

## An iterative method for a second order problem with nonlinear two-point boundary conditions

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### Abstract

A semi-linear second order ODE under a nonlinear two-point boundary condition is considered. Under appropriate conditions on the nonlinear term of the equation, we define a two-dimensional shooting argument which allows to obtain solutions for some specific situations by the use of Poincaré-Miranda's theorem. Finally, we apply this result combined with the method of upper and lower solutions and develop an iterative sequence that converges to a solution of the problem.

**Keywords:** Nonlinear two-point boundary conditions; upper and lower solutions; iterative methods.

**MSC(2000):** 34B15

## 1 Introduction

We study the semi-linear second order ODE

$$u''(t) + g(t, u(t), u'(t)) = 0, \quad 0 < t < T \quad (1)$$

under a nonlinear two-point boundary condition.

Problem (1) under various boundary conditions has been studied by many authors. In the pioneering work of Picard [18], the existence of a solution for the Dirichlet problem was proved by the well-known method of successive approximations, assuming that  $g$  is Lipschitz and  $T$  is small. These conditions have been improved by Hamel [9], for the special case of a forced pendulum equation (see also [13], [14]). For general  $g = g(\cdot, u)$ , the variational approach has been employed already in 1915 by Lichtenstein [12]. However, when  $g$  depends on  $u'$  the problem has non-variational structure, and different techniques are required. As a historical antecedent of the topological methods, we may mention the shooting method introduced in 1905 by Severini [20]; later on, more abstract topological tools have been applied, such as the Leray-Schauder degree theory. For an overview of the use of topological methods to this kind of problems, we refer the reader to [15].

The above-mentioned two-point boundary conditions, as well as some other standard ones, such as the Neumann or the Sturm-Liouville conditions, are *linear*;

it is worthy to mention, however, that the general nonlinear case

$$\phi(u(0), u(T), u'(0), u'(T)) = 0, \quad (2)$$

where  $\phi : \mathbb{R}^4 \rightarrow \mathbb{R}$  is continuous is very important in applications and, in recent years, a considerable number of works have been developed in this direction.

We shall study the existence of solutions of (1) under a particular case of condition (2): namely, nonlinear boundary conditions of the type

$$u'(0) = f_0(u(0)), \quad u'(T) = f_T(u(T)) \quad (3)$$

where  $f_0, f_T : \mathbb{R} \rightarrow \mathbb{R}$  are given continuous functions. The special case  $f_i(x) = a_i x + b_i$  for  $i = 0, T$  corresponds to the Sturm-Liouville conditions, and Neumann conditions when  $a_0 = a_T = 0$ . Our interest in (3) relies on some models in nonlinear beam theory, usually leading to fourth order problems [7], but that admit second order analogues (see e.g. [19]). The results in the present paper complement and extend those in [1].

The paper is organized as follows. In the second section, we impose a growth condition on  $g$ , which allows to prove the unique solvability of the associated Dirichlet problem. Furthermore, we prove that the trace mapping  $Tr : \mathcal{S} \rightarrow \mathbb{R}^2$  given by  $Tr(u) = (u(0), u(T))$ , where

$$\mathcal{S} := \{u \in H^2(0, T) : u''(t) + g(t, u(t), u'(t)) = 0\} \quad (4)$$

is a homeomorphism for the  $H^2$ -norm.

Then, we define a two-dimensional shooting argument, which proves to be successful with the aid of the Poincaré-Miranda theorem (see e.g. [11]) in some particular situations, which include the Sturm-Liouville boundary conditions. This generalizes some of the results in [2], and constitutes the main tool for our iterative method for problem (1)-(3), developed in the third section.

Our method, based on the existence of an ordered couple  $(\alpha, \beta)$  of a lower and an upper solution, has been successfully applied to different boundary value problems when  $g$  does not depend on  $u'$ . For general  $g$ , existence results can still be obtained if one assumes a Nagumo-Bernstein type condition (see [3], [16]). However, these results are usually proved by fixed point or degree arguments and, in consequence, they are non-constructive.

We shall assume instead a Lipschitz condition on  $u'$ , which is more restrictive, but allows the construction of a non-increasing (resp. non-decreasing) sequence of upper (lower) solutions that converges to a solution of the problem. Our method is slightly different from the monotone techniques known in the literature for linear boundary conditions, see e.g. [4], [17] among others (for upper and lower solutions in the reversed order, see [10]).

