Rate of growth for the eigenvalues of Schrödinger operators using Schatten-von Neumann S_p classes

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Julio, 2025

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Acknowledgments

This work has been made possible thanks to the support and collaboration of my family, my friends and my professors, because it is thanks to them that I became the man of today.

Abstract

This work aims to study the rate of growth of the eigenvalues of $\mathbf{H} = (-\Delta + V + I)^s$. To do so we make use of Weyl's inequality between singular values and eigenvalues, which provides a bound for the growth of eigenvalues of a compact operator in a Hilbert space, using some additional tools this will give us the rate of decay of the eigenvalues of $(-\Delta + V + I)^{-s}$, which serve as a direct tool to find the rate of growth of the eigenvalues of $\mathbf{H}$. In order to do this, basic concepts from spectral theory and operator theory are introduced, along with the notion of singular values and S_p classes.

Key words: Compact operators, S_p classes, singular values, eigenvalues, Weyl's Inequality.

Resumen

En este trabajo se estudia la taza de crecimiento de los valores propios de $\mathbf{H} = (-\Delta + V + I)^s$, para ello se hace uso de la desigualdad de Weyl entre valores propios y valores singulares de un operador compacto en un espacio de Hilbert, usando algunas herramientas adicionales se llegará a la tasa de decaímiento de los valores propios de $(-\Delta + V + I)^{-s}$, y como consecuencia de esto se llega a la tasa de crecimiento de los valores propios de $\mathbf{H}$. Para llegar a esto, se introducen conceptos básicos de la teoría de operadores y teoría espectral, para luego empezar el estudio de las clases S_p.

Palabras clave: Operadores compactos, clases S_p, valores singulares, valores propios, desigualdad de Weyl.

Introduction

In the 1900s, David Hilbert introduced the term spectral theory while developing his formulation of the spaces that now bear his name. These advances laid the foundation for the mathematical formulation of quantum mechanics and also enabled other great thinkers, such as Hermann Weyl and John von Neumann, to make significant contributions to mathematics. In "Über die asymptotische Verteilung der Eigenwerte" [10], Hermann Weyl showed some important inequalities that relate the singular values of a compact operator with its eigenvalues. John von Neumann developed much of the spectral theory of operators on Hilbert spaces, driven by the needs of quantum mechanics,. He published the famous book Mathematical Foundations of Quantum Mechanics [8].

Years later, together with Robert Schatten, he studied nuclear operators in Hilbert spaces, tensor products of Banach spaces, and studied trace-class operators, their ideals, and their duality with compact operators, as well as their preduality with bounded operators. By 1937, von Neumann had already published several results in this area, including important work on what we now know as the Schatten–von Neumann ideals.

The idea of this thesis is to study the rate of growth of the eigenvalues of $\mathbf{H} = (-\Delta + V + I)^s$. In order to do so we will not study the eigenvalues directly, instead, we will shift our focus to singular values and a certain class of operators that we will denote by S_p, which provide inequalities that help bound the rate of decay of the eigenvalues given certain conditions which we will illustrate with a simple example. To understand S_p classes one needs to know about the spectral theory of compact operators, it is for this reason that we divide this thesis in three chapters. The first chapter lays the foundations of functional analysis and spectral theory of bounded operators, by the end of this chapter we study the spectral theory of compact operators so that in chapter 2 we can begin talking about S_p classes, it is worth noting that we will not use most of the results in this chapter, it is mostly just to expose the results that arise when studying these classes of operators. After this exposition of S_p classes, we turn our attention to two important inequalities that we will use in order to finally study the rate of growth of the eigenvalues of $\mathbf{H} = (-\Delta + V + I)^s$, and once they are proved and ready to go, we proceed with the study of the rate of growth of the eigenvalues in question.

1

Topics in Functional Analysis

1.1 Linear Maps and Banach Spaces

1.1.1 Normed Spaces. In linear algebra the main setting for the theory is the finite dimensional vector space $\mathbb{R}^n$. Some topics that are studied in said course are linear transformations, norm and dot product, and most importantly; matrices. In this first chapter we talk about these same topics but in a more general setting for any vector space. We start by giving vector spaces some analytical properties such as norm, convergence and continuity. Followed by this we recall the definition of a linear transformation and imbue it with perhaps one of the most important notions which is that of being bounded. We then note quickly that the set of all bounded linear transformations from a vector space X to a vector space Y is a vector space. Additionally we give meaning to vector multiplication and give some definitions with respect to this. After this, we remember the definition of an invertible linear map and give some properties. We then end the chapter with the brief definition of a very important concept which is the dual space and make the distinction between algebraic dual and topological dual. Proofs in this first section will be omitted given that this is not the main setting of our study. The reader can refer to the following books [3], [5] and [4] for a more detailed study of these topics as we will just briefly go over the basic theory that we will need later.

We begin our study through basic notions of functional analysis by introducing the concept of normed spaces.

Definition 1.1.1. Let X be a vector space over a field $\mathbb{F}$ usually $\mathbb{R}$ or $\mathbb{C}$. A norm on X is a function $\|\cdot\| : X \to [0, \infty)$ such that:

1. For all $x \in X, \|x\| \geq 0$. If $x \in X$ and $\|x\| =, 0$ then $x = 0$.
2. For all $\alpha \in \mathbb{F}$ and all $x \in X, \|\alpha x\| = |\alpha| \, \|x\|$.
3. For all $x, y \in X, \|x + y\| \leq \|x\| + \|y\|$.

A normed space is a vector space X equipped with a norm.

A Banach space is a normed vector space where every Cauchy sequence associated with the norm converges.

Example 1.1.2.

1. Finite dimensional normed spaces are Banach spaces.
2. $(C[a,b], \|\cdot\|_\infty)$ is a Banach space.
3. Consider the space of all vectors with infinite number of components (real or complex) a_j where $|a_j|$ is bounded for all j. The norm is
$$\|x\|_\infty = \sup_j |a_j| .$$
This space is denoted as ℓ^∞ and it is a Banach space.
4. Similarly, consider the space of all vectors with infinite number of components (real or complex) a_j such that $\sum |a_j|^p < \infty$ for some fixed $p \geq 1$. The norm is
$$\|x\|_p = \left(\sum |a_j|^p\right)^{\frac{1}{p}} .$$
This space is denoted as ℓ^p and it is a complete normed space.

Definition 1.1.3. A normed linear space is called separable if it contains a countable set of points that is dense, meaning that its closure is the whole space.

1.1.2 Continuous linear operators. Now that we have seen what normed spaces are, we introduce linear operators and we mention a very important property which states that a linear operator is bounded if and only if it is continuous.

Definition 1.1.4. Let X and Y be vector spaces over $\mathbb{R}$ or $\mathbb{C}$. An operator $T : X \to Y$ is linear if it satisfies the following:

1. For all $x, y \in X, T(x+y) = T(x) + T(y)$.
2. For all $x \in X, \alpha \in \mathbb{F}, T(\alpha x) = \alpha T(x)$.

Example 1.1.5.

1. Let $X = Y = \ell^2$. Define the $left/right$ operators L/R as follows: if $(x)_{n\in\mathbb{N}} \in \ell^2$, then
$$L\left((x_1, x_2, x_3, ...)\right) = (x_2, x_3, x_4, ...),$$
$$R\left((x_1, x_2, x_3, ...)\right) = (0, x_1, x_2, x_3, ...).$$
It can be verified that L and R are linear operators.
2. The map $I : C[a,b] \to \mathbb{R}$ given by $I(x) = \int_a^b x(t)dt$ for all $x \in C[a,b]$ is a linear operator.
3. If $\bar{\cdot}$ denotes complex conjugation, then the complex conjugation map $z \mapsto \overline{z} : \mathbb{C} \to \mathbb{C}$ is not a linear operator, since the second condition is not satisfied.

From now on, we shall denote the set of all linear operators from a vector space X to a vector space Y as $\mathcal{L}(X, Y)$. If the linear operators go from X to X we will instead write $\mathcal{L}(X)$. Recall that this set is a vector space over the common field $\mathbb{F}$, where addition and scalar multiplication are defined as follows: Let $S, T \in \mathcal{L}(X, Y), \alpha \in \mathbb{F}$, then

$$(S+T)(x) = S(x) + T(x).$$

$$(S+T)(x) = S(x) + T(x) \tag{1.1}$$

$$(\alpha \cdot T)(x) = \alpha \cdot T(x).$$

Theorem 1.1.6. *Let X, Y be normed spaces and $T : X \to Y$ be a linear operator. Then, the following properties of T are equivalent.*

1. *T is continuous.*
2. *T is continuous at 0.*
3. *There exists an $M > 0$ such that $\|T(x)\| \leq M\|x\|$.*

We shall denote the set of all continuous linear operators from a normed space X to a normed space Y as $\mathcal{B}(X, Y)$. From now on, continuous linear operators and bounded linear operators will mean the same thing thanks to the theorem, and we will insist on using the notation $\mathcal{B}(x)$ for both. We will also refer to linear operators as operators for the sake of brevity.

Example 1.1.7.

1. Let $X, Y = \ell^2$ and recall the $left/right$ shift operators. We now show that they are continuous by showing that they are bounded. For all $(x_n)_{n\in\mathbb{N}} \in \ell^2$ we have

$$\|L(x)_{n\in\mathbb{N}}\|_2 \ = \sqrt{|x_2|^2 + |x_3|^2 + |x_4|^2 + ...} \leq \sqrt{|x_1|^2 + |x_2|^2 + |x_3|^2 + ...} = 1 \cdot \|(x)_{n\in\mathbb{N}}\|_2.$$

   So that $L \in \mathcal{B}(\ell^2, \ell^2)$.

   Similarly,

$$\|R(x)_{n\in\mathbb{N}}\|_2 \ = \sqrt{0 + |x_1|^2 + |x_2|^2 + |x_3|^2 + ...} = \|(x)_{n\in\mathbb{N}}\|_2.$$

   Thus, $R \in \mathcal{B}(\ell^2, \ell^2)$.

2. Let $L^1(\mathbb{R})$ be the space of complex valued Lebesgue integrable functions on $\mathbb{R}$ with the usual L^1-norm:

$$\|f\|_1 := \int_{-\infty}^{\infty} |f(x)|\, dx < \infty, \quad f \in L^1(\mathbb{R}).$$

   We define its Fourier transform as the function given by

$$\hat{f}(\xi) := \int_{-\infty}^{\infty} f(x) e^{-2\pi i \xi x} dx, \quad \xi \in \mathbb{R}.$$

Note that $\hat{f}$ is bounded because we have the following estimate

$$\left|\int_{-\infty}^{\infty} f(x)e^{-2\pi i\xi x}dx\right| \leq \int_{-\infty}^{\infty} |f(x)| \left|e^{-2\pi i\xi x}\right| dx = \int_{-\infty}^{\infty} |f(x)|\, dx = \|f\|_1 .$$

We now introduce a very important notion; the operator norm.

Theorem 1.1.8. *Let $T \in \mathcal{B}(X,Y)$. The map $\|\cdot\| : \mathcal{B}(X,Y) \to \mathbb{R}$. Given by*

$$\|T\| := sup\{\|Tx\| : x \in X, \|x\| \leq 1\}.$$

Defines a norm on $\mathcal{B}(X,Y)$.

From this definition it is immediate that for $T \in \mathcal{B}(X)$ and $x \in X$, $\|Tx\| \leq \|T\|\,\|x\|$.

Theorem 1.1.9. *Let X, Y be normed spaces. The set $\mathcal{B}(X,Y)$ is a Banach space if and only if Y is a Banach space.*

Notation 1. Let $T : X \to Y$ be linear map.

1. We will denote the range of T as $\operatorname{ran}(T)$.
2. The domain of T will be written as $D(T)$.
3. The nullspace or kernel of T will be denoted by $\ker(T)$.

1.1.3 The Algebra of continuous linear maps.

Let us take a moment from all the analytic properties to delve into some algebraic notions that will be important in the future. We will give sense to vector multiplication and talk about what it means for a vector space to be an algebra, as well as some other properties and definitions such as invertibility of operators. For this brief section we follow [5].

Definition 1.1.10. A vector space V which has associative vector multiplication and compatibility between addition and multiplication is called an Algebra, that is $(uv)w = u(vw)$, $u(v+w) = uv+uw$, and $(v+w)u = vu+wu$, for all $u, v, w \in V$. Additionally, we have that $\alpha(uv) = u(\alpha v) = (\alpha u)v$, for all $u, v \in V$ and $\alpha \in \mathbb{F}$. If there exists an element $e \in V$ such that for all $v \in V$ $ev = v = ve$, we say that $e \in V$ is a multiplicative identity

Definition 1.1.11. A normed algebra is an algebra V equipped with a norm $\|\cdot\|$ that satisfies $\|uv\| \leq \|u\|\|v\|$ for all $u, v \in V$. A Banach algebra is a normed algebra that is complete.

Let X, Y, Z be normed spaces and $T \in \mathcal{B}(X,Y)$, $S \in \mathcal{B}(Y,Z)$. Then, the composition $ST : X \to Z$ given by $(ST)(x) = S(Tx)$ for all $x \in X$, is a continuous linear transformation. Where continuity is given by noting that $\|(ST)(x)\| = \|S(Tx)\| \leq \|S\|\|Tx\| \leq \|S\|\|T\|\|x\|$.

This tells us that if we have a normed space X, the set $\mathcal{B}(X)$, not only has vector addition and scalar multiplication, it also has a multiplication operation between vectors given by composition of linear maps. Therefore, the set $\mathcal{B}(X)$ is a normed algebra with a multiplicative element given by the identity operator. Note also that if X is a Banach space, then $\mathcal{B}(X)$ is a Banach algebra.

The following results are about the existence of an inverse for bounded operators.

Definition 1.1.12. Let X be a normed space. We say that $A \in \mathcal{B}(X)$ is invertible if there exists an operator $B \in \mathcal{B}(X)$ such that $AB = I = BA$.

Proposition 1.1.13. *Let X be a normed space. If $A \in \mathcal{B}(X)$ is invertible, then there exists a unique $B \in \mathcal{B}(X)$ such that $AB = I = BA$. This unique inverse of A will be denoted by A^{-1}.*

Proposition 1.1.14. *If $A \in \mathcal{B}(X)$ is invertible, then A is bijective.*

Theorem 1.1.15. *Let X be a Banach space and $A \in \mathcal{B}(X)$ such that $\|A\| \leq 1$ then:*

1. *$I - A$ is invertible in $\mathcal{B}(X)$.*
2. *$(I - A)^{-1} = \sum\limits_{n=0}^{\infty} A^n$.*
3. *$\left\|(I - A)^{-1}\right\| \leq \frac{1}{1-\|A\|}$.*

1.1.4 Dual Space.

Definition 1.1.16. Let X be a normed space over $\mathbb{F}$. Then the normed space $\mathcal{B}(X, \mathbb{F})$ equipped with the operator norm is called the dual space of X. We will denote the dual space of X by X'. Elements of the dual space are called bounded linear functionals.

