

A categorical characterization of non-distributive lattices

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Abstract

In this paper we give a categorical characterization of non-distributive lattices and non-distributive modular lattices, in terms of finite limits and finite colimits.

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1 Introduction

The following characterization of the non-distributive and modular lattices is well known (see Theorems 3.5 (Dedekind) and 3.6 (Birkhoff)): a lattice is non-distributive if and only if it contains a sublattice isomorphic to one of the lattices M_3 or N_5 . A lattice is modular if and only if it does not contain a sublattice isomorphic to N_5 .

It is our goal to give an alternative and, in some sense, more geometric characterization for the non-distributive and modular lattices using a categorical approach.

This paper is organized in the following way: in section 2 we give some definitions and examples; in section 3 we state and prove our main results: a categorical characterization for non-distributive lattices (theorem 1) and for non-distributive modular lattices (theorem 2).

2 Basic notions

We begin by introducing the basic definitions and the notation that will be used throughout this paper. We will denote the pullback and pushout of two morphisms f and g by $pullback(f, g)$ and $pushout(f, g)$, respectively.

Definition 1. We say that the pullback $fx = gy$ (the pushout $xh = yl$, respectively) is trivial if $x \approx Id$ or $y \approx Id$.

We will denote by $\mathcal{C}_{\mathbb{P}}$ the category associated to the poset $(\mathbb{P}, \leq)$ (see [1]) and we will refer to a poset $(\mathbb{P}, \leq)$ simply by its underlying set $\mathbb{P}$.

Definition 2. A non-distributive lattice $\mathbb{P}$ will be called *exact* if it contains a sublattice $\widehat{\mathbb{P}}$ isomorphic to M_3 or N_5 , such that the following conditions are satisfied:

1. Every $p \in \mathbb{P}$ lies between two elements of $\widehat{\mathbb{P}}$ and
2. Every pair of elements of $\mathbb{P}$ is comparable if it lies between two consecutive elements of $\widehat{\mathbb{P}}$ or if the closed interval defined by them contains some element of $\widehat{\mathbb{P}}$.

Now we give the definitions of non-distributive category, and of modular category and give some examples to illustrate that our definitions lead to the expected results in some familiar cases.

Definition 3. A category $\mathcal{C}_{\mathbb{P}}$ will be called *non-distributive* if there exist three morphisms f_1, f_2 y f_3 with the same codomain, such that at least two of the pullbacks $f_i x_{ij} = f_j x_{ji}$ are non trivial, and furthermore every non trivial pullback having one of the following forms $\text{pullback}(f_i x_{il}, f_j x_{jk})$ or $\text{pullback}(f_i, f_j)$ is a pushout. Notice that in this situation, $x_{ii} = \text{id}$ for every i .

Definition 4. A category $\mathcal{C}_{\mathbb{P}}$ will be called *distributive* if it does not satisfy the conditions of definition 3, and $\mathcal{C}_{\mathbb{P}}$ will be called *modular* if for every morphisms f_1, f_2, f_3 with the same codomain, satisfying the conditions of definition 3, if two of the pullbacks $\text{pullback}(f_i, f_j)$ are non trivial then the third pullback is non trivial ($i \neq j$).

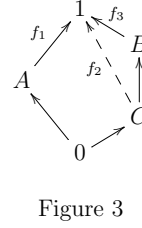
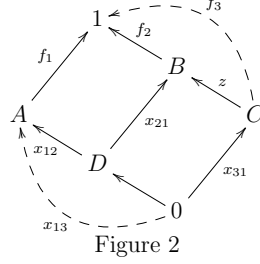
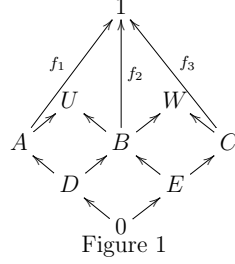
Remark 1. It is a straightforward verification that if $\text{pullback}(f_i, f_j)$ is non trivial then the corresponding pushout (x_{ij}, x_{ji}) is also non trivial. Additionally, it follows easily that if $\text{pullback}(f_i, f_j)$ is non trivial then f_i, f_j are not isomorphic. Consequently, if two of the pullbacks of the above definition are non trivial, then none of the morphisms f_1, f_2, f_3 is the identity morphism.