## 2 A continuum of solutions of (1)

For simplicity, let us assume that  $g$  is continuous, and write it as

$$g(t, u(t), u'(t)) = r(t)u'(t) + h(t, u(t), u'(t)),$$

with  $r \in W^{1,\infty}(0, T)$ . We shall assume that  $h$  satisfies a global Lipschitz condition on  $u'$ , namely

$$\left| \frac{h(t, u, A) - h(t, u, B)}{A - B} \right| \leq k < \frac{\pi}{T} \quad \text{for } A \neq B. \quad (5)$$

Furthermore, in this section we shall assume the following one-side growth condition on  $u$ :

$$\frac{h(t, u, A) - h(t, v, A)}{u - v} \leq c \quad (6)$$

for  $u \neq v$ , where the constant  $c \in \mathbb{R}$  satisfies

$$c + \frac{k\pi}{T} < \left(\frac{\pi}{T}\right)^2 + \frac{1}{2} \inf_{0 \leq t \leq T} r'(t). \quad (7)$$

Under these assumptions, the set  $\mathcal{S}$  of solutions of (1) defined by (4) is homeomorphic to  $\mathbb{R}^2$ . More precisely,

**Theorem 1.** *Assume that (5) and (6) hold and let  $x, y \in \mathbb{R}$ . Then there exists a unique solution  $u_{x,y}$  of (1) satisfying the non-homogeneous Dirichlet condition*

$$u_{x,y}(0) = x, \quad u_{x,y}(T) = y$$

Furthermore, the mapping  $Tr : (\mathcal{S}, \|\cdot\|_{H^2}) \rightarrow \mathbb{R}^2$  given by  $Tr(u) = (u(0), u(T))$  is a homeomorphism.

*Proof.* For fixed  $v \in H^1(0, T)$ , let  $u := \mathcal{T}v$  be defined as the unique solution of the linear problem

$$\begin{aligned} u'' &= -[rv' + h(\cdot, v, v')] \\ u(0) &= x, \quad u(T) = y. \end{aligned}$$

It is immediate that  $\mathcal{T} : H^1(0, T) \rightarrow H^1(0, T)$  is compact. Moreover, if  $S_\sigma : H^2(0, T) \rightarrow L^2(0, T)$  is the semilinear operator defined by  $S_\sigma u := u'' + \sigma[ru' + h(\cdot, u, u')]$ , with  $\sigma \in [0, 1]$ , then using (6) and (7) it is seen that the following a priori bound holds for any  $u, v \in H^2(0, T)$  with  $u - v \in H_0^1(0, T)$ :

$$\|u' - v'\|_{L^2} \leq \mu \|S_\sigma u - S_\sigma v\|_{L^2} \quad (8)$$

for some constant  $\mu$  independent of  $\sigma$ .

Hence, if  $u = \sigma \mathcal{T}u$  for some  $\sigma \in [0, 1]$ , then setting  $l_{x,y}(t) = \frac{y-x}{T}t + x$  we obtain:

$$\|u' - \sigma l'_{x,y}\|_{L^2} \leq \mu \|S_\sigma(\sigma l_{x,y})\|_{L^2} \leq C$$

for some constant  $C$  depending only on  $x$  and  $y$ . Existence of solutions follows from the Leray-Schauder Theorem. Uniqueness is an immediate consequence of (8) for  $\sigma = 1$ .

Thus,  $Tr$  is bijective, and its continuity is clear. On the other hand, if  $(x, y) \rightarrow (x_0, y_0)$ , then applying (8) to  $u = u_{x,y} - l_{x,y}$  and  $v = u_{x_0,y_0} - l_{x_0,y_0}$  it is easy to see that  $u_{x,y} \rightarrow u_{x_0,y_0}$  for the  $H^1$ -norm. As  $u_{x,y}$  and  $u_{x_0,y_0}$  satisfy (1), we conclude from (5) that also  $u''_{x,y} \rightarrow u''_{x_0,y_0}$  for the  $L^2$ -norm and so completes the proof.  $\square$

It is worth noticing that the previous result allows to define a two-dimensional shooting argument as follows: let  $\Theta : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be defined by

$$\Theta(x, y) = (u'_{x,y}(0) - f_0(x), u'_{x,y}(T) - f_T(y)).$$

From the previous theorem, we deduce that  $\Theta$  is continuous, and it is clear that, if  $\Theta(x, y) = (0, 0)$ , then  $u_{x,y}$  is a solution of the problem.

