$ and $w_{m_x} = 0$. Furthermore, $w_{m_x+k} = 1$ for $k = 1, 2, 3$ because if not, $w \leq y$. From the latter it follows that $w \notin F$.

If $w \in I$ is such that $x < w < z$, then $w_i = x_i$ for $i = 1, 2, \ldots, m_x$ and $w_{m_x+1} = 0$; if it were not so, we would have $w \geq z$. Furthermore, since $w > x$, there exists $k > m_x$ with $k \in \mathbb{N}$ such that $w_k = 1$, hence $w \notin F$ because $m_w > m_x, m_x \in J_w$ and $w_{m_x-1} = w_{m_x+1} = 0$, which contradicts condition $c)$ for w.

This yields $(y, z) - \{x\} \subseteq (I - F)$ and therefore F is a set of isolated points.

2. Let E be the set of the points $x \in I$ such that x satisfies the following conditions:

$i)$ J_x is an infinite set.
$ii)$ For all $i \in \mathbb{N}$, $x_{i+k} = 0$, for some $k \in \{0, 1, 2\}$.
$iii)$ For $i \in J_x$, either $i - 1 \in J_x$ or $i + 1 \in J_x$.

Let us see that E is an uncountable set. Indeed, suppose $E = \left\{e^k\right\}_{k=1}^{\infty}$ is a countable set. Define $c = 0, c_1c_2 \cdots c_n \cdots$ in the following way: $c_{4i} = 0$, for all $i \in \mathbb{N}$. For every $k \geq 0$ we choose $j_k \in \{1, 2, 3\}$ such that $e^k_{4k+j_k} = 0$, this j_k exists due to $ii)$. Define $c_{4k+j_k} = 1$ and besides:

If $j_k = 1$, define $c_{4k+2} = 1$ and $c_{4k+3} = 0$; if $j_k = 2$, define $c_{4k+1} = 1$ and $c_{4k+3} = 0$; if $j_k = 3$, define $c_{4k+1} = 0$ and $c_{4k+2} = 1$.

From the definition of c, we have that $c \in E$ and $c \neq e^k$, for all $k \geq 0$, which is a contradiction.

Finally observe that $E \subset \overline{F}$. Let $\varepsilon > 0$ fixed, $y \in E$ with $y = 0, y_1y_2 \cdots$. Take $k \in \mathbb{N}$ large enough such that $y_k = 1, y_{k-1} = 0$ and $\dfrac{1}{2^k} < \varepsilon$. Define $w = 0, w_1w_2 \cdots$ where $w_i = y_i$, if $i = 1, 2, \ldots, k$ and $w_i = 0$ if $i > k$. From the definition of w we obtain that J_w is a finite set, $m_w = k, m_w - 1 = k - 1 \notin J_w$ and $b)$ and $c)$ are satisfied, therefore $w \in F$ and we get $|y - w| < \dfrac{1}{2^k} < \varepsilon$. Thus $E \subset \overline{F}$ and hence E is an uncountable set.

Finally we bestow a sketch of the set in the real line, where $F = \cup_{n \in \mathbb{N}} A_{2n-1}$ and A_{2n-1} is the subset of F with its elements having $(2n - 1)$ times the digit 1 in its binary expansion.

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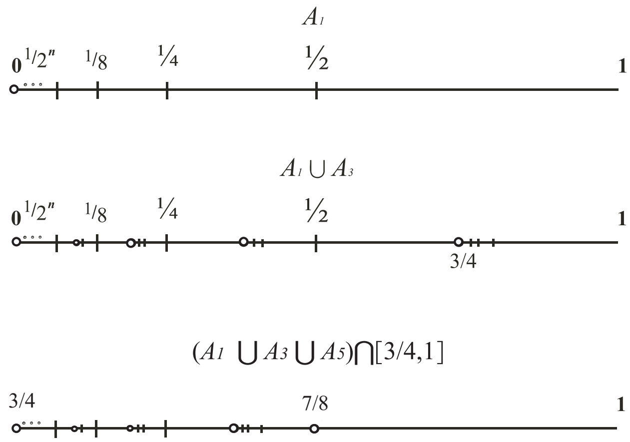

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