Definition 1.1.17. Let X, Y be normed spaces and $T \in \mathcal{B}(X, Y)$. We define the dual operator of T, as $T' \in \mathcal{B}(Y', X')$, by $(T'\varphi)(x) = \varphi(Tx)$, for all $x \in X$ and $\varphi \in Y'$.

We will not pay too much attention to dual spaces of general normed spaces, since our structure to work on will be Hilbert spaces and later on we will have an important result which tells us that the dual space of a Hilbert space is itself and this will make things somewhat easier to understand. If the reader is interested, a rich study regarding dual spaces can be found in [4].

1.2 Hilbert Spaces

We follow our analytical study of vector spaces by defining the inner product and give some definitions and examples which are very similar to what one would expect in the euclidean space $\mathbb{R}^n$. However, if we consider a complex vector space, we get a much richer theory out of inner products which will lead to very important consequences in spectral theory. We begin with some definitions, some results and note immediately how an inner product defines a norm, which allows us to use all the theory that we had mentioned so far. We move on to study everything orthogonality related such as, orthonormality, orthogonal complement and projections. We end the chapter with the Riesz representation theorem and we define adjoints of bounded operators, study some of their important properties and mention some other types of operators. In this section some proofs will be given with respect to the more relevant results. This section on Hilbert spaces is based on [5] and [3].

1.2.1 Inner Product Spaces. We begin our study regarding Hilbert spaces by defining some notions that are ver similar to what one would encounter when studying $\mathbb{R}^n$.

Definition 1.2.1. Let X be a vector space over $\mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$). A function $\langle\cdot,\cdot\rangle : X \times X \to \mathbb{F}$ is called an inner product on the vector space X if:

1. For all $x \in X, \langle x, x\rangle \geq 0$. If $x \in X$ and $\langle x, x\rangle = 0$, then $x = 0$.

2. For all $x_1, x_2, y \in X, \langle x_1 + x_2, y\rangle = \langle x_1, y\rangle + \langle x_2, y\rangle$.

   For all $x, y \in X$ and all $\alpha \in \mathbb{F}, \langle \alpha x, y\rangle = \alpha \langle x, y\rangle$

3. For all $x, y \in X, \langle x, y\rangle = \overline{\langle y, x\rangle}$ (Recall that $\bar{\cdot}$ denotes complex conjugation).

A vector space X endowed with an inner product is called an inner product space. Note that 2 and 3 imply that the inner product is antilinear with respect to the second variable.

Remark 1.2.2. From now on we will consider the base field for all vector spaces as the space of complex numbers, unless stated otherwise.

Theorem 1.2.3. *Let X be an inner product space, then for all $x, y \in X, |\langle x, y\rangle| \leq \|x\| \|y\|$. Where equality holds if and only if x, y are linearly dependent.*

This result is usually referred to as the Cauchy–Schwarz inequality. and it is what allows inner product spaces to be normed spaces with the norm given by the following proposition.

Proposition 1.2.4. *Every inner product space is automatically a normed space, with the norm given by*

$$\|x\| = \sqrt{\langle x, x\rangle}, x \in X.$$

Definition 1.2.5. Let X be an inner product space and $T \in \mathcal{B}(X)$ we say that T is an isometry if $\langle x, x\rangle = \langle Tx, Tx\rangle$ for all $x \in X$.

Definition 1.2.6. An Inner product space which is a Banach space with the induced norm is called a Hilbert space.

This next result gives way to a generalized form of the Pythagorean theorem.

Theorem 1.2.7. *Let X be an inner product space, then for all $x, y \in X$ we have*

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

If x, y are orthogonal then $\|x + y\|^2 = \|x\|^2 + \|y\|^2$.

More generally, if $x_1, x_2, ..., x_n$ are pairwise orthogonal, the we have

$$\left\| \sum_{k=1}^{n} x_k \right\|^2 = \sum_{k=1}^{n} \|x_k\|^2$$

Example 1.2.8.

1. The vector space ℓ^2 is a Hilbert space with the inner product
$$\langle x, y \rangle = \sum x_n y_n^*$$
for $(x_n), (y_n) \in \ell^2$.

2. The space L^2 of all functions square integrable in the Lebesgue sense in some domain in $\mathbb{R}^n$. This space is complete.

This next result proves the continuity of inner products.

Theorem 1.2.9. *Let X be an inner product space and suppose that (x_n) and (y_n) are sequences in X which converge to $x, y \in X$ respectively. Then*

$$\lim_{n \to \infty} \langle x_n, y_n \rangle = \langle x, y \rangle .$$

Proof. Let $\epsilon > 0$. Consider $N_1 \in \mathbb{N}$ such that for all $n > N_1$,

$$\|x_n - x\| < \frac{\epsilon}{2\left(\|y\| + 1\right)}.$$

Additionally, given that x_n is convergent, it is bounded, so that $M := \sup_{n \in \mathbb{N}} \|x_n\| < \infty$ is well defined. Moreover, let $N_2 \in \mathbb{N}$ be such that for all $n > N_2$,

$$\|y_n - y\| < \frac{\epsilon}{2\left(M + 1\right)}.$$

With this in mind, for all $n > \max\{N_1, N_2\}$, we have

$$\begin{aligned}
|\langle x_n, y_n \rangle - \langle x, y \rangle| &= |\langle x_n, y_n \rangle - \langle x_n, y \rangle + \langle x_n, y \rangle - \langle x, y \rangle| \\
&= |\langle x_n, y_n - y \rangle - \langle x_n - x, y \rangle| \\
&\leq |\langle x_n, y_n - y \rangle| + |\langle x_n - x, y \rangle| \\
&\leq \|x_n\| \|y_n - y\| + \|x_n - x\| \|y\| \\
&\leq M \frac{\epsilon}{2\left(M + 1\right)} + \frac{\epsilon}{2\left(\|y\| + 1\right)} \|y\| \\
&\leq \epsilon
\end{aligned}$$

So that $\langle x_n, y_n \rangle$ converges with limit $\langle x, y \rangle$. □

1.2.2 Orthogonality and Orthonormal sets.. Even though measuring angles in infinite dimensional vector spaces might not be possible, it is still possible to talk about orthogonality in inner product spaces, and this subsection will do just that, notice that many of these results are what one would encounter in linear algebra when studying $\mathbb{R}^n$.

Definition 1.2.10. Let X be an inner product space.

1. $x, y \in X$ are said to be orthogonal if $\langle x, y \rangle = 0$.
2. A subset S of X is said to be an orthonormal set if for all $x, y \in S$,
   (a) $x \neq y, \rightarrow \langle x, y \rangle = 0$ and
   (b) $\langle x, x \rangle = 1$.

Theorem 1.2.11. *Let $\{x_1, x_2, x_3, ...\}$ be a linearly independent subset of an inner product space X. Define*

$$u_1 = \frac{x_1}{\|x_1\|},$$

$$u_n = \frac{x_n - \langle x_n, u_1 \rangle u1 - ... - \langle x_n, u_{n-1} \rangle u_{n-1}}{\|x_n - \langle x_n, u_1 \rangle u1 - ... - \langle x_n, u_{n-1} \rangle u_{n-1}\|}$$

Then $\{u_1, u_2, u_3, ...\}$ is an orthonormal set in X and for all $n \in \mathbb{N}$,

$$span\{x_1, ..., x_n\} = span\{u_1, ..., u_n\}.$$

This next inequality is known as Bessel's inequality and it is a very important tool in the setting of orthonormal sets, as it provides information about the elements of the space in question. Furthermore, it is also used in many proofs.

Theorem 1.2.12. *Let u_n an orthonormal set in an inner product space X. Then for all $x \in X$*

$$\sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2 \leq \|x\|^2.$$

Proof. For $N \in \mathbb{N}$ we have

$$\begin{aligned}
0 &\leq \left\| x - \sum_{n=1}^{N} \langle x, u_n \rangle u_n \right\|^2 \\
&= \langle x, x \rangle - 2\mathrm{Re} \left\langle x, \sum_{n=1}^{N} \langle x, u_n \rangle u_n \right\rangle + \left\langle \sum_{n=1}^{N} \langle x, u_n \rangle u_n, \sum_{m=1}^{N} \langle x, u_m \rangle u_m \right\rangle \\
&= \|x\|^2 - 2\mathrm{Re} \sum_{n=1}^{N} \left(\overline{\langle x, u_n \rangle} \langle x, u_n \rangle \right) + \sum_{n=1}^{N} \sum_{m=1}^{N} \langle x, u_n \rangle \overline{\langle x, u_m \rangle} \langle u_n, u_m \rangle \\
&= \|x\|^2 - 2 \sum_{n=1}^{N} |\langle x, u_n \rangle|^2 + \sum_{n=1}^{N} |\langle x, u_n \rangle|^2 \\
&= \|x\|^2 - \sum_{n=1}^{N} |\langle x, u_n \rangle|^2 \\
\to & \sum_{n=1}^{N} |\langle x, u_n \rangle|^2 \leq \|x\|^2
\end{aligned}$$

Since N was arbitrary, we can let it grow indefinetely to arrive at our result. □

Example 1.2.13.

1. In ℓ^2, $\{e_i := (0, \cdots, 0, \underset{i\text{th term}}{1}, 0, \cdots) : i \in \mathbb{N}\}$ is an orthonormal set.
   If δ_{ij} denotes the Kronecker delta, that is,
   $$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$
   then we have $\langle e_i, e_j \rangle = \delta_{ij}$.

2. Consider $C[0,1]$ with the usual inner product.
   For $n \in \mathbb{Z}$ let $T_n \in C[0,1]$ be given by
   $$T_n(t) = e^{2\pi i n t}, \quad t \in [0,1].$$
   Let us check that $\{T_n : n \in \mathbb{Z}\}$ is an orthonormal set. First, if $m, n \in \mathbb{Z}$ and $n \neq m$, then we have
   $$\int_0^1 T_n(t)\overline{T_m(t)}\, dt = \int_0^1 e^{2\pi i n t} e^{-2\pi i m t}\, dt = \int_0^1 e^{2\pi i (n-m)t}\, dt.$$
   $$= \left. \frac{e^{2\pi i (n-m)t}}{2\pi i (n-m)} \right|_0^1 = \frac{1-1}{2\pi i (n-m)} = 0.$$
   Now, for all $n \in \mathbb{Z}$,
   $$\langle T_n, T_n \rangle = \int_0^1 |T_n(t)|^2\, dt = \int_0^1 1\, dt = \int_0^1 1\, dx = 1.$$

So $\{\cdots, e^{-4\pi it}, e^{-2\pi it}, e^0 = 1, e^{2\pi it}, e^{4\pi it}, \cdots\}$ is an orthonormal set in $C[0,1]$.

3. Consider the following sequence P_n, $n \geq 0$ of polynomials that satisfy

$$P_n(t) = \frac{1}{n!2^n}\left(\frac{d}{dt}\right)^n (t^2-1)^n.$$

This is known as Rodrigues' Formula and the polynomials P_n that solve this equation are known as the polynomials of Legendre. It can be verified that they have the following properties:

(a) P_n is a polynomial of degree n, and $P_n(1) = 1$.

(b) The polynomials are orthogonal:

$$\int_{-1}^{1} P_n(t)P_m(t)\,dt = 0 \quad \text{for } n \neq m.$$

(c) The norm of P_n is given by

$$\int_{-1}^{1} P_n(t)^2\,dt = \frac{2}{2n+1}.$$

1.2.3 Orthogonal complement and Orthogonal Projection. Since in inner products we have the notion of orthogonality, it is only natural to talk about concepts like orthogonal complement and orthogonal projection, in this general setting, given that we are not necessarily working with angles, we will have to carefully use the definition of orthogonality. One of the important properties of the orthogonal complement that we will mention briefly is that it is a closed vector space. This property will be crucial when discussing several results later on.

Definition 1.2.14. Let Y be a subspace of an inner product space X. The orthogonal complement $Y^\perp$ of Y is defined by

$$Y^\perp := \{x \in X : \langle x, y\rangle = 0, \forall y \in Y\}$$

If $N \subset M$ are subspaces of a Hilbert space H, we denote the set of all the vectors in M orthogonal to N as $M_N^\perp$.

Proposition 1.2.15. *If Y is a subspace of an inner product space X, then $Y^\perp$ is a closed subsapce of X.*

Proof. First note that $Y^\perp$ is nonempty since $\langle 0, y\rangle = 0$ for all $y \in Y$ so that $0 \in Y^\perp$. If $x_1, x_2 \in Y^\perp$ and $\alpha \in \mathbb{C}$ then $\langle \alpha(x_1+x_2), y\rangle = \alpha\langle x_1+x_2, y\rangle = \alpha\langle x_1, y\rangle + \alpha\langle x_2, y\rangle = \alpha\cdot 0 + \alpha\cdot 0 = 0$. Meaning that $\alpha(x_1+x_2) \in Y^\perp$. Finally, let (x_n) be a sequence in $Y^\perp$ that converges in to $x \in X$. This implies that for all $y \in Y$,

$$\langle x, y\rangle = \left\langle \lim_{n\to\infty} x_n, y\right\rangle = \lim_{n\to\infty}\langle x_n, y\rangle = 0.$$

From this we can conclude that $x \in Y^{\perp}$ and that $Y^{\perp}$ is a closed subspace of X.

$\square$

These next results give way to the notion of orthogonal projection, which mirrors the definition in say $\mathbb{R}^2$ when one has that the path which minimizes the distance from a point to a line which does not contain said point is precisely the orthogonal projection, which is another line that forms an angle of 90 degrees with the other line.

Theorem 1.2.16. *Let X be an inner product space, $x \in X$ and $Y = span\{u_1, ..., u_n\} \subseteq X$, where $\{u_1, ..., u_n\}$ is an orthonormal basis for Y.*

Define $y_* = \sum_{k=1}^{n} \langle x, u_k \rangle u_k$.

Then $y_* \in Y$ is such that for all $y \in Y, \|x - y\| \geq \|x - y_*\|$.