**Example 1.** Assume that (5) and (6) hold, and that

$$h(t, u, 0) \operatorname{sgn}(u) < 0 \quad \text{for } |u| \geq M, \quad (9)$$

$$f_0(M^+) \geq 0 \geq f_0(M^-), \quad f_T(M^+) \leq 0 \leq f_T(M^-) \quad (10)$$

for some constants  $M^- \leq -M < M \leq M^+$ . Then (1)-(3) admits at least one solution.

In particular, the result holds for the Sturm-Liouville conditions

$$u'(0) = a_0 u(0) + b_0, \quad u'(T) = a_T u(T) + b_T, \quad a_0 > 0 > a_T. \quad (11)$$

Furthermore, in this case the solution is unique, provided that  $c < 0$  in (6).

Indeed, it follows from (9) that  $u_{x,y}$  cannot attain in  $(0, T)$  neither a maximum value larger than  $M$ , nor a minimum value smaller than  $-M$ . Moreover, for  $M^- \leq y \leq M^+$  we obtain:

$$u_{M^+,y}(0) = M^+ \geq y = u_{M^+,y}(T), \quad u_{M^-,y}(0) = M^- \leq y = u_{M^-,y}(T).$$

Thus,  $u'_{M^+,y}(0) \leq 0 \leq u'_{M^-,y}(0)$ , and hence  $\Theta_1(M^+, y) \leq 0 \leq \Theta_1(M^-, y)$ . In the same way, it follows that  $\Theta_2(x, M^+) \geq 0 \geq \Theta_2(x, M^-)$  for  $M^- \leq x \leq M^+$ . By the Poincaré-Miranda's generalized intermediate value theorem, we conclude that  $\Theta$  has at least one zero  $(x, y) \in [M^-, M^+] \times [M^-, M^+]$ .

On the other hand, if  $u$  and  $v$  are solutions of (1)-(11), then

$$(u - v)'' + (r + \psi)(u - v)' + h(\cdot, u, v') - h(\cdot, v, v') = 0$$

where

$$\psi = \frac{h(\cdot, u, u') - h(\cdot, u, v')}{u' - v'} \in L^\infty(0, T).$$

Next, take  $p(t) = e^{\int_0^t (r(s) + \psi(s)) ds}$ , multiply the previous equality by  $(u - v)p$  and integrate. We obtain:

$$\begin{aligned} 0 &= p(u' - v')(u - v) \Big|_0^T - \int_0^T p(u' - v')^2 + \int_0^T p[h(\cdot, u, v') - h(\cdot, v, v')](u - v) \\ &\leq p(T)a_T(u - v)^2(T) - a_0(u - v)^2(0) - \int_0^T p(u' - v')^2 + c \int_0^T p(u - v)^2. \end{aligned}$$

Hence, for  $c < 0$  it is seen that  $u = v$ .

### 3 Iterative sequences of upper and lower solutions

In this section we shall construct solutions of (1) under the two-point boundary condition (3) by an iterative method, based upon the existence of upper and lower solutions.

Let us recall that  $(\alpha, \beta)$  is an ordered couple of a lower and an upper solution for (1) if  $\alpha \leq \beta$  and

$$\alpha'' + g(\cdot, \alpha, \alpha') \geq 0 \geq \beta'' + g(\cdot, \beta, \beta').$$

Existence results under various boundary conditions in presence of an ordered couple of a lower and an upper solution are known (see e. g. [6]). In our particular case, we shall assume the boundary constraints:

$$\alpha'(0) - f_0(\alpha(0)) \geq 0 \geq \beta'(0) - f_0(\beta(0)),$$

$$\alpha'(T) - f_T(\alpha(T)) \leq 0 \leq \beta'(T) - f_T(\beta(T))$$

and a Nagumo type condition adapted from [5]:

$$|g(t, u, v)| \leq \psi(|v|), \quad \text{for } \alpha(t) \leq u \leq \beta(t), m \leq |v| \leq M \quad (12)$$

where  $\psi : [0, +\infty) \rightarrow (0, +\infty)$  is continuous and satisfies:

$$\int_m^M \frac{1}{\psi(t)} dt > T,$$

and

$$m = \min \left\{ \frac{|\alpha(0) - \beta(T)|}{T}, \frac{|\alpha(T) - \beta(0)|}{T}, \max_{\alpha(0) \leq s \leq \beta(0)} |f_0(s)|, \max_{\alpha(T) \leq s \leq \beta(T)} |f_T(s)| \right\}$$

$$M > \max\{\|\alpha'\|_\infty, \|\beta'\|_\infty, m\}.$$

Then, the following existence result can be obtained as in [1]:

**Theorem 2.** *Assume there exists an ordered couple  $(\alpha, \beta)$  of a lower and an upper solution as before, and that (12) holds. Then the boundary value problem (1)-(3) admits at least one solution  $u$ , with  $\alpha \leq u \leq \beta$ .*

*Sketch of the proof.* The proof follows the outline of the standard results on the subject. Let  $P(t, u) = \max\{\alpha(t), \min\{u, \beta(t)\}\}$  and  $Q(v) = \text{sgn}(v)\min\{|v|, M\}$ , and apply Schauder's Theorem in order to obtain a solution of the problem

$$\begin{aligned} u''(t) - \lambda u(t) &= -g(t, P(t, u(t)), Q(u'(t))) - \lambda P(t, u(t)), \\ u'(0) &= f_0(P(0, u(0))), \quad u'(T) = f_T(P(T, u(T))) \end{aligned}$$

for some fixed  $\lambda > 0$ . It is easy to see that  $\alpha \leq u \leq \beta$ , and hence  $P(t, u(t)) = u(t)$  for every  $t$ . Furthermore, if we suppose that for example  $u'(t_1) = M$ , then there exists  $t_0$  such that  $u'(t_0) = m$  and  $m < u'(t) < M$  for  $t$  between  $t_0$  and  $t_1$ . Hence

$$T < \int_m^M \frac{1}{\psi(s)} ds = \int_{t_0}^{t_1} \frac{u''(t)}{\psi(u'(t))} dt \leq |t_1 - t_0|,$$

a contradiction. The same conclusion holds if we suppose  $u'(t_1) = -M$ ; thus,  $|u'(t)| < M$  and the proof is complete.  $\square$

**Example 2.** *The previous result applies when (9) and (10) hold: indeed, in this case it is clear  $(M^-, M^+)$  is an ordered couple of a lower and an upper solution. Thus, conditions (5) and (6) in example 1 can be dropped.*

*Also, we may consider the forced pendulum equation with friction*

$$u'' + ru' + \sin u = \theta,$$

*and assume that the forcing term  $\theta$  is a measurable function satisfying:*

$$-1 \leq \theta(t) \leq 1 \quad \forall t \in [0, T].$$

*Then  $\alpha \equiv \frac{\pi}{2}$  and  $\beta \equiv \frac{3\pi}{2}$  are respectively a lower and an upper solution. Hence, (1)-(3) has a solution for any continuous  $f_0$  and  $f_T$  such that*

$$f_0\left(\frac{\pi}{2}\right) \leq 0 \leq f_0\left(\frac{3\pi}{2}\right)$$

*and*

$$f_T\left(\frac{\pi}{2}\right) \geq 0 \geq f_T\left(\frac{3\pi}{2}\right).$$

Our last result is concerned with the construction of solutions by iteration, provided that  $h$  and  $f$  satisfy some stronger assumptions.

Let us firstly establish the following auxiliary lemmas:

**Lemma 1.** *Assume that (5) holds and let  $\lambda$  be a positive constant satisfying  $\lambda \geq k\frac{\pi}{T} - \left(\frac{\pi}{T}\right)^2 - \frac{1}{2}\text{infr}'$ . Then for any  $z, \theta \in C([0, T])$  the equation*

$$u'' + ru' + h(\cdot, z, u') - \lambda u = \theta$$

*is uniquely solvable under the Sturm-Liouville conditions (11). Furthermore, the mapping  $\mathcal{K} : C([0, T])^2 \rightarrow C([0, T])$  given by  $\mathcal{K}(z, \theta) = u$  is compact.*

*Proof.* Existence and uniqueness follow as a particular case of example 1, with  $\bar{g}(\cdot, u, u') = ru' + \bar{h}(\cdot, u, u')$ , where

$$\bar{h}(\cdot, u, u') = h(\cdot, z, u') - \lambda u - \theta.$$

Indeed, it is clear that  $\bar{h}$  satisfies (5) and (6) with  $c = -\lambda$ . Moreover,

$$\bar{h}(t, u, 0) \operatorname{sgn}(u) = (h(t, z(t), 0) - \theta(t)) \operatorname{sgn}(u) - \lambda|u| < 0$$

when  $|u| > \|h(\cdot, z, 0) - \theta\|_\infty$ . Thus, (9) is also satisfied.

