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Totally geodesic surfaces and the Hopf's conjecture

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Abstract

Let g be a riemannian metric on $S^2 \times S^2$. In this paper we will show that if $(S^2 \times S^2, g)$ contains a totally geodesic torus, then $S^2 \times S^2$ does not have positive sectional curvature. Then, we use the formula for the second variation of energy to rule out a family of metrics from having positive sectional curvature.
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1. Introduction

Let $M = S^2 \times S^2 = \{(x, y) \in \mathbf{R}^3 \times \mathbf{R}^3 : |x| = |y| = 1\}$, for any $(x, y) \in M$ we have that $T_{(x,y)}M = \{(v, w) \in \mathbf{R}^3 \times \mathbf{R}^3 : \langle v, x\rangle = \langle w, y\rangle = 0\}$. In this paper we will denote by $\langle\ .,.\ \rangle$ the inner product on euclidean spaces. A riemannian metric on M is defined by means of providing an inner product on every T_mM, $m \in M$. More precisely, we can *locally* define a riemannian metric on a subset U of M by a smooth function $g : U \subset M \to \mathbf{R}^{36}$, such that, after identifying $\mathbf{R}^{36}$ with the 6 by 6 square matrices, we have that, for every $m \in M$ the function g satisfies:

(i) The matrix $g(m)$ is symmetric and all its eigenvalues are positive.

(ii) If $m = x + y$ with $x \in \mathbf{R}^3 \times \{0\}$ and $y \in \{0\} \times \mathbf{R}^3$ then $g(m)x = x$ and $g(m)y = y$ here $g(m)$ is thought as a 6 by 6 matrix and x and y as vectors in $\mathbf{R}^6$.

It is clear that a matrix $g(m)$ that satisfies (i) and (ii) defines an inner product in $\mathbf{R}^6$. The reason of these conditions is that given an inner product on T_mM there is a unique way to extend it to an inner product on $\mathbf{R}^6$ such that (i) and (ii) are satisfied. Also notice that given a metric g that satisfies (i) and (ii) we have an inner product defined on every T_mM for all m, namely, for every $X, Y \in T_mM$ the inner product is given by $\langle X, g(m)Y\rangle$. In general we will write $g(X, Y)(m) = \langle X, g(m)Y\rangle$, we will call this number the inner product of the vector X and Y using the metric g. We will explain in the next section how by using a metric g we can define a function $K : E \to \mathbf{R}$, where

$$E = \{(m, \sigma) : m \in M \quad \text{and} \quad \sigma \subset T_m M \quad \text{is a 2 dimensional subspace } \}$$

This function K is call the sectional curvature function and as we mention before, this function depends on the metric. More than 35 years ago, Hopf conjectured that for any riemannian metric on M, there exists an element on $(m, \sigma) \in E$ such that $K(m, \sigma) \leq 0$. In this paper we will first make use of the Gauss-Bonnet theorem to show that if (M, g) contains a totally geodesic torus, then (M, g) does not have positive sectional curvature. i.e. the Hopf's conjecture holds true for this metric.

Given a diffeomorphism $\phi : M \to M$ we will say that the metric g is invariant under ϕ if for every $m \in M$ and every pair of vectors $X, Y \in T_m M$ we have that $g(X, Y)(m) = g(d\phi_m(X), d\phi_m(Y))(\phi(m))$. Here $d\phi : T_m M \to T_{\phi(m)} M$ is the differential of the function ϕ. Since ϕ is a diffeomorphism, then this differential defines a vector space isomorphism. Notice that the metric g is invariant under ϕ if and only if ϕ is an isometry. Let us consider the following family of rotations $\phi_\alpha : M \to M$, where

$$\phi_\alpha(x, y) = (x_1 \cos\alpha + -x_1 \sin\alpha + x_2 \cos\alpha, x_2 \sin\alpha, x_3, y_1, y_2, y_3) \quad (1)$$

Here $(x, y) = (x_1, x_2, x_3, y_1, y_2, y_3)$. We finish the paper by showing that if a metric on M is invariant under any of the diffeomorphisms defined in (1), then this metric does not have positive sectional curvature, i.e. the Hopf's conjecture holds true for this metric.