Theorem 1.2.17. *Let H be a Hilbert space, C a closed, convex, nonempty subset of H, and $x \in H$. Then:*

1. *There exists a unique $c_* \in C$ such that for all $c \in C, \|x - c_*\| \leq \|x - c\|$.*
2. *The point $c_* \in C$ can be characterised by the following property: for all*
$$c \in C, Re\langle x - c_*, c - c_* \rangle \leq 0.$$

Theorem 1.2.18. *Let Y be a closed subspace of a Hilbert space H and let $x \in H$. Then there exists a unique $y_* \in Y$ such that for all $y \in Y, \|x - y_*\| \leq \|x - y\|$. The point $y_* \in Y$ can be characterised by: for all $y \in Y, \langle x - y_*, y \rangle = 0$. We write $x - y_* \perp Y$.*

Definition 1.2.19. Let Y be a closed subspace of the Hilbert space H, and let $x \in H$. Then the unique $y_* \in Y$ is called the orthogonal projection of x to Y. We will denote the orthogonal projection of x to Y by $P_Y x$.

Now that we have formally defined the orthogonal projection we give some important properties of this operator

Theorem 1.2.20. *Let Y be a closed subspace of the Hilbert space H.*

1. *The map $x \mapsto P_Y x : H \mapsto H$ belongs to $\mathcal{B}(H)$.*
2. *$ran(P_Y) = Y$.*
3. *$\|P_Y\| = 1$ except if $Y = \{0\}$, in which case $P_Y = 0$ and $\|P_Y\| = 0$.*
4. *$ker(P_Y) = Y^{\perp}$.*
5. *Every $x \in H$ has a unique decomposition $x = y + z$, with $y \in Y, z \in Y^{\perp}$.*
6. *$P_{Y^{\perp}} = I - P_Y$.*

7. *$P^2{}_Y = P_Y$.*

8. *P_Y is symmetric, that is, for all $x, x' \in H, \langle P_Y x, x' \rangle = \langle x, P_Y x' \rangle$.*

Proof.

1. Let $x_1, x_2 \in H$. Then $P_Y x_1, P_Y x_2 \in Y$, and so is their sum.
   Moreover, for all $y \in Y$,
   $$\langle x_1 + x_2 - (P_Y x_1 + P_Y x_2), y \rangle = \langle x_1 - P_Y x_1, y \rangle + \langle x_2 - P_Y x_2, y \rangle = 0 + 0 = 0.$$
   Thus $P_Y(x_1 + x_2) = P_Y x_1 + P_Y x_2$.

   Let $x \in H$ and $\alpha \in \mathbb{K}$. Then $P_Y x \in Y$ and so $\alpha P_Y x \in Y$ too.
   Moreover, for all $y \in Y$,
   $$\langle \alpha x - \alpha P_Y x, y \rangle = \alpha \langle x - P_Y x, y \rangle = \alpha \cdot 0 = 0.$$
   So $P_Y(\alpha x) = \alpha P_Y x$.
   Hence $P_Y \in L(H)$. Next, we'll show continuity of P_Y.
   For $x \in X$, and all $y \in Y$, $\langle x - P_Y x, y \rangle = 0$, that is, $x - P_Y x \in Y^\perp$.
   Thus
   $$\|x\|^2 = \|x - P_Y x + P_Y x\|^2 = \|x - P_Y x\|^2 + \|P_Y x\|^2 \geq \|P_Y x\|^2,$$
   showing that $P_Y \in \mathcal{B}(H)$ and that $\|P_Y\| \leq 1$.

2. For all $x \in H$, $P_Y x \in Y$, and so $\operatorname{ran} P_Y \subseteq Y$. Now suppose that $y' \in Y$. Then $y' - P_Y y' \in Y^\perp$, but for all $y \in Y$, we have $\langle y' - P_Y y', y \rangle = 0$.
   In particular, with $y := y' - P_Y y'$, $\|y' - P_Y y'\|^2 = \langle y' - P_Y y', P_Y y \rangle = 0$, showing that $P_Y y' = y'$.
   Hence $Y \subseteq \operatorname{ran} P_Y$.

3. If $Y = \{0\}$, then $P_Y = 0$, and so $\|P_Y\| = 0$.
   Suppose that $Y \neq \{0\}$. Then there exists a $y' \in Y \setminus \{0\}$, and as we had seen above, $P_Y y' = y'$. So $\|P_Y\| \|y'\| \geq \|P_Y y'\| = \|y'\|$, giving $\|P_Y\| \geq 1$. Also, we had established in part (1) that $\|P_Y\| \leq 1$. Thus $\|P_Y\| = 1$.

4. Let $x \in \ker P_Y$. Then $P_Y x = 0$. For all $y \in Y$,
   $$0 = \langle x - P_Y x, y \rangle = \langle x - 0, y \rangle = \langle x, y \rangle,$$
   and so $x \in Y^\perp$. Thus $\ker P_Y \subseteq Y^\perp$.
   Let $x \in Y^\perp$. Then for all $y \in Y$, $\langle x - 0, y \rangle = \langle x, y \rangle = 0$. Thus $P_Y x = 0$. Hence $Y^\perp \subseteq \ker P_Y$ too.

5. If $x \in X$, then $x - P_Y x \in Y^\perp$. Thus, taking $y := P_Y x$ and $z := x - P_Y x$, we get $x = y + z$ with $y \in Y$ and $z \in Y^\perp$. Also, if $y' \in Y$ and $z' \in Y^\perp$ such that $x = y' + z'$, then $y + z = y' + z'$,

and so we obtain that $Y \cap Y^{\perp} = \{0\}$. Hence $y - y' = 0 = z' - z$, that is, $y = y'$ and $z = z'$.

6. We have seen that $x = y + z$, where $y = P_Y x \in Y$ and $z = x - P_Y x \in Y^{\perp}$. As $Y^{\perp}$ is also a closed subspace and using the fact that $Y \subset (Y^{\perp})^{\perp}$, it follows from part (5) that $P_{Y^{\perp}} x = z$. Hence $P_{Y^{\perp}} = I - P_Y$.

7. We have that $P_Y x = P_Y x + 0$, where $P_Y x \in Y$ and $0 \in Y^{\perp}$. Using part (5) this means that $P_Y(P_Y x) = P_Y x$.

8. Let $x, z \in H$. It follows that

$$\begin{aligned}
\langle P_Y x, z \rangle &= \langle P_Y x, P_Y z + P_{Y^{\perp}} z \rangle \\
&= \langle P_Y x, P_Y z \rangle + \langle P_Y x, P_{Y^{\perp}} z \rangle \\
&= \langle P_Y x, P_Y z \rangle \\
&= \langle x - P_{Y^{\perp}} x, P_Y z \rangle \\
&= \langle x, P_Y z \rangle - \langle P_{Y^{\perp}} x, P_Y z \rangle \\
&= \langle x, P_Y z \rangle .
\end{aligned}$$

□

Proposition 1.2.21. *Let Y be a closed subspace of a Hilbert space H. Then $(Y^{\perp})^{\perp} = Y$.*

Just like in finite dimensions we have the notion of orthonormal basis. Additionally we give some crucial properties that help us relate elements form the whole space with elements that come from the orthonormal basis.

Proposition 1.2.22. *For any subset $Y \neq \emptyset$ of a Hilbert space H, the span of Y is dense in H if and only if $Y^{\perp} = \{0\}$.*

Definition 1.2.23. Let X be an inner product space. A set $B \subseteq X$ is called an orthonormal basis for X if:

1. $span(B)$ is dense in X, and
2. B is an orthonormal set.

Theorem 1.2.24. *Let X be an inner product space with a countable orthonormal basis $\{u_1, u_2, u_3, ...\}$. Then:*

1. *For all $x \in X, x = \sum_{n=1}^{\infty} \langle x, u_n \rangle u_n$.*
2. *For all $x, y \in X, \langle x, y \rangle = \sum_{n=1}^{\infty} \langle x, u_n \rangle \langle y, u_n \rangle^*$.*
3. *For all $x \in X, \|x\|^2 = \sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2$.*

Proof. 1. Let $X \in X$ and $\epsilon > 0$. Since span $\{u_1, u_2, u_3, ...\}$ is dense in X, there exists an $N \in \mathbb{N}$ and scalars $c_1, ..., c_N \in \mathbb{C}$ such that

$$\left\| x - \sum_{k=1}^{N} c_k u_k \right\| < \epsilon.$$

Let $n > N$ and $Y = \text{span}\{u_1, ..., u_n\}$. Since $\sum_{k=1}^{N} c_k u_k \in Y$, we get

$$\left\| x - \sum_{k=1}^{N} \langle x, u_k \rangle u_k \right\| \leq \left\| x - \sum_{k=1}^{N} c_k u_k \right\|$$

Which means that $\sum_{k=1}^{N} \langle x, u_k \rangle u_k$ converges to $x \in X$ when taking the limit as $n \to \infty$.

2. If $x, y \in X$, by (1) we have that

$$x_n := \sum_{k=1}^{n} \langle x, u_k \rangle u_k \to x \quad \text{as} \quad n \to \infty$$

and

$$y_n := \sum_{k=1}^{n} \langle y, u_k \rangle u_k \to y \quad \text{as} \quad n \to \infty.$$

Taking this into acount we now calculate $\langle x, y \rangle$.

$$\begin{aligned} \langle x, y \rangle &= \left\langle \lim_{n\to\infty} x_n, \lim_{n\to\infty} y_n \right\rangle = \lim_{n\to\infty} \langle x_n, y_n \rangle \\ &= \lim_{n\to\infty} \left\langle \sum_{k=1}^{n} \langle x, u_k \rangle u_k, \sum_{j=1}^{n} \langle y, u_j \rangle u_j \right\rangle \\ &= \lim_{n\to\infty} \sum_{k=1}^{n} \sum_{j=1}^{n} \langle x, u_k \rangle \overline{\langle y, u_j \rangle} \langle u_k, u_j \rangle \\ &= \lim_{n\to\infty} \sum_{k=1}^{n} \langle x, u_k \rangle \overline{\langle y, u_k \rangle}. \end{aligned}$$

3. This is readily seen from (2) by taking $y = x$.

□

Theorem 1.2.25. *If $X = H$ is a Hilbert space with a countable orthonormal basis $\{u_1, u_2, ..\}$, then for all $(c_n)_{n\in\mathbb{N}} \in \ell^2$, we have $\sum_{n=1}^{\infty} c_n u_n \in H$.*

Proof. Let H be a Hilbert space and $(c_n) \in \ell^2$. Then, $(\sum)_{k=1}^{n} |c_k|^2$ is a Cauchy sequence. We will now show that $(\sum_{k=1}^{n} c_k u_k)$ is a Cauchy sequence. Let $n > m$, we thus have,

$$\begin{aligned}\left\|\sum_{k=1}^{n} c_k u_k - \sum_{k=1}^{m} c_k u_k\right\|^2 &= \left\|\sum_{k=m+1}^{n} c_k u_k\right\|^2 = \sum_{k=m+1}^{n} |c_k|^2 \\ &= \left|\sum_{k=m+1}^{n} |c_k|^2 - \sum_{k=m+1}^{m} |c_k|^2\right|.\end{aligned}$$

Since we are in a Hilbert space, this is also a convergent sequence in H. As a consequence, $\sum_{n=1}^{\infty} c_n u_n \in H$. □

This theorem is basically telling us that all Hilbert spaces with a countable orthonormal basis are isomorphic to ℓ^2.

1.2.4 Riesz Representation Theorem.

Let H be a Hilbert space, and $y \in H$. Then the map $\varphi_y : H \mapsto \mathbb{C}$ given by

$$x \mapsto \langle x, y\rangle : H \mapsto \mathbb{C}$$

is a continuous linear transformation on H. This tells us that φ_y is an element of the dual space $H' = \mathcal{B}(H, \mathbb{C})$. We will now see that every element of the dual space is of this space.

Theorem 1.2.26. *If H is a Hilbert space, and $\varphi \in \mathcal{B}(H, \mathbb{C})$, then there exists a unique $y \in X$ such that $\varphi = \varphi_y$, that is, for all $x \in H, \varphi(x) = \langle x, y\rangle$.*

1.2.5 Adjoints of bounded operators. Let H, K be Hilbert spaces, and $T \in \mathcal{B}(H, K)$. Then there exists a unique operator $T^* = \mathcal{B}(K, H)$ such that for all $h \in H$ and $k \in K$, $\langle Th, k\rangle = \langle h, T^*k\rangle$.

This T^* is called the adjoint of T.

In the context of bounded operators, the adjoint operator plays a very similar role to thata of the transpose matrix studied in linear algebra.

Theorem 1.2.27. *Let H, K be Hilbert spaces, $T, S \in \mathcal{B}(H, K)$ and $\alpha \in \mathbb{C}$. Then:*

1. $(T^*)^* = T$.
2. $\|T^*\| = \|T\|$.
3. $(\alpha T)^* = \alpha^* T$.
4. $(S + T)^* = S^* + T^*$.
5. *If $T_1 \in \mathcal{B}(H_1, K)$ and $T_2 \in \mathcal{B}(H_2, K)$ then $(T_2 T_1)^* = T_1^* T_2^*$ where H_1, H_2, K are Hilbert spaces.*

Proposition 1.2.28. *If H is a Hilbert space and $T \in \mathcal{B}(H)$ is invertible in $\mathcal{B}(H)$, then T^* is invertible in $\mathcal{B}(H)$ and $(T^*)^{-1} = (T^{-1})^*$.*

This next result is very elegant since it involves many concepts that we have seen so far and is used in many proofs

Theorem 1.2.29. *Let H, K be Hilbert spaces and $T \in \mathcal{B}(H, K)$. Then:*

1. $ker(T) = (ran(T^*))^\perp$.
2. $\overline{ran(T)} = (ker(T^*))^\perp$.

Proof. 1. Let $h \in \ker(T)$, this means that $Th = 0$. if $k \in K$ then $\langle h, T^*k\rangle = \langle Th, k\rangle = \langle 0, k\rangle = 0$. Meaning that $h \in (\operatorname{ran}(T^*))^\perp$, so that $\ker(T) \subset (\operatorname{ran}(T^*))^\perp$. Now let $h \in (\operatorname{ran}(T^*))^\perp$. It follows that for all $k \in K$, $\langle h, T^*k = 0 = \langle Th, k\rangle\rangle$, letting $k = Th$ yields $\langle Th, Th\rangle = 0$, which tells us that $Th = 0$, so that $h \in \ker(T)$ and $\ker(T) \supset (\operatorname{ran}(T^*))^\perp$. Thus $\ker(T) = (\operatorname{ran}(T^*))^\perp$.