Let  $(z, \theta)$  tend to  $(z_0, \theta_0)$ , and set  $u = \mathcal{K}(z, \theta)$ ,  $u_0 = \mathcal{K}(z_0, \theta_0)$ . Then

$$(u - u_0)'' + (r + \psi)(u - u_0)' - \lambda(u - u_0) = h(\cdot, z, u_0') - h(\cdot, z_0, u_0') + \theta - \theta_0$$

where  $\psi = \frac{h(\cdot, z, u_0') - h(\cdot, z_0, u_0')}{u' - u_0'}$ . Hence, continuity of  $\mathcal{K}$  is a consequence of the following estimate, which is valid for any  $w$  satisfying (11) with  $b_0 = b_T = 0$  and some constant  $c$  depending only on  $k$ :

$$\|w\|_{H^1} \leq c\|w'' + (r + \psi)w' - \lambda w\|_{L^2}.$$

Indeed, this is easily deduced by applying the Cauchy-Schwartz inequality to the integral  $\int_0^T pLw.w$ , where  $Lw = w'' + (r + \psi)w' - \lambda w$  and  $p(t) = e^{\int_0^t (r(s) + \psi(s)) ds}$ , and the fact that  $0 < m \leq p \leq M$  for some  $m$  and  $M$  depending only on  $k$ . Finally, compactness of  $\mathcal{K}$  follows from the imbedding  $H^1(0, T) \hookrightarrow C([0, T])$ .  $\square$

**Remark 1.** *In the previous proof, an analogous estimate can be also obtained for the  $H^2$ -norm of  $w$ . This implies the compactness of  $\mathcal{K}$ , but now regarded as an operator from  $C([0, T])^2$  to  $C^1([0, T])$ . More generally, one might consider also  $a_i$  and  $b_i$  as variables for  $i = 0, T$ : in this case,  $\mathcal{K}$  could be seen as a compact operator from  $\mathbb{R}^4 \times C([0, T])^2$  to  $C^1([0, T])$ .*

**Lemma 2.** *Let  $\phi \in L^\infty(0, T)$  and assume that  $w'' + \phi w' - \lambda w \geq 0$  (in the weak sense) for some  $\lambda \geq 0$ , and*

$$w'(0) - a_0 w(0) \geq 0 \geq w'(T) - a_T w(T)$$

*with  $a_0 > 0 > a_T$ . Then  $w \leq 0$ .*

*Proof.* If  $w(0), w(T) \leq 0$ , the result is the well-known maximum principle for Dirichlet conditions.

If for example  $w(0) > 0$ , then restricting  $w$  up to its first zero if necessary, it suffices to consider only the case  $w \geq 0$ . Taking  $p(t) = e^{\int_0^t \phi(s) ds}$ , it is observed that  $(pw')' \geq \lambda pw \geq 0$ . Thus,  $pw'$  is nondecreasing on  $[0, T]$ , and hence

$$0 \geq p(T)a_T w(T) \geq p(T)w'(T) \geq p(0)w'(0) \geq p(0)a_0 w(0) > 0,$$

a contradiction. The proof is similar when  $w(T) > 0$ .  $\square$

In order to define our iterative scheme, we shall assume that  $f_0$  and  $f_T$  satisfy a one-side Lipschitz condition:

(F) There exists a positive constant  $R$  such that

$$f_0(y) - f_0(x) \leq R(y - x)$$

if  $\alpha(0) \leq x < y \leq \beta(0)$ , and

$$f_T(y) - f_T(x) \geq -R(y - x)$$

if  $\alpha(T) \leq x < y \leq \beta(T)$ .