2. Preliminaries:

So far we have defined the manifold $M = S^2 \times S^2$. Let us define the space of vector fields on M by

$$\Gamma(M) = \{X : M \to \mathbf{R}^6 : X \text{ is smooth and } \quad X(m) \in T_m M \text{for all } m \in M\}$$

The Levi Civita connection is the only map $\nabla : \Gamma(M) \times \Gamma(M) \to \Gamma(M)$ such that for any X, Y and Z in $\Gamma(M)$ we have that

1. $\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z$
2. $\nabla_{fX+Y} Z = f\nabla_X Z + \nabla_Y Z$ for every smooth function $f : M \to \mathbf{R}$
3. $\nabla_X(fY) = X(f)Y + f\nabla_X Y$ for every smooth function $f : M \to \mathbf{R}$, here $X(f) : M \to \mathbf{R}$ is the directional derivative of f in the direction X.

4. $X(g(Y,Z)) = g(\nabla_X Y, Z) + g(X, \nabla_X Z)$

5. $(\nabla_X Y - \nabla_Y X)(f) = X(Y(f)) - Y(X(f))$ for every smooth function $f : M \to \mathbf{R}$.

The vector field defined in 5 is called the bracket of X and Y and it is denoted by $[X,Y]$; i.e. $[X,Y]$ is the only vector field such that $[X,Y](f) = X(Y(f)) - Y(X(f))$ for every smooth function f. Given a metric g, there is a unique map that satisfies (1) through (5) [1].
We will need the following well known propositions which are also proved in [1].

Proposition 2.1: *Given $m \in M$ and $v \in T_m M$ there exists a unique geodesic $\gamma : [0,\infty) \to M$ such that $\gamma(0) = m$ and $\gamma'(0) = v$. We will denote the point $\gamma(t)$ by $\exp(m, tv)$.*

Proposition 2.2: *If $c : [a,b] \to M$ is a smooth curve on M then the mapping from $T_{c(a)}M \to T_{c(b)}M$ obtained by parallel transporting the vectors in $T_{c(a)}M$ along the curve c is an isometry. Here we are considering $T_{c(a)M}$ and $T_{c(b)}M$ with the metric induced by metrics $g(c(a))$ and $g(c(b))$.*

Proposition 2.3: (Formula for the second variation). *Let $\gamma : [0,a] \to M$ be a geodesic and let $f : (-\epsilon, \epsilon) \times [0,a] \to M$ be a smooth variation of γ. Let E be the energy function of the variation. Then*

$$\frac{1}{2}E'' = \int_0^a \{g(V',V') - g(R(\gamma',V)\gamma',V)dt$$
$$- g(\frac{D}{ds}\frac{\partial f}{\partial s}, \gamma')(0,0) + g(\frac{D}{ds}\frac{\partial f}{\partial s}, \gamma')(0,a))$$

Here V is the vector field along γ defined by $V = \frac{\partial f}{\partial s}(0,t)$, $\frac{D}{dt}$ is the covariant derivative along γ and $\frac{D}{ds}$ is the covariant derivative along a curve $s \to f(t,s)$.

Proposition 2.4: *If N_1 and N_2 are two disjoint closed submanifolds of a compact riemannian manifold M then the distance between N_1 and N_2 is assumed by a geodesic γ perpendicular to both N_1 and N_2.*

Once we have the Levi Civita connection on M, the curvature tensor is defined by

$$R(X,Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z$$

For any $X, Y, Z \in \Gamma(M)$. Since R is a tensor, then the value of $R(X,Y)Z$ at $m \in M$ only depends on the values of X, Y and Z at $m \in M$ [1], therefore it makes sense to talk about $R(X,Y)Z$ when X, Y, Z are not vector fields but vectors in T_mM.

Given a plane $\sigma \in T_mM$, the sectional curvature of σ is defined by $K(\sigma) = g(R(X,Y)X,Y)$ where $\{X,Y\}$ is any orthonormal base of σ with respect to the inner product g. Notice that as we mention before, K can be seen as a map from E to $\mathbf{R}$.