2. Notice that if in the first equation we take orthogonal complement we obtain

$$(\ker(T))^\perp = ((\operatorname{ran}(T^*))^\perp)^\perp = \overline{\operatorname{ran}(T^*)}$$

Replacing T with T^* gives us our desired result

$$(\ker(T^*))^\perp = \overline{\operatorname{ran}(T^*)^*} = \overline{\operatorname{ran}(T)}.$$

□

Theorem 1.2.30. *Let Y be a closed subspace of a Hilbert space H, and let P_Y be the orthogonal projection operator onto Y. Then $(P_Y)^* = P_Y$.*

Here we give some additional names to operators that satisfy certain properties.

Definition 1.2.31. Let H be a Hilbert space. An operator $T \in \mathcal{B}(H)$ is called

1. Self-adjoint or Hermitian if $T^* = T$.
2. Unitary if T is bijective and $T^* = T^{-1}$.
3. Normal if $TT^* = T^*T$.
4. Positive if $\langle Tx, x\rangle \geq 0$ for all $x \in H$.

It is immediate that if T is self adjoint or unitary, T is normal. The converse is not necessarily true. If T is positive we denote it by $T \geq 0$.

Theorem 1.2.32. *Let H be a Hilbert space and $T \in \mathcal{B}(H)$, then*

1. *If T is self adjoint then $\langle Tx, x\rangle$ is real for all $x \in H$.*
2. *If H is complex and $\langle Tx, x\rangle$ is real for all $x \in H$ then T is self adjoint.*

Theorem 1.2.33. *Let H be a Hilbert space and $T, S \in \mathcal{B}(H)$. The product TS is self adjoint if and only if $TS = ST$.*

Theorem 1.2.34. *Let H be a Hilbert space and $U, V \in \mathcal{B}(H)$ such that U, V are unitary, then*

1. *U is isometric, thus $\|Ux\| = \|x\|$ for all $x \in H$.*
2. *$\|U\| = 1$ provided that $H \neq \{0\}$.*
3. *$U^* = U^{-1}$ is also unitary.*
4. *UV is unitary.*
5. *U is normal.*

We finish this section by introducing the concept of the square root of a bounded operator, which will play a very important role later on when we additionally impose that the operator is compact

Definition 1.2.35. For every positive operator T there is one and only positive operator S such that $T = S^2$; we write $S = \sqrt{T}$. In particular, for every operator T, the operator $|T| = \sqrt{T^*T}$ is well defined. Clearly, $\sqrt{T^*T} = \sqrt{TT^*}$ if and only if T is normal. If T is of finite rank then both T^*T and TT^* and therefore also $|T|$ and $|T^*|$ are of finite rank.

1.3 Compact Operators

In this section we introduce the concept of compactness of an operator. These type of operators are very important in analysis since they resemble that of finite dimensional operators, and they help solve many differential equations. We mainly focus on the definition and give some of the important results that arise when studying these operators. The theory described in this section comes from [5] and [4].

1.3.1 Definition and Properties.

Definition 1.3.1. Let X, Y be normed spaces. A linear transformation $T : X \to Y$ is said to be compact if for every bounded sequence $(x_n)_{n\in\mathbb{N}}$ contained in X, $(Tx_n)_{n\in\mathbb{N}}$ has a convergent subsequence.

We will denote the set of all compact operators from X to Y by $\mathcal{K}(X, Y)$.

Example 1.3.2. Let H be a Hilbert space. Choose two fixed, non-zero vectors $u, v \in H$. Define the operator $T : H \to H$ by:

$$T(x) = \langle x, u\rangle v$$

This operator takes any vector x, computes its inner product with the fixed vector u (which results in a scalar), and then multiplies that scalar by the fixed vector v.

We will prove this using the definition of compactness directly.

Let $(x_n)_{n\in\mathbb{N}}$ be any bounded sequence in H, then there exists a positive real number M such that $\|x_n\| \leq M$ for all $n \in \mathbb{N}$.

We want to show that the sequence $(Tx_n)_{n\in\mathbb{N}}$ has a convergent subsequence. The elements of this sequence are:

$$Tx_n = \langle x_n, u\rangle v$$

For each n, the term $\langle x_n, u\rangle$ is a scalar. Let us denote this scalar by $c_n \in \mathbb{C}$. The sequence of images is therefore $Tx_n = c_n v$.

Using the Cauchy-Schwarz inequality, we can analyze the sequence of scalars (c_n):

$$\begin{aligned} |c_n| = |\langle x_n, u\rangle| &\leq \|x_n\| \cdot \|u\| \\ &\leq M \cdot \|u\| \end{aligned}$$

Since M and $\|u\|$ are fixed constants, we have shown that $(c_n)_{n\in\mathbb{N}}$ is a bounded sequence of complex numbers.

By the Bolzano-Weierstrass Theorem, every bounded sequence in $\mathbb{C}$ has a convergent subsequence. Therefore, there must exist a subsequence $(c_{n_k})_{k\in\mathbb{N}}$ that converges to some scalar $c \in \mathbb{C}$.

$$\lim_{k\to\infty} c_{n_k} = c$$

Now, consider the corresponding subsequence of (Tx_n), which is $(Tx_{n_k})_{k\in\mathbb{N}}$. We will show that this subsequence converges to the vector cv.

$$Tx_{n_k} = c_{n_k} v$$

Let us look at the limit of the norm of the difference:

$$\begin{aligned} \lim_{k\to\infty} \|Tx_{n_k} - cv\| &= \lim_{k\to\infty} \|c_{n_k} v - cv\| \\ &= \lim_{k\to\infty} \|(c_{n_k} - c)v\| \\ &= \lim_{k\to\infty} |c_{n_k} - c| \cdot \|v\| \\ &= 0 \cdot \|v\| = 0 \end{aligned}$$

This shows that the subsequence (Tx_{n_k}) converges to cv, which makes T a compact operator.

Theorem 1.3.3. *Let X, Y be normed spaces and $T : X \to Y$ be a linear transformation. Then the following are equivalent:*

1. *T is compact.*
2. *The closure $\overline{T(B)}$ of the image under T of the closed unit ball, $B := \{x \in X : \|x\| \leq 1\}$, is compact.*

It follows from 2 that if $dim X = \infty$ then the identity operator is not compact.

Proposition 1.3.4. *Let X, Y be normed spaces, then $\mathcal{K}(X, Y)$ is a subspace of $\mathcal{B}(X, Y)$.*

The following results showcase the relationship between operators with finite dimensional range and compact operators in different settings for an ambient space X.

Theorem 1.3.5. *Let X be a normed space and Y be an inner product space. Suppose that $T \in \mathcal{B}(X,Y)$ is such that $ran(T)$ is finite dimensional. Then T is compact.*

Proof. Let $\{u_1, \cdots, u_m\}$ be an orthonormal basis for $ran(T)$. Let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence in X. Suppose that $M > 0$ is such that $\|x_n\| \leq M$ for all $n \in \mathbb{N}$. We want to show that $(Tx_n)_{n\in\mathbb{N}}$ has a convergent subsequence. (We will show that

$$\langle Tx_{n_k}, u_1\rangle \overset{k\to\infty}{\longrightarrow}, \cdots, \langle Tx_{n_k}, u_m\rangle \overset{k\to\infty}{\longrightarrow} \alpha_m \quad \text{and that} \quad Tx_{n_k} \to \sum_{\ell=1}^{m} \alpha_\ell u_\ell$$

for some subsequence $(x_{n_k})_{k\in\mathbb{N}}$.) For all $n \in \mathbb{N}$ and each $\ell \in \{1, ..., m\}$,

$$|\langle Tx_n, u_\ell\rangle| \leq \|Tx_n\| \|u_\ell\| \leq \|T\| M.$$

This implies that

$(x_n)_{n\in\mathbb{N}}$ has some subsequence $(x_n^{(1)})_{n\in\mathbb{N}}$ such that $\left\langle Tx_n^{(1)}, u_1\right\rangle \overset{n\to\infty}{\longrightarrow} \alpha_1$.

$(x_n^{(1)})_{n\in\mathbb{N}}$ has some subsequence $(x_n^{(2)})_{n\in\mathbb{N}}$ such that $\left\langle Tx_n^{(2)}, u_1\right\rangle \overset{n\to\infty}{\longrightarrow} \alpha_2$.

$(x_n^{(m-1)})_{n\in\mathbb{N}}$ has some subsequence $(x_n^{(m)})_{n\in\mathbb{N}}$ such that $\left\langle Tx_n^{(m)}, u_1\right\rangle \overset{n\to\infty}{\longrightarrow} \alpha_m$.

With this in mind, we now show that $(Tx_n^{(m)})_{n\in\mathbb{N}}$ converges to $\alpha_1 u_1 + \cdots + \alpha_m u_m$. As

$$\begin{aligned}
\left\|Tx_n^{(m)} - (\alpha_1 u_1 + \cdots + \alpha_m u_m)\right\|^2 &= \left\|\sum_{\ell=1}^{m} \left\langle Tx_n^{(m)}, u_\ell\right\rangle u_\ell - \sum_{\ell=1}^{m} \alpha_\ell u_\ell\right\|^2 \\
&= \left\|\sum_{\ell=1}^{m} (\left\langle Tx_n^{(m)}, u_\ell\right\rangle - \alpha_\ell) u_\ell\right\|^2 \\
&= \sum_{\ell=1}^{m} \left|\left\langle Tx_n^{(m)}, u_\ell\right\rangle - \alpha_\ell\right|^2 \overset{n\to\infty}{\longrightarrow} 0,
\end{aligned}$$

it follows that $(Tx_n^{(m)})_{n\in\mathbb{N}}$ is a convergent subsequence of the sequence $(Tx_n)_{n\in\mathbb{N}}$. Consequently T is compact.

□

These next results talk about limits of compact operators being compact operators.

Theorem 1.3.6. *Let X be a normed space, Y a Banach space, and $(T_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{K}(X,Y)$ that converges in $\mathcal{B}(X,Y)$ to $T \in \mathcal{B}(X,Y)$. Then T is compact.*

Corollary 1.3.7. *Let X be a normed space, Y a Hilbert space, and $(T_n)_{n\in\mathbb{N}}$ be a sequence of finite rank operators in $\mathcal{B}(X,Y)$ that converges in $\mathcal{B}(X,Y)$ to $T \in \mathcal{B}(X,Y)$. Then T is compact.*

We continue the study of compact operators by mentioning a very important algebraic concept, which is that of being an ideal of an algebra.

Definition 1.3.8. An ideal I of an algebra R is a subset I having the properties:

1. $0 \in I$.

2. If $a, b \in I$, then $a + b \in I$.

3. If $a \in I$, and $r \in R$, then $ar \in I$ and $ra \in I$.

Theorem 1.3.9. *Let H be a Hilbert space. Then we have:*

1. *If $T \in \mathcal{K}(H)$ and $S \in \mathcal{B}(H)$, then TS is compact.*

2. *If $T \in \mathcal{B}(H)$ is compact, then T^* is compact.*

3. *If $T \in \mathcal{B}(H)$ and $S \in \mathcal{K}(H)$, then ST is compact.*

Thus, the compact operators form a two sided ideal in the algebra of bounded operators.

Proof. 1. Let (x_n) be a bounded sequence in H and let $M > 0$ such that $\|x_n\| \leq M$ for all $n \in N$. Since $S \in \mathcal{B}(H)$, it follows that (Sx_n) is also a bounded sequence given that $\|Sx_n\| \leq \|S\| \|x_n\| \leq \|S\| M$. Now, as T is compact we can see that $(T(Sx_n)) = (TSx_n)$ has a convergent subsequence. We conclude that TS is compact.

2. By the result that we just proved, we immediately have that TT^* is compact. Let (x_n) be a bounded sequence in H and $\|x_n\| \leq M$ for all n. This implies that (TT^*x_n) has some convergent subsequence $(TT^*x_{n_k})$. So that for a given $\epsilon > 0$

$$\begin{aligned} 0 \leq \left\|T^*x_{n_k} - T^*_{n_l}\right\|^2 &= \langle TT^*(x_{n_k} - x_{n_l}), (x_{n_k} - x_{n_l})\rangle \\ &\leq \|TT^*(x_{n_k} - x_{n_l})\| \, \|x_{n_k} - x_{n_l}\| \end{aligned}$$

Using the boundedness of the right side term and the compactness of $TT^*x_{n_k}$ we can make both of them as small as we want. This tells us that $(T^*x_{n_k})$ is a Cauchy sequence and given that we are in a Hilbert space, it is convergent. As a consequence, T^* is compact.

3. If T is compact we use the fact that T^* is also compact. We also know that $S^* \in \mathcal{B}(H)$ so that T^*S^* is compact by the first item. We now use the second item to conclude that $(T^*S^*)^* = S^{**}T^{**} = ST$ is compact. ☐

We finish this section with a very important theorem on compact operators which will be used when we start talking about singular values.

Theorem 1.3.10. *Let T be compact. Then $\||T| u\| = \|Tu\|$ for all $u \in H$. There exists an isometric operator U from $\overline{ran(|T|)}$ onto $\overline{ran(T)}$ such that $T = U|T|$, and $|T| = U^{-1}T$. The representation*

$T = U\,|T|$ is called the polar decomposition of T.

Proof. Let $T \in \mathcal{K}(H)$. Let $A = (T^*T)^{1/2} = |T|$ and u in H we then have,

$$\|Tu\|^2 = \langle Tu, Tu\rangle = \langle u, T^*Tu\rangle = \left\langle u, A^2u\right\rangle = \langle Au, Au\rangle = \|Au\|^2. \tag{1}$$

With this in mind we define the operator U on the range of A as

$$U : Au \mapsto Tu. \tag{2}$$

To see that U is well defined note that if $Av = Au$ then $A(u-v) = 0$, so that $\|A(u-v)\| = 0$ which means that $\|T(u-v)\| = 0$ so that $T(u-v) = 0$ and $Tu = Tv$.

It further follows from (1) that U is an isometry on the range of A. Let $\operatorname{ran}(A)$ be the range of A, and define U to be zero on the orthogonal complement of $\operatorname{ran}(A)$:

$$Un = 0 \quad \text{for } n \perp \operatorname{ran}(A).$$

Since $\langle Un, v\rangle = \langle n, U^*v\rangle = 0$ for $n \perp \operatorname{ran}(A)$, and for all v, it follows that U^* maps H into the orthogonal complement of $\operatorname{ran}(A)^{\perp}$; thus the range of U^* lies in the closure of $\operatorname{ran}(A)$. We claim that

$$U^*Uw = w \quad \text{for } w \in \overline{\operatorname{ran}(A)}. \tag{3}$$

To see this we will prove that $U^*U = I$ on $\operatorname{ran}(A)$ which will then extend by continuity to $\overline{\operatorname{ran}(A)}$. Let $w \in \operatorname{ran}(A)$, using the fact that U is an isometry on $\operatorname{ran}(A)$ and the properties of inner products we have that $\|Uw\|^2 = \langle Uw, Uw\rangle = \langle w, U^*U\rangle\, w$ and $\|w\|^2 = \langle w, w\rangle$ so that $\langle w, U^*Uw\rangle = \langle w, w\rangle$ which implies that $\langle w, (U^*U - I)w\rangle = 0$ for all $w \in \operatorname{ran}(R)$. From this it follows that $U^*U - I$ must be the zero operator on $\operatorname{ran}(R)$ so that $U^*U = I$ on $\operatorname{ran}(R)$ and $U^*Uw = w$.