In virtue of Lemma 1, if (5) holds then for  $\lambda = \min\{R, k\frac{\pi}{T} - (\frac{\pi}{T})^2 - \frac{1}{2}\inf r'\}$ , we may define the compact operator  $\mathcal{T} : C([0, T]) \rightarrow C([0, T])$  given by  $\mathcal{T}v = u$ , where  $u$  is the unique solution of the problem

$$u'' + ru' + h(\cdot, v, u') - \lambda u = -\lambda v$$

satisfying the following Sturm-Liouville condition:

$$u'(0) - Ru(0) = f_0(v(0)) - Rv(0), \quad u'(T) + Ru(T) = f_T(v(T)) + Rv(T).$$

From Remark 1, we observe, moreover, that the set  $\mathcal{T}(\{v : \alpha \leq v \leq \beta\})$  is bounded for the  $C^1$ -norm. In particular, this implies the existence of a constant  $M = M(R)$  such that if  $u = \mathcal{T}v$  for some  $v$  lying between  $\alpha$  and  $\beta$ , then  $\|u'\|_\infty \leq M$ . This suggests to consider the following Lipschitz condition on  $h$ :

(H)

$$|h(t, u, A) - h(t, v, A)| \leq R|u - v|$$

for  $u, v$  such that  $\alpha(t) \leq u < v \leq \beta(t)$  and  $|A| \leq M(R)$ .

**Remark 2.** Conditions (F) and (H) are trivially satisfied if  $f_0$ ,  $f_T$  and  $h$  are  $C^1$  functions, and  $\frac{\partial h}{\partial u}$  is bounded with respect to  $u'$ .

**Theorem 3.** Assume there exists an ordered couple  $(\alpha, \beta)$  of a lower and an upper solution as before. Further, assume that (5), (H) and (F) hold. Set  $\lambda$  as before, and define the sequences  $\{\underline{u}_n\}$  and  $\{\bar{u}_n\}$  recursively by

$$\underline{u}_0 = \alpha, \quad \bar{u}_0 = \beta$$

and

$$\bar{u}_{n+1} = \mathcal{T}\bar{u}_n, \quad \underline{u}_{n+1} = \mathcal{T}\underline{u}_n$$

Then  $(\underline{u}_n, \bar{u}_n)$  is an ordered couple of a lower and an upper solution. Furthermore,  $\{\bar{u}_n\}$  (resp.  $\{\underline{u}_n\}$ ) is non-increasing (non-decreasing) and converges to a solution of the problem.

*Proof.* Let us firstly prove that  $\alpha \leq \bar{u}_1 \leq \beta$ . From the definition,

$$\bar{u}_1'' + r\bar{u}_1' + h(\cdot, \beta, \bar{u}_1') - \lambda\bar{u}_1 = -\lambda\beta \geq -\lambda\beta + \beta'' + r\beta' + h(\cdot, \beta, \beta').$$

Hence, setting

$$\psi = \frac{h(\cdot, \beta, \bar{u}_1') - h(\cdot, \beta, \beta')}{\bar{u}_1' - \beta'} \in L^\infty(0, T)$$

we deduce that

$$(\bar{u}_1 - \beta)'' + (r + \psi)(\bar{u}_1 - \beta)' - \lambda(\bar{u}_1 - \beta) \geq 0.$$

On the other hand,

$$\bar{u}_1'(0) - R\bar{u}_1(0) = f_0(\beta(0)) - R\beta(0)$$

and

$$\bar{u}_1'(T) + R\bar{u}_1(T) = f_T(\beta(T)) + R\beta(T).$$

Thus,

$$(\bar{u}_1 - \beta)'(0) - R(\bar{u}_1 - \beta)(0) = 0 = (\bar{u}_1 - \beta)'(T) - R(\bar{u}_1 - \beta)(T),$$

and from Lemma 2 we obtain that  $\bar{u}_1 \leq \beta$ .

In the same way,

$$\bar{u}_1'' + r\bar{u}_1' + h(\cdot, \beta, \bar{u}_1') - \lambda\bar{u}_1 \leq -\lambda\beta + \alpha'' + r\alpha' + h(\cdot, \alpha, \alpha')$$

and hence

$$(\bar{u}_1 - \alpha)'' + (r + \psi)(\bar{u}_1 - \alpha)' - \lambda(\bar{u}_1 - \alpha) \geq 0$$

where

$$\psi = \frac{h(\cdot, \alpha, \bar{u}_1') - h(\cdot, \alpha, \alpha')}{\bar{u}_1' - \alpha'} \in L^\infty(0, T).$$

Also

$$\bar{u}_1'(0) - R\bar{u}_1(0) = f_0(\beta(0)) - R\beta(0) \leq f_0(\alpha(0)) - R\alpha(0)$$

and

$$\bar{u}_1'(T) + R\bar{u}_1(T) = f_T(\beta(T)) + R\beta(T) \geq f_T(\alpha(T)) + R\alpha(T),$$

and we conclude that  $\bar{u}_1 \geq \alpha$ .