In the following examples we illustrate that our definitions are minimal, in the sense that if one of the conditions of definition 3 is not satisfied, then it is possible to give an example of distributive lattice whose associated category satisfies the remaining conditions of the definition.

Example 1. (A) If $\mathbb{P}$ is a chain then $\mathcal{C}_{\mathbb{P}}$ is a distributive category since every pullback is trivial. (B) If $\mathbb{P}$ is the lattice 2^2 (associated to the power set of a two element set) then $\mathcal{C}_{\mathbb{P}}$ is a distributive category since there exists only one non trivial pullback.

Example 2. (A) The category of Figure 1 is distributive since $\text{pullback}(f_1, f_2)$ is not a pushout; however, $\text{pullback}(f_i, f_j)$ ($i, j \in \{1, 2, 3\}, i \neq j$) are three non trivial pullbacks. (B) The category of Figure 2 is distributive even though two of the three non trivial pullback (f_i, f_j) ($i \neq j$) are pushouts, since we have that $\text{pullback}(f_1 x_{12}, f_3)$ is not a pushout. (C) The category $\mathcal{C}_{N_5}$ (See Figure 3) is

non-distributive and non-modular since $\text{pullback}(f_1, f_2)$ and $\text{pullback}(f_1, f_3)$ are two non trivial pullbacks which are pushouts. (D) The category associated to M_3 is non-distributive and modular. In particular, $\mathcal{C}_{M_3}$ has exactly three non trivial pullbacks which are pushouts.



3 Non-distributive lattices and non-distributive categories

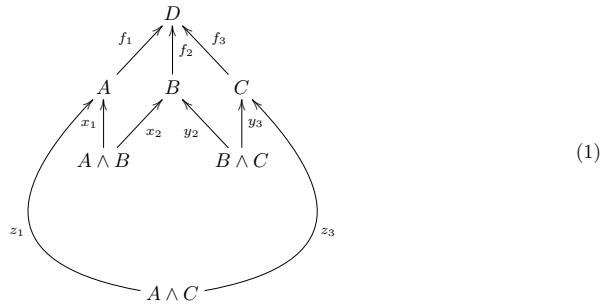
First, let us recall that a category $\mathcal{C}$ is *cartesian* if it has finite limits and finite colimits. In particular, it has final object 1 and initial object 0 (see [1]).

Now we come to the first one of our main results: the categorical characterization of non-distributive lattices.

Theorem 1. *A cartesian category $\mathcal{C}_{\mathbb{P}}$ is non-distributive if and only if $\mathbb{P}$ is a non-distributive lattice.*

Demostración. ($\Rightarrow$) The finite coproducts and finite products in $\mathcal{C}_{\mathbb{P}}$ correspond to the finite supremum and infimum in $\mathbb{P}$, thus $\mathbb{P}$ is a lattice. We will consider now two cases:

Case 1: Let f_1, f_2, f_3 be three morphisms satisfying the conditions of definition 3 and assume that the following three non trivial pullbacks are pushouts:



A is not comparable to B ; otherwise $\text{pullback}(f_1, f_2)$ would be trivial. Analogously, B is not comparable to C , and A is not comparable to C . Let us prove

now that $A \wedge B \wedge C$ must be equal to at least two elements of the set

$$\{A \wedge B, A \wedge C, B \wedge C\}.$$

Indeed, if $A \wedge B \wedge C \neq A \wedge B$ and $A \wedge B \wedge C \neq B \wedge C$, from the above diagram we have that $A \wedge B \wedge C < A \wedge B$ and $A \wedge B \wedge C < B \wedge C$.