This means that

$$U^*Uw - w \perp \operatorname{ran}(A).$$

On the other hand, we have shown that U^* maps H into $\overline{\operatorname{ran}(A)}$. So we see that for w in $\overline{\operatorname{ran}(A)}$, $U^*Uw - w$ both belong to $\overline{\operatorname{ran}(A)}$ and is ortho to it so that (3) follows. □

1.4 Some Spectral Theory

One of the most important aspects of linear transformations and matrix theory is that of the diagonalization of matrices. In finite dimensions, one looks for the self adjoint property to achieve this. However, since our study of operators is mostly in infinite dimensions, several problems will arise when wanting to find eigenvalues for operators. In this section we will begin with the core definitions of spectral theory for general bounded operators in normed spaces and give some examples to illustrate what can possibly happen when calculating the spectrum in infinite dimensions. We then move on to the spectrum of adjoint operators which have some special properties and we end with the spectrum of compact operators. By combining the theory of self adjoint operators and compact operators, we will get a similar result to finite dimensions and the most important theorem of spectral theory; the spectral theorem. The material of this section was taken from [3], [4] and [5].

1.4.1 Definitions and Examples. We start by giving the definition of eigenvalues, eigenvectors, spectrum and resolvent set, as well as some examples.

Definition 1.4.1. Let X be a normed space and $T \in \mathcal{B}(X)$. Then, $\lambda \in \mathbb{C}$ is called an eigenvalue of T if there exists a nonzero vector $x \in X$ such that $Tx = \lambda x$. This nonzero vector is called an eigenvector of T, which corresponds to the eigenvalue λ.

Definition 1.4.2. Let X be a normed space and $T \in \mathcal{B}(X)$. We say that $\lambda \in \mathbb{C}$ belongs to the spectrum $\sigma(T)$ of T if $\lambda I - T$ is not invertible in $\mathcal{B}(X)$.

The set

$$\rho(T) := \mathbb{C} \setminus \sigma(T) = \{\lambda \in \mathbb{C} : \lambda I - T \in \mathcal{B}(X) \text{ is invertible in } \mathcal{B}(X)\}.$$

is called the resolvent set of T.

The set $\sigma_p(T)$ of all eigenvalues of T is called the point spectrum of T.

Theorem 1.4.3. *Let X be a Banach space and $T \in \mathcal{B}(X)$. then*

1. $\sigma(T) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \leq \|T\|\}$.
2. *$\rho(T)$ is an open subset of $\mathbb{C}$.*
3. *$\sigma(T)$ is a compact subset of $\mathbb{C}$.*
4. *$\sigma(T)$ is nonempty.*

Example 1.4.4.

1. Let $R \in \mathcal{B}(\ell^2)$ and suppose there exists $\lambda \in \mathbb{C}$ such that $Rx = \lambda x$ for some $(x_n)_{n\in\mathbb{N}} \in \ell^2$. Then

$$R(x_1, x_2, ...) = (0, x_1, x_2, ...) = (\lambda x_1, \lambda x_2, ...).$$

If $\lambda \neq 0$, it is immediate that $x_1 = x_2 = ... = 0$, meaning that $(x_n)_{n\in\mathbb{N}}$ would have to be the zero vector. If we now let $\lambda = 0$ then we also arrive at the conclusion that $(x_n)_{n\in\mathbb{N}}$ is the zero vector. This means that R has no eigenvalues and that $\sigma(R) = \{0\}$.

2. Now let us turn our atention to the left shift operator $L \in \mathcal{B}\left(\ell^2\right)$. Let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, $\lambda \in \mathbb{D}$ and $x = (1, \lambda, \lambda^2, \lambda^3, ...)$, since $|\lambda| < 1$, if follows that

$$\sum_{n=0}^{\infty} |\lambda^n|^2 = \sum_{n=0}^{\infty} \left(|\lambda|^2\right)^n = \frac{1}{1 - |\lambda|^2} < \infty$$

so that $x \in \ell^2$ and $x \neq 0$. From this we get that x is an eigenvector of L given that

$$Lx = L(1, \lambda, \lambda^2, \lambda^3, ...) = (\lambda, \lambda^2, \lambda^3, ...) = \lambda(1, \lambda, \lambda^2, ...) = \lambda x.$$

With this we get that each point in the open unit disk is an eigenvalue of L, that is, $\{z \in \mathbb{C} : |z| < 1\} \subset \sigma_p(L)$.

We know that $\|L\| \leq 1$, which means that $\sigma(L) \subset \{z \in \mathbb{C} : |z| \leq 1\}$. From the above calculation we know that $\{z \in \mathbb{C} : |z| < 1\} \subset \sigma_p(L) \subset \sigma(L)$. Given that $\sigma(L)$ is closed, we must have that $\{z \in \mathbb{C} : |z| \leq 1\} \subset \sigma(L)$, so that $\sigma(L) = \{z \in \mathbb{C} : |z| \leq 1\}$.

To fully determine $\sigma_p(L)$, let $\lambda \in \sigma_p(L)$ with its corresponding eigenvector $x = (x_n)_{n \in \mathbb{N}}$. By definition this means that $(x_2, x_3, ...) = L(x_1, x_2, x_3, ...) = \lambda(x_1, x_2, x_3, ...)$. Thus $\lambda x_n = x_{n+1}$ for all n, and after induction we get that $x_n = \lambda^{n-1} x_1$ for all n. Since $x \in \ell^2$ we get that

$$0 \neq \|x\|_2^2 = |x_1|^2 \sum_{n=0}^{\infty} \left(|\lambda|^2\right)^n < \infty,$$

which means that $x_1 \neq 0$, and the sum converges. In particular this means that $|\lambda| < 1$ and we get the inclusion $\sigma_p(L) \subset \{z \in \mathbb{C} : |z| < 1\}$.

3. Consider the Volterra integral operator defined by $V : L^2[0, 1] \to L^2[0, 1]$ by

$$Vf(x) = \int_0^x f(y)dy.$$

We will show that $\sigma(V) = \{0\}$.

If $\lambda \neq 0$, the equation $Vf = \lambda f$ implies that

$$f(x) = \frac{1}{\lambda} \int_0^x f(t)dt.$$

since $f \in L^2$, for $x < y$ we get

$$\begin{aligned} |f(y) - f(x)| &= \frac{1}{|\lambda|} \left| \int_x^y f(t)dt \right| \\ &\leq \frac{1}{|\lambda|} \int_x^y |f(t)|\, dt = \frac{1}{|\lambda|} \int_0^1 |f(t)| \cdot 1_{[x,y]}(t)dt \\ &\leq \frac{\|f\|_2}{|\lambda|} \left(\int_0^1 \left(1_{[x,y]}\right)^2 \right)^{\frac{1}{2}} = \frac{\|f\|_2}{|\lambda|} \sqrt{y - x}. \end{aligned}$$

With this we get that f satisfies the Hölder condition for continuity. Additionally, the fundamental theorem of calculus ensures that f is differentiable so that $f = \lambda f'$, which means that $f(t) = ce^{\frac{t}{\lambda}}$. However, $f(0) = 0$, this tells us that f must be 0 and λ cannot be an eigenvalue. If $\lambda = 0$ then $f = 0$.

Finally, since the spectrum is nonempty we get that $\sigma(V) = \{0\}$.

More generally, if we consider

$$K(f(x)(a) = \int_0^s K(s,t)f(t)dt,$$

where $K(s,t)$ is continuous in both entries and $t \leq s$, it can be shown in a very similar way that the spectrum of K is made up only by the single point 0.

Operators of this type are called Volterra operators and they are also compact.

Definition 1.4.5. Let X be a Banach space, and $T \in \mathcal{B}(X)$. The scalar

$$r_\sigma(T) := \sup_{\lambda \in \sigma(T)} |\lambda|,$$

is called the spectral radius of T.

Proposition 1.4.6. *If X is a Banach space and $T \in \mathcal{B}(X)$. Then*

1. $r_\sigma(T) \leq \|T\|$.
2. $r_\sigma(T) = \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}}$.

The second result is called the Gelfand-Beurling Formula.

It follows from this proposition that if we have a normal operator T. Then $r_\sigma(T) = \|T\|$.

We end this first part of spectral theory by mentioning the spectral mapping theorem which is a very important tool when studying eigenvalues.

Theorem 1.4.7 (Spectral Mapping Theorem)**.** *If X is a Banach space, $T \in \mathcal{B}(X), p = c_0 + c_1 z + ... + c_d z^d \in \mathbb{C}[z]$, and $p(T) := c_0 I + c_1 T + ... + c_d T^d$, then we have*

$$\sigma(p(T)) = p(\sigma(T)) := \{p(\lambda) : \lambda \in \sigma(T)\}.$$

1.4.2 Spectral Theory for self adjoint operators.

These next two results are analogous to what one would have in finite dimensions and showcase the nice behavior of the eigenvalues and eigenvectors of self-adjoint operators.

Theorem 1.4.8. *Let H be a Hilbert space and $T \in \mathcal{B}(H)$ such that $T = T^*$. Then $\sigma(T) \subseteq \mathbb{R}$.*

Proof. Let us show that given $\lambda \in \mathbb{C} \setminus \mathbb{R}, \lambda I - T$ is invertible. To this end, we will prove that it is one to one and onto. Let $\lambda = a + bi$, with $a, b \in \mathbb{R}$ and $b \neq 0$. To see that it is one to one, note that $\|(\lambda I - T)x\|^2 = \|(aI - T)x\|^2 - 2\mathrm{Re}\,(ib\,\langle x, (aI - T)x\rangle) + \|ibx\|^2$ for $x \in H$. Since $(aI - T)^* = aI - T$ we have that $\langle x, (aI - T)x\rangle$ is real so that $-2\mathrm{Re}\,(ib\,\langle x, (aI - T)x\rangle)$ is 0, meaning that $\|(\lambda I - T)x\|^2 = \|(aI - T)x\|^2 + |b|^2 \|x\|^2 \geq |b|^2 \|x\|^2$. If $(\lambda I - T)x = 0$, then $|b|^2 \|x\|^2$ is also 0 which implies that $x = 0$. From this we can conclude that $\lambda I - T$ is one to one if $\mathrm{Im}(\lambda) \neq 0$.

Next we will show that the range of $\lambda I - T$ is closed. Let (y_n) be a sequence in $\lambda I - T$ that converges to $y \in H$. This implies that there exists another sequence $(x_n) \in H$ such that $(\lambda I - T)x_n = y_n$. However, as we saw above $\|(\lambda I - T)x\| \geq |b| \|x\|$. If we let $x = x_n - x_m$ we get that $\|y_n - y_m\| \geq |b| \|x_n - x_m\|$. Since (y_n) is Cauchy, we have that (x_n) is Cauchy as well. The completeness of H tells us that there is some $x \in H$ to which (x_n) converges, so that

$$(\lambda I - T)x = (\lambda I - T)\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} (\lambda I - T)x_n = \lim_{n\to\infty} y_n = y.$$

From this we see that y belongs to the range of $\lambda I - T$. As a consequence, the range of $\lambda I - T$ is closed. Finally, note that

$$\begin{aligned} \mathrm{ran}(\lambda I - T) &= \overline{\mathrm{ran}(\lambda I - T)} = (\ker(\lambda I - T)^*)^\perp \\ &= (\ker(\lambda^* I - T))^\perp = \{0\}^\perp = H. \end{aligned}$$

Meaning that T is onto. From the definition of the spectrum it is immediate that pure complex numbers are not part of the spectrum. □

Theorem 1.4.9. *Let H be a Hilbert space and $T \in \mathcal{B}(H)$ be such that $T = T^*$. Then the eigenvectors corresponding to distinct eigenvalues are orthogonal.*

Proof. Since $\sigma_p(T) \subset \sigma(T) \subset \mathbb{R}$, let λ_1, λ_2 be distinct real eigenvalues corresponding to the eigenvectors v_1, v_2 respectively. Then

$$\begin{aligned} \lambda_1 \langle v_1, v_2 \rangle &= \langle \lambda_1 v_1, v_2 \rangle = \langle T v_1, v_2 \rangle = \langle v_1, T^* v_2 \rangle = \langle v_1, T v_2 \rangle \\ &= \langle v_1, \lambda_2 v_2 \rangle = \lambda_2^* \langle v_1, v_2 \rangle = \lambda_2 \langle v_1, v_2 \rangle. \end{aligned}$$

So that $(\lambda_1 - \lambda_2) \langle v_1, v_2 \rangle = 0$ which means that $\langle v_1, v_2 \rangle = 0$ giving us the orthogonality of these eigenvectors. □

Lemma 1.4.10. If $T = T^* \in \mathcal{B}(H)$, then $\|T\| = sup_{\|x\|=1} |\langle Tx, x \rangle|$.

1.4.3 Spectral Theory for Compact Operators. In finite dimensions we needed self-adjointness for the spectral theorem to take place, however, since we are now in infinite dimensions we must be careful. Thankfully we already discussed operators which get us somewhat closer to finite dimension. By requiring a bounded operator to be compact and self-adjoint the spectral theorem can take place in an infinite dimensional Hilbert space.

We begin by giving two results that we will use for proving the spectral theorem.

Lemma 1.4.11. If H is a nontrivial Hilbert space and $T = T^* \in \mathcal{K}(H)$, then either $\|T\|$ or $-\|T\|$ is an eigenvalue of T.