On the other hand,

$$\bar{u}_1'' + r\bar{u}_1' + h(\cdot, \bar{u}_1, \bar{u}_1') = (\lambda - R)(\bar{u}_1 - \beta) + [h(\cdot, \bar{u}_1, \bar{u}_1') + R\bar{u}_1] - [h(\cdot, \beta, \bar{u}_1') + R\beta] \leq 0,$$

and we deduce that  $\bar{u}_1$  is an upper solution of the problem. Inductively, it follows that  $\bar{u}_n$  is an upper solution for every  $n$ , with  $\alpha \leq \bar{u}_{n+1} \leq \bar{u}_n$ , which by Dini's

theorem implies that  $\bar{u}_n$  converges uniformly to a function  $\bar{u}$ . From the definition of  $\{\bar{u}_n\}$ ,

$$\bar{u}_{n+1}'' + r\bar{u}_{n+1}' + h(\cdot, \bar{u}_n, \bar{u}_{n+1}') \rightarrow 0$$

uniformly. Moreover, from Lemma 1 and Remark 1 we know that  $\{\bar{u}_n\}$  is bounded in  $H^2(0, T)$ , and it follows easily that

$$\bar{u}'' + r\bar{u}' + h(\cdot, \bar{u}, \bar{u}') = 0.$$

Thus,  $\bar{u}$  is a solution of the problem. The proof for  $\underline{u}_n$  is analogous. Moreover, if we assume as inductive hypothesis that  $\underline{u}_n \leq \bar{u}_n$ , then

$$\begin{aligned} & \bar{u}_{n+1}'' + r\bar{u}_{n+1}' + h(\cdot, \bar{u}_n, \bar{u}_{n+1}') - \lambda\bar{u}_{n+1} = -\lambda\bar{u}_n \\ & \leq -\lambda\underline{u}_n = \underline{u}_{n+1}'' + r\underline{u}_{n+1}' + h(\cdot, \underline{u}_n, \underline{u}_{n+1}') - \lambda\underline{u}_{n+1}. \end{aligned}$$

In the same way as before, we may define

$$\psi = \frac{h(\cdot, \underline{u}_n, \bar{u}_{n+1}') - h(\cdot, \underline{u}_n, \underline{u}_{n+1}')}{\bar{u}_{n+1}' - \underline{u}_{n+1}'} \in L^\infty(0, T),$$

and hence for  $w = \bar{u}_{n+1} - \underline{u}_{n+1}$  we deduce:

$$w'' + (r + \psi)w' - \lambda w \leq h(\cdot, \underline{u}_n, \bar{u}_{n+1}') - h(\cdot, \bar{u}_n, \bar{u}_{n+1}') \leq -R(\underline{u}_n - \bar{u}_n) \leq -Rw.$$

From Lemma 2, we conclude that  $w \geq 0$ , i.e.  $\underline{u}_{n+1} \leq \bar{u}_{n+1}$ .  $\square$

**Remark 3.** *It is interesting to observe that, even if (5) is somewhat too restrictive, some condition regarding the growth of  $h$  with respect to  $u'$  is required. We may recall, for instance, the following example by Habets and Pouso [8] for the mean curvature operator:*

$$\left( \frac{u'}{\sqrt{1+u'^2}} \right)' = u + a,$$

where the function  $a \in L^\infty(0, T)$  is defined by

$$a(t) = 2[\chi_{[0, \frac{T}{2}]}(t) - \chi_{(\frac{T}{2}, T]}(t)] = \begin{cases} 2 & 0 \leq t \leq \frac{T}{2} \\ -2 & \frac{T}{2} < t \leq T \end{cases}$$

Under conditions (11) with  $b_0 = b_T = 0$ , it is seen that  $\alpha = -3$  and  $\beta = 3$  is an ordered couple of a lower and an upper solution, but the equation has no solutions when  $T > 2\sqrt{2}$ . However, here

$$h(\cdot, u, u') = (u + a) \left( \sqrt{1 + u'^2} \right)^{3/2},$$

and (9) is satisfied. This explains the need of the Nagumo condition, or at least a similar one, in Theorem 2.

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