Without loss of generality, if $A \wedge B \wedge C \neq A \wedge B$ and $A \wedge B \wedge C \neq B \wedge C$ then $\text{pullback}(f_1 x_1, f_3 y_3)$ would be non trivial; taking into account that $\mathcal{C}_{\mathbb{P}}$ is non-distributive, this pullback would be a pushout and this would imply that B and D are equal which contradicts that f_2 is not an identity. Therefore $A \wedge B \wedge C$ must be equal to at least two elements of the set $\{A \wedge B, A \wedge C, B \wedge C\}$.

If $A \wedge B \wedge C = A \wedge B = B \wedge C = A \wedge C$ then the lattice in (1) becomes the one shown in Figure 4 and is therefore isomorphic to M_3 . If now $A \wedge B \wedge C = A \wedge C = A \wedge B \neq B \wedge C$, then $A \wedge B \wedge C < B \wedge C$, and consequently, the lattice in (1) reduces to the one shown in Figure 5 and is now therefore isomorphic to N_5 . The cases $A \wedge B \wedge C \neq A \wedge B$, and $A \wedge B \wedge C \neq A \wedge C$, are completely analogous to the previous situation.

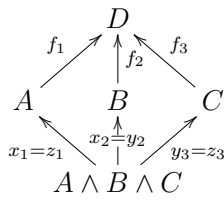


Figure 4

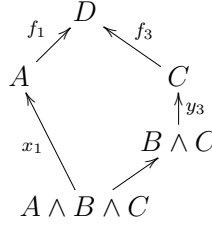


Figure 5

Case 2: We have the non trivial lattice of Figure 6 and the trivial $\text{pullback}(f_1, f_3)$.

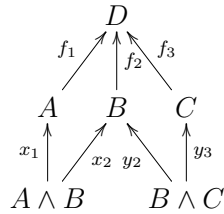


Figure 6

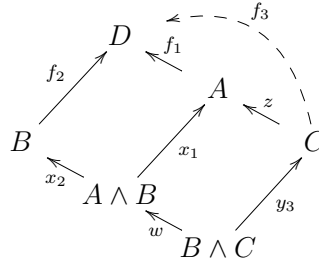


Figure 7

This implies $A < C$ or $C < A$. We can assume, without loss of generality, $C < A$ and now the diagram in Figure 6 has the shape shown in Figure 7 (Notice that $A \wedge B \wedge C = B \wedge C$ since $A \wedge C = C$; $f_1 z = f_3$.)

Since $A \wedge B \wedge C = B \wedge C$, we have that $\text{pullback}(f_3, f_2x_2)$ is $f_2x_2w = f_1zy_3$. If this pullback is non trivial then, by our hypothesis, the above diagram corresponds also to $\text{pushout}(w, y_3)$, and consequently $(A \wedge B) \vee C = D$, but since $(A \wedge B) \vee C \leq A$, we have $A = D$ and this contradicts the fact that f_1 is not an isomorphism. Therefore the pullback is trivial; now, since $\text{pullback}(f_2, f_3)$ is non trivial, we have $B \wedge C < C$ which in turn yields $A \wedge B = B \wedge C$ and the sublattice $\{B \wedge C, B, C, A, D\}$

is isomorphic to N_5 .

($\Leftarrow$) This follows at once from the definitions. $\square$

Now we are ready for our second main result: a categorical characterization of modular lattices.

Theorem 2. *The non-distributive category $\mathcal{C}_{\mathbb{P}}$ is modular if and only if $\mathbb{P}$ is a non-distributive and modular lattice (i.e., it does not contain a sublattice isomorphic to N_5).*

Demostración. It is a straightforward consequence of Theorem 1 and definition 4). $\square$

Corollary 2. *The category $\mathcal{C}_{\mathbb{P}}$ is modular if and only if $\mathbb{P}$ is a modular lattice.*

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