Proof. Suppose that T is nonzero so that the proof is non trivial. From the previous lemma we have that there is a sequence of vectors with unitary norm in H such that $|\langle Tx'_n, x'_n\rangle| \to \|T\| \neq 0$ as $n \to \infty$. However, since $\langle Tx'_n, x'_n\rangle$ is real, $\langle Tx'_n, x'_n\rangle$ must be $|\langle Tx'_n, x'_n\rangle|$ or $-|\langle Tx'_n, x'_n\rangle|$. Therefore, for infinitely many n, we have that $\langle Tx'_n, x'_n\rangle$ converges to $\|T\|$ or to $-\|T\|$. With this in mind let λ be $\|T\|$ or $-\|T\|$. Thus $\langle Tx_n, x_n\rangle \overset{n\to\infty}{\longrightarrow} \lambda$ where $\lambda = \|T\|$ or $-\|T\|$. With this in mind, note that

$$\begin{aligned} 0 \leq \|Tx_n - \lambda x_n\|^2 &= \|Tx_n\|^2 - 2\lambda \langle Tx_n, x_n\rangle + \lambda^2 \\ &\leq \|T\|^2 - 2\lambda \langle Tx_n, x_n\rangle + \lambda^2 \\ &= \lambda^2 - 2\lambda \langle Tx_n, x_n\rangle + \lambda^2 \end{aligned}$$

Letting $n \to \infty$ we see that the whole expression goes to 0.

So $Tx_n - \lambda x_n \overset{n\to\infty}{\longrightarrow} 0$. As T is compact, there is a subsequence, say $(Tx_n)_{k\in\mathbb{N}}$ of $(Tx_n)_{n\in\mathbb{N}}$ that converges, say, to $y \in H$. Then $(x_{n_k})_{k\in\mathbb{N}}$ converges to y/λ, because $\lambda x_{n+k} = T_{x_k} - (T_{x_{n_k}} - \lambda x_{n_k}) \overset{k\to\infty}{\longrightarrow} y - 0 = y$. Thanks to the continuity of T, we obtain

$$y = \lim_{k\to\infty} T_{x_{n_k}} = T\left(\lim_{k\to\infty} x_{n_k}\right) = T\left(\frac{1}{\lambda}y\right) = \frac{1}{\lambda}Ty.$$

Hence $Ty = \lambda y$, and $y \neq 0$ since $\|y\| = \lim_{k\to\infty} \|\lambda x_{n_k}\| = |\lambda| = \|T\| \neq 0$.

$\square$

Lemma 1.4.12. Let H be a Hilbert space, $T = T^* \in \mathcal{B}(H)$, and Y be a $T-$invariant closed subspace of H. Then:

1. $Y^\perp$ is also $T-$invariant.
2. The restriction $T\big|_{Y^\perp} : Y^\perp \to Y^\perp$ of T to the Hilbert space $Y^\perp$ is also self-adjoint.
3. If T is in addition compact, then $T\big|_{Y^\perp}$ is also compact.

Proof. 1. Let $z \in Y^\perp$. Since Y is $T-$invariant we have that for all $y \in Y, Ty \in Y$, so that $\langle Tz, y\rangle = \langle z, T^*y\rangle = \langle z, Ty\rangle = 0$. Meaning that $Tz \in Y^\perp$.

2. Recall that $Y^\perp$ is a Hilbert given that it is a closed subspace of a Hilbert space. As we just proved that $Y^\perp$ is $T-$invariant, we have that the restriction $T\big|_{Y^\perp} : Y^\perp \to Y^\perp$ is well defined. Since for $z_1, z_2 \in Y^\perp$ T and $T\big|_{Y^\perp}$ coincide, the result easily follows by using the fact that $T = T^*$.

3. For this last item, suppose that T is compact and that $(z_n)_{n\in\mathbb{N}}$ is a bounded sequence in $Y^\perp$. From this assumption we get that $\|z_n\|_{Y^\perp} = \|z_n\|$ so that (z_n) is a bounded sequence in H. Additionally, since we are in $Y^\perp$, the sequence (Tz_n) coincides with the sequence of the images of the restriction. Let us call the restriction $\hat{T}$. By the compactness of T, $\left(\hat{T}z_n\right)$ has a convergent subsequence $(T_{n_k})_{k\in\mathbb{N}}$ in H. Moreover, $\left(\hat{T}x_{n_k}\right)$ is Cauchy in $Y^\perp$ since it is Cauchy in H. The final touch is given by the completeness of $Y^\perp$ to give the convergence of $\left(\hat{T}x_{n_k}\right)$ in $Y^\perp$. $\square$

Theorem 1.4.13. *Let H be a Hilbert space and $T = T^* \in \mathbb{K}(H)$ have infinite rank. Then there exists orthonormal eigenvectors $u_n, n \in \mathbb{N}$ with corresponding eigenvalues $\lambda_n, n \in \mathbb{N}$, such that $\lim_{n\to\infty} \lambda_n = 0$, and for all $x \in H$,*

$$Tx = \sum_{n=1}^{\infty} \lambda_n \langle x, u_n \rangle u_n.$$

Proof. Let $H_1 := H$ and $T_1 := T$. By lemma, there exists and eigenvalue $\lambda_!$ of T and a corresponding eigenvector u_1 such that $\|u_1\| = 1$ and $|\lambda_1| = \|T_1\|$. Define H_2 as $H_2 = (\text{span}\{u_1\})^{\perp}$. Then H_2 is a closed subspace of H_1 and it is also $T-$invariant. Let $T_2 = T\big|_{H_2}$. We thus have that T_2 is self adjoint and compact. Additionaly, there exists and eigenvalue λ_2 of T_2 with a corresponding eigenvector u_2 such that $\|u_2\| = 1$ and $|\lambda_2| = \|T_2\|$. So

$$|\lambda_2| = |\langle T_2 u_2, u_2 \rangle| = |\langle T u_2, u_2 \rangle| \le \|T\| = |\lambda_1|.$$

From this it is immediate that $\{u_1, u_2\}$ is orthonormal and $Tu_1 = \lambda_1 u_1$ and $Tu_2 = \lambda_2 u_2$. We continue this process by letting $H_3 := (\text{span}\{u_1, u_2\})^{\perp}$. With this we have that H_3 is a closed subspace of H, $H_3 \subset H_2$ and since span$\{u_1, u_2\}$ is $T-$invariant, it follows that $TH_3 \subset H_3$. Similarly to last step, let $T_3 = T\big|_{H_3}$. T_3 is thus self adjoint and compact, so that there exists an eigenvalue λ_3 of T_3 with a corresponding eigenvector $u_3 \in H_3$, such that $\|u_3\| = 1$ and $|\lambda_3| = \|T_3\|$. It follows that

$$|\lambda_3| = |\langle T_3 u_3, u_3 \rangle| = |\langle T_2 u3, u_3 \rangle| \le \|T_2\| = |\lambda_2|.$$

If we continue this process indefinetely, we get a sequence $\lambda_1, \lambda_2, \lambda_3, ...$ of eigenvalues of T, with a corresponding set of eigenvector $u_1, u_2, u_3, ...$ such that

$$\lambda_{n+1} = \|T_{n+1}\| \le \|T_n\| = |\lambda_n|, \quad n = 1, 2, 3, ...$$

The process would stop if for some n, $H_n := (\text{span}\{u_1, ..., u_{n-1}\})^{\perp}$ became $\{0\}$. However, as we will show now, this is not possible because T is assumed to be of infinite rank. To this end, suppose on the contrary that $H_n = 0$.

For arbitrary $x \in H$ we have $x - \sum_{k=1}^{n-1} \langle x, u_k \rangle u_k \in (\text{span}\{u_1, ..., u_{n-1}\})^{\perp}$. By our assumption it follows that $x = \sum_{k=1}^{n-1} \langle x, u_k \rangle u_k$. Applying T on both sides yields $Tx = \sum_{k=1}^{n-1} \langle x, u_k \rangle Tu_k$. This means that ran$T$ is generated by $Tu_1, ..., Tu_n$, which is a clear contradiction since T is assumed to have infinite rank. With this, we have successfully constructed an infinite sequence of eigenvectors $u_1, u_2, u_3, ...$ with eigenvalues $\lambda_1, \lambda_2, \lambda_3, ...$.

We now wish to show that $(|\lambda_n|)$ converges to 0. Given that it is decreasing, it must converge to $\epsilon := \inf_{n\in\mathbb{N}}$. If $\epsilon > 0$, then for $n \neq m$,

$$\|Tu_n - tu_m\|^2 = \|\lambda_n u_n - \lambda_m u_m\|^2 = \lambda_n^2 + \lambda_m^2 \ge 2\epsilon^2.$$

But this contradicts the compactness of T, because the sequence (Tu_n) should have some convergent subsequence and hence Cauchy. Consequently, $(|\lambda_n|)$ converges to 0, so that (λ_n) also converges to

0.

Finally, for all $x \in H$ we have $x - \sum_{k=1}^{n-1} \langle x, u_k \rangle u_k \in H_n$, which means

$$\begin{aligned}\left\| Tx - \sum_{k=1}^{n-1} \lambda_k \langle x, u_k \rangle u_k \right\| &= \left\| T\left(x - \sum_{k=1}^{n-1} \langle x, u_k \rangle u_k \right) \right\| \\ &= \left\| T\left(x - \sum_{k=1}^{n-1} \langle x, u_k \rangle u_k \right) \right\| \\ &\leq \|T_n\| \left\| x - \sum_{k=1}^{n-1} \langle x, u_k \rangle u_k \right\| \\ &\leq |\lambda_n| \, \|x\| \overset{n\to\infty}{\longrightarrow} 0.\end{aligned}$$

Hence, $Tx - \sum_{k=1}^{n-1} \lambda_k \langle x, u_k \rangle u_k$ for all $x \in H$. $\square$

This next theorem is also very important in displaying the key properties of the spectrum of a compact operator.

Theorem 1.4.14. *Let H be an infinite dimensional Hilbert space and $K \in \mathcal{K}(H)$, then*

1. *$\sigma(K) \setminus \{0\} = \sigma_p(K) \setminus \{0\}$, and $\sigma(K)$ is countable.*
2. *0 is the only accumulation point of $\sigma(K)$.*
3. *For all $\lambda \in \sigma_p(K) \setminus \{0\}$, $dimN(\lambda I - K) = dimN(\lambda^* I - K^*) < \infty$.*

Proposition 1.4.15. *A diagonal operator T is compact if and only if $\lim_{n\to\infty} \lambda_n = 0$.*

This next example is a nice application of the spectral mapping theorem when studying the spectrum of an operator.

Example 1.4.16.

1. Recall the Fourier transform given by $\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i \xi x} dx$. It can be shown that the inverse Fourier transform is given by
$$f(x) = \int_{-\infty}^{\infty} f(\xi) e^{2\pi i \xi x} d\xi, \quad \xi \in \mathbb{R}.$$
If we let $T := f(x) \to f(-x)$ denote the map that changes x with $-x$ in the inverse Fourier transform, it follows that $(\hat{f})^2 = T$, and since $T^2 = I$, this means that $(\hat{f})^4 = I$. By the spectral mapping theorem it follows that the spectrum of F lies on the set that consists of the fourth roots of the unity: $\pm 1, \pm i$.

2

Schatten-Von neumann Class and Singular values

2.0.1 Singular values of an operator. This chapter provides the core theory with respect to singular values. As we mentioned before, many results here are not used in chapter 3 when we do our estimations for the rate of decay of the eigenvalues of $H = (-\Delta + V + I)^{-s}$, but we still wanted to include them so the reader can gain insight on some of the results that can be found when studying S_p classes.

In the previous section we mentioned the spectral theorem for self-adjoint compact operators, but as one can notice, we are asking two very specific conditions for an operator which are not the easiest to encounter. To overcome this difficulty, consider a compact operator T that is not self-adjoint and recall the polar decomposition $T = V|T|$ where $|T| = \sqrt{T^*T}$. Now that we have a compact and self-adjoint operator, by the spectral theorem for compact operators there exists an orthonormal sequence $\{e_n\} \in H$ such that

$$|T|\, x = \sum_n \lambda_n \langle x, e_n \rangle e_n, \quad x \in H,$$

Let $\sigma_n = Ve_n$ for each n, clearly $\{\sigma_n\}$ is still an orthonormal set and we get

$$Tx = \sum_n \lambda_n \langle x, e_n \rangle \sigma_n, \quad x \in H.$$

This is called the canonical decomposition for a compact operator T.

From now on, the eigenvalues of the operator $|T|$ will be denoted by μ, and the sequence $\{\mu_n\}$ arranged in a non-increasing way is called the sequence of singular values of T. The quantity μ_n is said to be the nth singular value of T.

With this in mind, the canonical decomposition for a compact operator T will look like this:

$$Tx = \sum_n \mu_n \langle x, e_n \rangle \sigma_n.$$

In this chapter we will focus mainly on the theory described in [11], for more results regarding singular values and the study of the S_p classes, the reader may consult [2],[4],[7] and [9].

2.1 Schatten-von Neumann classes

With this in mind, given $0 < p < \infty$, we define the Schatten-von Neumann $p-$class of H, denoted by $S_p(H)$ or simply S_p, to be the space of all compact operators T on H with its singular value sequence $\{\mu_n\}$ belonging to ℓ^p. We will focus mainly on the range $1 \leq p < \infty$. For these values S_p is a Banach space with the norm

$$\|T\|_p = \left(\sum_n |\mu_n|^p\right)^{\frac{1}{p}}.$$

It is also worth mentioning that the two most important cases of p are $p = 1$ and $p = 2$ for which S_1 is called the trace class of H and S_2 is called the Hilbert-Schmidt class.

To see that the Schatten-von Neumann classes do not define a norm for $0 < p < 1$, recall that for these values of p the function x^p is concave on $(0, \infty)$ which means that given nonnegative numbers a, b one has $(a + b)^p \leq a^p + b^p$. Using this fact and 2.1.13 one gets that S_p is not Banach for $0 < p < 1$. Additionaly, 2.1.13 is key to prove that for $p \geq 1$, S_p is Banach.

Singular values are a powerful generalization of eigenvalues, particularly useful for operators that are not self-adjoint or not square (in the matrix sense). While eigenvalues are defined for square matrices and self-adjoint operators, singular values can be defined for any compact operator on a Hilbert space. This provides an alternative analytical tool when direct eigenvalue analysis is difficult or inapplicable, such as for non-normal operators or rectangular matrices. Beyond theoretical insights, singular values are practically applied in various fields, including determining the rank of a matrix, constructing orthonormal bases, solving least-squares problems, and approximating data or images (e.g., for denoising) by retaining only the largest singular values.

These classes form a hierarchy ($S_p \subseteq S_q$ for $p \leq q$), allowing for a precise characterization of how rapidly an operator's singular values decay.

The idea of this section is to provide some general results for operators that belong in S_p classes.

In the following results, let $\{\sigma_n\}$ the orthonormal set that appears in the canonical decomposition of a compact operator T.

Theorem 2.1.1 (Hilbert-Schmidt norm). *Let $T \in \mathcal{K}(H)$ with singular values $\{\mu_n\}$, then*

$$\sum_n |\mu_n|^2 = \sum_{n=1}^{\infty} \|Te_n\|^2 = \sum_{n,m=1}^{\infty} |\langle Te_n, e_m\rangle|^2.$$

for any orthonormal basis $\{e_n\}$ of H.

Proof. Let

$$Tx = \sum_k \mu_k \langle x, u_n\rangle \sigma_k$$

be the canonical decomposition of T.

First, let us compute $\|Te_k\|^2$, keeping in mind that $\{\sigma_n\}$ is an orthonormal basis, as we are about to use the generalized Pythagorean theorem.

$$\|Te_k\|^2 = \left\|\sum_n \mu_n \langle e_k, u_n\rangle \sigma_n\right\|^2 = \sum_n |\mu_n \langle e_k, u_n\rangle|^2 = \sum_n \mu_n^2 |\langle e_k, u_n\rangle|^2 .$$

Now we want to sum over k,

$$\sum_k \|Te_k\|^2 = \sum_k \sum_n \mu_n^2 |\langle e_k, u_n\rangle|^2$$

given that the sums are absolutely convergent, we get

$$\sum_k \|Te_k\|^2 = \sum_n \mu_n^2 \sum_k |\langle e_k, u_n\rangle|^2$$

Note that thanks to Parseval's inequality, the sum on the right is equal to 1, thus

$$\sum_n^\infty \|Te_n\|^2 = \sum_n^\infty |\mu_n|^2 .$$

For the other equality, fix n so that Te_n is a fixed vector in H, thanks to this, we can express Te_n in terms of the orthonormal basis $\{e_m\}$ as

$$Te_n = \sum_m \langle Te_n, e_m\rangle e_m.$$

In a similar fashion as done above, we calculate $\|Te_n\|^2$.

$$\|Te_n\|^2 = \left\|\sum_m \langle Te_n, e_n\rangle e_m\right\|^2 = \sum_m |\langle Te_n, e_m\rangle|^2 .$$

Summing over n yields the desired equality.

This first theorem is pretty much the infinite dimensional version of the Frobenius norm for matrices, which simplifies the task of calculating the S_p norm in the special case of $p = 2$, by just choosing an orthonormal basis and summing the squared norms of the images of the elements of said orthonormal basis.

□

Theorem 2.1.2. *If $T \geq 0 \in \mathcal{K}(H)$ with the canonical decomposition*

$$Tx = \sum_n \mu_n \langle x, e_n\rangle e_n, \quad x \in H,$$

then

$$\sum_n \mu_n = \sum_{k=1}^{\infty} \langle T\sigma_k, \sigma_k \rangle$$

for any orthonormal basis $\{\sigma_k\}$ *of* H.

Proof. If $\{\sigma_k\}$ is an orthonormal basis for H, then

$$\langle T\sigma_k, \sigma_k \rangle = \sum_n \mu_n \left|\langle \sigma_k, e_n \rangle\right|^2.$$

The rest of the proof is done the same way as the previous theorem, sum over k, change the order of summation thanks to absolute convergence, use Parseval's identity to simplify and conclude.

□

This result highlights the importance of basis independence for the sum of the diagonal elements, which is the foundation upon which the general definition of the trace is built in the following theorem.

We will need the following result before we move on to the theorem regarding the trace.

Proposition 2.1.3. *If* $T \in S_p$*, then there are four positive operators* $T_i \in S_p$ *such that*

$$T = (T_1 - T_2) + i(T_3 - T_4).$$

Theorem 2.1.4. *If* $T \in S_1$ *then the series* $\sum_{k=1}^{\infty} \langle Te_k, e_k \rangle$ *converges absolutely for any orthonormal basis* $\{e_k\}$ *of* H *and the sum is independent of the choice of the orthonormal basis. We call this value the trace of* T *and denote it by* $tr(T)$.

Proof. If T is positive, then $\sum_{k=1}^{\infty} \langle Te_k, e_k \rangle = \sum_n \mu_n = \|T\|_1 < +\infty$ for any orthonormal basis $\langle e_k \rangle$ by the previous theorem, where $\{\mu_n\}$ is the singular value sequence of T which is independent of the orthonormal basis $\{e_k\}$.

In general, we can write $T = (T_1 - T_2) + i(T_2 - T_4)$ with each T_i positive and in S_1. This implies that each series $\sum_{k=1}^{\infty} \langle T_i e_k, e_k \rangle$ converges and the sum is independent of the orthonormal basis $\{e_k\}$. Hence $\sum_{k=1}^{\infty} \langle Te_k, e_k \rangle$ is absolutely convergent and the sum is independent of the choice of the orthonormal basis. □

The basis-independence guaranteed by this theorem ensures that physical predictions do not depend on the arbitrary choice of a measurement basis, which is a physical necessity.

Lemma 2.1.5. Suppose $T \in \mathcal{K}(H) \geq 0$ and $p > 0$, then $T \in S_p$ if and only if $T^p \in S_1$. Moreover, $\|T\|_p^p = \|T^p\|_1$.

Proof. Since T is positive it is self adjoint, and since it is also compact we get that the singular values coincide with the eigenvalues $\lambda_n = \mu_n$.

Thus, if $\{u_n\}$ are the corresponding orthonormal eigenvectors, we can write

$$Tx = \sum_n \mu_n \langle x, u_n \rangle u_n$$

If we apply T p times, with p being a positive integer, induction gives us

$$T^p x = \sum_n \mu_n^p \langle x, u_n \rangle u_n.$$

This tells us that the singular values of T^p are μ_n^p.

Now, assume that $T \in S_p$. By definition this means that

$$\sum_n \mu_n^p < \infty,$$

note that μ_n^p are precisely the singular values of T^p so this tells us directly by definition that $T^p \in S_1$ since the sum of the singular values of T^p is finite.

Conversely, if $T \in S_1$ we have by definition that

$$\sum_n \mu_n^n < \infty,$$

which also means that T must be in S_p.

For the final equality note that

$$\|T\|_p^p = \left(\left(\sum_n \mu_n^p\right)^{1/p}\right)^p = \sum_n \mu_n^p = \|T^p\|_1 .$$

$\square$

Theorem 2.1.6. *Let $T \in \mathcal{K}(H)$ and $p \geq 1$, then $T \in S_p$ if and only if $|T|^p = (T^*T)^{p/2} \in S_1$ if and only if $T^*T \in S_{\frac{p}{2}}$. Moreover, $\|T\|_p^p = \||T|\|_p^p = \||T|^p\|_1 = \|T^*T\|_{\frac{p}{2}}$*

Proof. Since $|T|$ satisfies the conditions for the previous lemma, it is immediate that $|T| \in S_p$ if and only if $|T|^p \in S_1$ and that $\||T|\|_p^p = \||T|^p\|_1$. Also note that the singular values μ_n of T are the same singular values of $|T|$ since both of them are exactly the eigenvalues of $|T|$. Using a similar reasoning to last proof, we would then arrive at the fact that that $T \in S_p$ if and only if $|T|^p \in S_p$ since the singular values of $|T|^p$ would be μ_n^p and using the definition of S_p class membership would then give us the desired result. So that $\|T^p\|_1 = \||T|^p\|_1$, which by the previous lemma implies that $\|T\|_p^p = \||T|^p\|_1$.

Additionally, $|T|^p = (T^*T)^{p/2} \in S_1$ by definition means that

$$\sum_n \mu_n \left(T^*T\right)^{p/2} < \infty$$

which is equivalent to saying that $T^*T \in S_{p/2}$. Thus, $\|T^*T\|_{p/2} = \||T|^p\|_1$. Putting all of this together gives us our desired chain of equalities

$$\|T\|_p^p = \||T|\|_p^p = \||T|^p\|_1 = \|T^*T\|_{\frac{p}{2}}.$$

□

The two results previously mentioned give alternatives when doing calculations regarding S_p class membership.

Theorem 2.1.7. *Let $T \in \mathcal{K}(H)$ and $p \geq 1$, then $T \in S_p$ if and only if $\sum_n |\langle Te_n, e_n\rangle|^p < \infty$ for all orthonormal sets $\{e_n\}$. Moreover,*

$$\|T\|_p = \sup\left\{\left(\sum_n |\langle Te_n, e_n\rangle|^p\right)^{\frac{1}{p}} : \{e_n\} \ \ \textit{orthonormal}\right\}.$$

Theorem 2.1.8. *Let $T \in \mathcal{K}(H)$ and $p \geq 1$, then $T \in S_p$ if and only if $\sum_n |\langle Te_n, \sigma_n\rangle|^p < \infty$ for all orthonormal sets $\{e_n\}$ and $\{\sigma_n\}$. Moreover,*

$$\|T\|_p = \sup\left\{\left(\sum_n |\langle Te_n, \sigma_n\rangle|^p\right)^{\frac{1}{p}} : \{e_n\}, \{\sigma_n\} \ \ \textit{orthonormal}\right\}.$$

Theorem 2.1.9. *Let $T \in \mathcal{K}(H)$ and $p \geq 2$, then $T \in S_p$ if and only if $\sum_n \|Te_n\|^p < \infty$ for all orthonormal sets $\{e_n\}$. Moreover,*

$$\|T\|_p = \sup\left\{\left(\sum_n \|Te_n\|^p\right)^{\frac{1}{p}} : \{e_n\} \ \ \textit{orthonormal}\right\}.$$

Remark 2.1.10. To summarize what is going on in these three Theorems, here are some key aspects to keep in mind:

1. Regarding inner products, in Theorem 2.1.7 only diagonal inner products $\langle Te_n, e_n\rangle$ are used in the sum, while in Theorem 2.1.9 both diagonal and off-diagonal terms $\langle Te_n, \sigma_n\rangle$ are considered in the sum. In Theorem 2.1.8, instead of using inner products, we are considering the sums of the norm of Te_n.

2. The range for p in Theorems 2.1.7 and 2.1.8 are $p \geq 1$ whereas in Theorem 2.1.9 the range is $p \geq 2$.

These Theorems provide a powerful characterization of the S_p norm and are very useful, since they provide a way to check for S_p membership and calculate the norm without computing singular values. The supremum is taken over all possible orthonormal bases, which means that if you can find just one basis for which the sum is infinite, you know the operator is not in S_p. Conversely, to prove it is in S_p, you need to show the sum is finite and bounded for all bases. The proofs can be consulted in [11].

Lemma 2.1.11. Let T be a positive compact operator and $\{\mu_n\}$ be the sequence of eigenvalues of T arranged in decreasing order and repeated according to multiplicity; then for any $n \geq 1$,

$$\mu_{n+1} = \min_{x_1,\ldots,x_n} \max \{\langle Tx, x\rangle : \|x\| = 1,\ x \perp x_i.\ 1 \leq i \leq n\}.$$

It is readily seen that $\mu_1 = \max\{\|Tx\| : \|x\| = 1\} = \|T\|$.

As a consequence of this we also get that $r_\sigma \leq \mu_1$.

Theorem 2.1.12 (Courant-Fischer-Weyl min-max principle)**.** *Suppose $T \in \mathcal{K}(H)$ and $\{\mu_n\}$ is the singular value sequence of T, then for each $n \geq 0$ we have*

1. $\mu_{n+1} = \inf\{\|T - F\| : F \in \mathcal{F}_n\}$, *where $\mathcal{F}_n$ is the set of all linear operators on H with rank less than or equal to n.*
2. $\mu_{n+1} = \min_{x_1,\ldots,x_n} \max\{\|Tx\| : \|x\| = 1, \langle x, x_i\rangle = 0,\ 1 \leq i \leq n\}$

The first part of this theorem says that μ_{n+1} is the error in the best possible rank-n approximation of T.

The second part gives a min-max criterion to find the $(n+1)$ singular value.

Corollary 2.1.13. *Let $T_1, T_2 \in \mathcal{K}(H)$ and $\{\mu_n(T)\}$ is the singular value sequence of T, then for any $m, n \geq 0$ we have*

1. $\mu_{n+m+1}(T_1 + T_2) \leq \mu_{n+1}(T_1) + \mu_{m+1}(T_2)$
2. $\mu_{n+m+1}(T_1T_2) \leq \mu_{n+1}(T_1)\mu_{m+1}(T_2)$.

3

Rate of growth of eigenvalues using inequalities on singular values

The goal of this chapter is to provide a relationship between singular values and eigenvalues through two powerful inequalities first established by Hermann Weyl, that create a profound link between the eigenvalues of a compact operator T and its singular values. These inequalities will be the our tools for studying the rate of growth of the eigenvalues of $\mathbf{H} = (-\Delta + V + I)^s$.

The inequalities we are interested in are the following

$$\prod_{j=1}^{n} |\lambda_j|(T) \leq \prod_{j=1}^{n} \mu_j(T).$$

and

$$\sum_{n=1}^{N} |\lambda_n(T)|^p \leq \sum_{n=1}^{N} \mu_n(T)^p.$$

The proof of the first inequality will be taken from [4], since it uses theory that we have seen so far. There are some other ways to prove this result, but we believe that this is the most understandable one since we do not have to include extra theory.

To prove the second inequality, we will base the proof on [7], which deals with majorization results. The proof for $p = 1$ can also be found in [4]. If the reader wishes, a proof that does not use majorization can be found in [6].

We begin with a lemma that will be used to prove the first inequality that we are interested in.

Lemma 3.0.1. Let $T \in S_1$ and B a bounded operator. Then

1. $\|T\|_1 = \|T\|_1$.
2. $\|BT\|_1 \leq \|B\| \|T\|_1$.
3. $\|TB\|_1 \leq \|B\| \|T\|_1$.

4. If S is also in S_1, then $T+S \in S_1$ and

$$\|T+S\|_1 \leq \|T\|_1 + \|S\|_1 .$$

This theorem is basically telling us that S_1 is a two sided ideal in the algebra of bounded operators.

Proof. To prove 2 and 3 note that it is immediate that

$$\langle T^* B^* BTu, u\rangle = \|BTu\|^2 \leq \|B\|^2 \|Tu\|^2 = \|B\|^2 \langle T^* Tu, u\rangle ,$$

which means that

$$(BT)^* BT \leq \|B\|^2 T^* T.$$

Since the jth eigenvalue is a monotonic function we get that

$$\mu_j^2 (BT) \leq \|B\|^2 \mu_j^2 (T) .$$

□

Additionally, since the singular values of adjoint operators are the same, we can also deduce from this inequality that

$$\mu_j (TB) = \mu_j (B^* T^*) \leq \|B^*\| \mu_j (T^*) = \|B\| \mu_j (T) \tag{3.1}$$

Theorem 3.0.2. *[Weyl's multiplicative inequality] Let $T \in \mathcal{K}(H)$ with nonzero eigenvalues $\lambda_1, \lambda_2, \ldots$ and singular values μ_j both arranged in decreasing order with their multiplicities included. For any n the following inequality holds:*

$$\prod_1^n |\lambda_j| \leq \prod_1^n \mu_j (T) .$$

Proof. Let E_N be the space spanned by the first N eigenvectors of T, and let P_N be the orthogonal projection onto E_N. Additionally, let T_N be the restriction of T to the invariant subspace E_N. By polar decomposition, we have the following equality, where A_N is the absolute value of T_N.

$$T_N = U_N A_N. \tag{3.2}$$

Since the eigenvalues λ_j are nonzero, T_N is invertible. Then so is U_N, and therefore U_N is unitary. Taking the determinant of (3.2) gives

$$|\det T_N| = \det A_N.$$

Since the determinant of a matrix is the product of its eigenvalues, we can rewrite this identity as

$$\prod_{j=1}^N |\lambda_j| = \prod_{j=1}^N \lambda_j(A_N). \tag{3.3}$$

Remember that here λ_j with $j = 1, \ldots, N$ are the eigenvalues of T_N.

Now consider the operator TP_N that acts on the whole space H, on the space E_N it will act as T_N and on the orthogonal complement of E_N it will be 0. It follows that the absolute value of TP_N is A_N on E_N, and zero on $E_N^{\perp}$. It follows that

$$\lambda_j(A_N) = s_j(TP_N), \quad j = 1, \ldots, N.$$

We appeal now to inequality (3.1): $s_j(TB) \leq \|B\| s_j(T)$. Apply this to $B = P_N$; we obtain

$$s_j(TP_N) \leq s_j(T). \tag{3.4}$$

Since we have shown that $\lambda_j(A_N) = s_j(TP_N)$, setting (3.4) into the right side of (3.3) yields the desired result. $\square$

Now that we are done with this inequality, we move towards the second one by giving some preliminary results, which come from majorization theory. We do not provide the proofs as we are just interested in using them as tools to prove the second inequality we mentioned regarding eigenvalues and singular values.

Definition 3.0.3. Let a_n be an infinite sequence of numbers such that $a_n \to 0$ as $n \to \infty$. Consider a_n^* as the sequence defined by $a_1^* = \max|a_i|$, $a_1^* + a_2^* = \max_{i \neq j}(|a_i| + |a_j|)$, etc. So that $a_1^* \geq a_2^* \geq \cdots$ and the sets of a_i^* and $|a_i|$ are identical, counting multiplicities.

Lemma 3.0.4. Let a_n, b_n be infinite sequences of numbers such that $a_n, b_n \to 0$ as $n \to \infty$, and define a_n^* and b_n^* as above, then

$$\sum_n |a_n b_n| \leq \sum_n a_n^* b_n^*.$$

Theorem 3.0.5. *Let $a_1 \geq a_2 \geq \cdots \geq a_N \geq 0$ and let b be a point in $\mathbb{C}^N$ with*

$$\sum_{j=1}^k b_j^* \leq \sum_{j=1}^k a_j$$

for $k = 1, ..., N$. Then there exists points $a^{(1)}, ..., a^{(m)} \in \mathbb{C}^N$ with $\left(a^{(\ell)}\right)^* = a$ and $0 \leq \lambda_\ell \leq 1$, $\sum_{\ell=1}^m \lambda_\ell = 1$ so that

$$b = \sum_{\ell=1} \lambda_\ell a^{(\ell)}.$$

In particular, if $\sum_{j=1}^k b_j^* \leq \sum_{j=1}^k a_j$ holds and Φ is a function on $([0,\infty])^N$ so that $\phi(c) = \Phi(c_1^*, ..., c_n^*)$ is convex on $\mathbb{C}^N$, then

$$\phi(b) \leq \phi(a).$$

Corollary 3.0.6. *Let $a_1 \geq a_2 \geq \cdots \geq a_N \geq 0$, $b_1 \geq \cdots b_N \geq 0$. Suppose that*

$$\prod_{j=1}^k b_j \leq \prod_{j=1}^k a_j$$

for $k = 1, ..., N$. Then for any continuous, monotone increasing function g on $[0, \infty)$ with $t \to g(e^t)$

convex, we have that

$$\sum_{j=1}^{k} g(b_j) \leq \sum_{j=1}^{k} g(a_j)$$

Theorem 3.0.7. *For any compact operators T and S and all N,*

$$\prod_{n=1}^{N} \mu_n(TS) \leq \prod_{n=1}^{N} \mu_n(T)\mu_n(S).$$

Theorem 3.0.8. *Let ϕ be a non-negative monotone increasing function on $[0,\infty)$ so that $t \to \phi(e^t)$ is convex. Then for any compact T,*

$$\sum \phi\left(|\lambda_n(T)|\right) \leq \sum \phi\left(\mu_n(T)\right).$$

Moreover, if S is another compact operator, then

$$\sum \phi\left(\mu_n(TS)\right) \leq \sum \phi\left(\mu_n(T)\mu_n(S)\right).$$

Proof. The proof of this theorem follows from 3.0.2, 3.0.6, and 3.0.7. □

With these results, we have the tools required to prove

$$\sum_{j=1}^{k} |\lambda_n\left(T\right)|^p \leq \sum_{j=1}^{k} \mu_n\left(T\right)^p.$$

First note that the conditions for 3.0.6 are satisfied by singular values and the absolute value of eigenvalues for a compact operator T. In particular, if we let $b_j = |\lambda_j(T)|$ and $a_j = \mu_j(T)$, and choose $g(x)$ as x^p, we get

$$\sum_{j=1}^{k} g\left(|\lambda_n(T)|\right) \leq \sum_{j=1}^{k} g\left(\mu_n(T)\right) \Longleftrightarrow \sum_{j=1}^{k} |\lambda_n\left(T\right)|^p \leq \sum_{j=1}^{k} \mu_n\left(T\right)^p.$$

3.1 A small look into Schrödinger operators

A Schrödinger operator usually has the form $-\Delta + V(x)$, where Δ is the laplacian which represents kinetic energy and $V(x)$ is a potential function which represents an external force. These operators act on a suitable space of functions which is usually $L^2(\mathbb{R}^d)$. The eigenvalues correspond to possible energy levels of a quantum system.

Weyl's multiplicative inequality (3.0.2), as well as (3) play an essential role in the study of Schrödinger operators by providing a bound that connects the eigenvalues and singular values of compact operators. Since Schrödinger operators are often compact in physically relevant settings, these inequalities help estimate their spectral properties.

To better illustrate how studying singular values can provide information regarding the spectral properties of Schrödinger operators we will give some concrete examples for $\mathbf{H} = (-\Delta + V + I)^s$. It is worth noting that determining whether or not a Schrödinger operator is compact or not, is no simple task, and even when one has a compact Schrödinger operator, proving S_p class membership is also very difficult and is an ongoing field of research. Nevertheless, if S_p class membership is achieved by a specific Schrödinger operator, much information can be extracted to help study the physical system.

We begin with an important result from real analysis.

Theorem 3.1.1. *Let $\{a_n\}_n^\infty$ with $a_n > 0,\ \forall n$ be such that $\sum_n^\infty a_n < \infty$. If $a_1 > a_2 > \ldots > a_n > \ldots$ then*

$$\lim_{n\to\infty} na_n = 0$$

Proof. Given that the sum of terms converges, we have that for any $\epsilon > 0$, there exists an integer N such that for any integers $m > n > N$, we have:

$$\sum_{k=n+1}^{m} a_k < \epsilon.$$

Let $M = \lfloor n/2 \rfloor$. We consider the sum of terms from $M+1$ to n.

$$S = a_{M+1} + a_{M+2} + \cdots + a_n = \sum_{k=M+1}^{n} a_k$$

This leads to the following inequality:

$$\sum_{k=M+1}^{n} a_k \geq \underbrace{a_n + a_n + \cdots + a_n}_{\text{at least } n/2 \text{ terms}} \geq \frac{n}{2} a_n$$

Combining our results and using the fact that $a_n > 0$, we have:

$$0 \leq \frac{n}{2} a_n \leq \sum_{k=M+1}^{n} a_k$$

So that:

$$2 \lim_{n\to\infty} 0 \leq \lim_{n\to\infty} na_n \leq 2 \lim_{n\to\infty} \sum_{k=M+1}^{n} a_k$$

Given that the left and right and terms are zero, we can conclude by the squeeze theorem that

$$\lim_{n\to\infty} na_n = 0.$$

□

This theorem and (3) are the tools that we need to study the rate of growth of the eigenvalues of $\mathbf{H}$, which as a consequence will also serve to study the rate of growth of said eigenvalues. To achieve this we will illustrate first with a very simple example.

Example 3.1.2 (Eigenvalues of the 1D Harmonic Oscillator)**.** The shifted harmonic oscillator on $L^2(\mathbb{R})$ is

$$\mathbf{H}_{\text{osc}} \;=\; -\frac{d^2}{dx^2} + x^2 + 1.$$

Its eigenvalues and eigenfunctions are

$$\lambda_n = n + \tfrac{3}{2}, \quad \mathbf{H}_{\text{osc}}\, h_n = \lambda_n\, h_n, \quad n = 0, 1, 2, \ldots,$$

where $\{h_n\}$ are the Hermite functions forming an orthonormal basis.

The operator $\mathbf{H}_{\text{osc}}^{-s}$ has eigenvalues

$$\lambda_n^{-s} \;=\; (n + \tfrac{3}{2})^{-s},$$

Since these form a sequence to zero and we are working with a diagonal operator, we thus have that $\mathbf{H}_{\text{osc}}$ is compact. Moreover it is self-adjoint and positive, so the singular values are just the absolute value of the eigenvalues. Since λ_n are positive, they coincide with the singular values. which form a decreasing sequence tending to zero.

By definition, $\mathbf{H}_{\text{osc}}^{-s}$ is in S_p if and only if

$$\sum_{n=0}^{\infty} \mu_n = \sum_{n=0}^{\infty} (n + \tfrac{3}{2})^{-sp} < \infty.$$

This series converges exactly when $sp > 1$. In particular, if we want $\mathbf{H}_{\text{osc}}^{-s}$ to be in S_1 we would need $p > 1$. Given a fixed p we would then have to look for $s > \frac{1}{p}$ for $\mathbf{H}_{\text{osc}}^{-s}$ to be in S_p

Now we would like to study the rate of decay of the eigenvalues $\mathbf{H}_{\text{osc}}^{-s}$.

First, assume that $sp > 1$, so that $\mathbf{H}_{\text{osc}}^{-s} \in S_p$, we know that by definition this means that

$$\sum_{n=0}^{\infty} \mu_n = \sum_{n=0}^{\infty} (n + \tfrac{3}{2})^{-sp} < \infty,$$

which by (3) tells us that

$$\sum_{n=0}^{\infty} \lambda_n = \sum_{n=0}^{\infty} (n + \tfrac{3}{2})^{-sp} < \infty.$$

Since the eigenvalues form a decreasing sequence, we are ready to use (3.1.1), so that

$$\lim_{n \to \infty} n(n + \tfrac{3}{2})^{-sp} = 0$$

implies that for a given $\epsilon > 0$, there exists $N > 0$ such that if $n > N$ then $\left|n(n + \tfrac{3}{2})^{-sp}\right| < \epsilon$.

In this example we related the theory of S_p classes with the rate of decay of the eigenvalues of an operator, however, we gained information on the singular values using the eigenvalues as a starting point, so we did not really need to involve S_p classes here if we wanted information for the eigenvalues of $\mathbf{H}_{\text{osc}}$.

In the next example, we will showcase a class of Schrödinger operators which currently do not have a known formula for their eigenvalues, but thanks to the theory of S_p classes and many advanced tools, it is possible to find their rate of decay, and as a direct consequence it is also possible to find the order of the rate of growth.

Example 3.1.3. In this example we will look at the much more general case of the anharmonic oscillator on $L^2(\mathbb{R}^n)$ which is given by $\mathbf{T}^{-s} = ((-\Delta)^\ell + |x|^{2k} + I)^{-s}$, where k, ℓ are integers ≥ 1. Thanks to corollary 5.4 in [1] it is known that if $s > \frac{(k+\ell)n}{2k\ell p}$, with $1 \leq p < \infty$, then $\mathbf{T}^{-s} \in S_p(L^2(\mathbb{R}^n))$. Additionally, thanks to corollary 5.6 in [1], we get that $\lambda_n(\mathbf{T}^{-s}) = o(n^{\frac{-1}{p}})$ as $n \to \infty$.

We will give some insight following the outline given by the article as to why this is true.

First assume that the condition $s > \frac{(k+\ell)n}{2k\ell p}$ holds, so that $\mathbf{T}^{-s} \in S_p(L^2(\mathbb{R}^n))$, by definition this means that

$$\sum_{n=1}^{\infty} \mu_n(\mathbf{T}^{-s})^p < \infty,$$

thanks to (3) we have that $\sum_{n=1}^{\infty} |\lambda_n(\mathbf{T}^{-s})|^p < \infty$. Since conditions are met, we appeal to (3.1.1) to conclude that

$$\lim_{n\to\infty} n\lambda_n(\mathbf{T}^{-s})^p = 0,$$

which by definition means that for any given $\epsilon > 0$, there exists an $N > 0$ such that if $n > N$, then $|n\lambda_n(\mathbf{T}^{-s})^p| < \epsilon$. In other words, $\lambda_n(\mathbf{T}^{-s}) = o(n^{\frac{-1}{p}})$.

In particular, if we let $s = 1$, and $p > \frac{(k+\ell)n}{2k\ell}$, we get

$$\lambda_j\left((-\Delta)^\ell + |x|^{2k} + 1\right)^{-1} = o(j^{-\frac{1}{p}}), \quad \text{as } j \to \infty.$$

To see how this translates to the rate of growth of the eigenvalues of $((-\Delta)^\ell + |x|^{2k})$ we must use the formal definition of little o notation: For every $L \in \mathbb{N}$ there exists $L_0 \in \mathbb{N}$ such that

$$Lj^{\frac{1}{p}} \leq \lambda_j\left((-\Delta)^\ell + |x|^{2k}\right), \quad \text{for } j \geq L_0.$$

This implies that the eigenvalues $\lambda_j\left((-\Delta)^\ell + |x|^{2k}\right)$ have a growth of order at least

$$j^{\frac{1}{p}}, \quad \text{as } j \to \infty.$$

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