Let $S \subset M$ be a smooth surface. For any $m \in S$ we have that T_mS is a 2 dimensional subspace of $\mathbf{R}^6$ that is contained in T_mM. S is totally geodesic, if for every pair of vector fields X and Y on $\Gamma(M)$ with the property that $X(m) \in T_mS$ and $Y(m) \in T_mS$ for all $m \in S$, we have that $\nabla_X Y(m) \in T_mS$. For surface $S \in M$, we have a riemannian metric h induced by M, namely, for every pair of tangent vectors in $v, w \in T_mS$, the inner product of v and w is given by $h(v,w)(m) = g(v,w)(m)$. This make sense because v, w are also elements in T_mM. We can define the space of vector field on S, $\Gamma(S)$ in the exact same way we defined $\Gamma(M)$, just change M by S in the definition of $\Gamma(M)$ above. We can also define the Levi Civita connection $D : \Gamma(S) \times \Gamma(S) \to \Gamma(S)$ in the same way and, once we have defined D, we can define the curvature tensor $R_S : \Gamma(S) \times \Gamma(S) \times \Gamma(S) \to \Gamma(S)$ just by changing ∇ by D and M by S in the definition of ∇ above. The Theorem Egregium of Gauss for surfaces tells us that the Gauss curvature of S can be found just in terms of the metric, in this case we have that if $\kappa(m)$ denotes the Gauss curvature of S at $m \in S$ then $\kappa(m) = h(R_S(X,Y)X,Y)(m)$ where $\{X,Y\}$ is any orthonormal base of T_mS (with respect to the inner product h). Given a vector $w \in T_mM$ with $m \in S$ we will denote by w^T the only vector in T_mS that satisfies that $g(v, w - w^T)(m) = 0$ for every $v \in T_mM$, i.e. w^T is the tangential component of w on T_mS with respect to the metric g. We will need the following well known lemma in the proof of the main theorem.

Lemma 2.1: *If $X, Y : S \to \mathbf{R}^6$ are tangent vector fields on S and $\bar{X}, \bar{Y} : M \to \mathbf{R}^6$ are tangent vector fields on M such that $\bar{X}(m) = X(m)$ and $\bar{Y}(m) = Y(m)$ for every $m \in S$, then*

$$D_X Y(m) = (\nabla_{\bar{X}} \bar{Y})^T(m) \quad \text{for every } m \in S \tag{2}$$

Proof: The lemma follows after checking that if, for every pair of vector fields X, Y on S, we define the Levi Civita connection using the formula

(2) then the properties 1 through 5 are satisfied. This lemma is an exercise in [1].

3. Main results

In this section we will state and prove the main results of this paper. Let us start with the first theorem.

Theorem 3.1: *Let g be a metric on M. If there exists a totally geodesic surface $S \subset M$ which is topologically a torus, then the function $K : E \to \mathbf{R}$ can not be everywhere positive.*

Proof: Let us compute the Gauss curvature κ of S at $m_0 \in S$. Let $\bar{X}$ and $\bar{Y}$ be tangent vector fields on M such that: **(i)** if we define $X(m) = \bar{X}(m)$ and $Y(m) = \bar{Y}(m)$ for every $m \in S$ then X and Y define tangent vector fields on S, **(ii)** $\{X(m_0), Y(m_0)\}$ is an orthonormal base of $T_{m_0}S$ with respect to the metric h and **(iii)** $[X,Y](m_0)$ vanishes. Notice that these 3 conditions can easily be achieved by choosing X and Y vector fields coming from a local parametrization of S at $m_0 \in S$, then extending these vector fields to S and later extending the vector fields to M. Since S is totally geodesic, we have that $D_X Y(m) = (\nabla_{\bar{X}} \bar{Y}(m))^T = \nabla_{\bar{X}} \bar{Y}(m)$ for every $m \in M$ because of lemma 2.1. Using this observation we have that

$$\begin{aligned}
\kappa(m_0) =& h(R_S(X,Y)X,Y)(m_0) \\
=& h(D_Y D_X X - D_X \nabla_Y X + D_{[X,Y]}X, Y)(m_0) \\
=& g(\nabla_{\bar{Y}}(\nabla_{\bar{X}}\bar{X})^T - \nabla_{\bar{X}}(\nabla_{\bar{Y}}\bar{X})^T, Y)(m_0) \qquad (3) \\
=& g(\nabla_{\bar{Y}}\nabla_{\bar{X}}\bar{X} - \nabla_{\bar{X}}\nabla_{\bar{Y}}\bar{X}, Y)(m_0) \\
=& K(m_0, T_{m_0}S)
\end{aligned}$$

In the last step above we used that, by the definition of $X, \bar{X}$ and $Y, \bar{Y}$, the vector $[X,Y](m)$ in $T_m S$ equals the vector $[\bar{X}, \bar{Y}](m)$ in $T_m M$ for every $m \in S$. By the Gauss-Bonnet theorem we have that $\int_S \kappa = 4\pi(1 - \text{genus of } S) = 0$, therefore for some $m_0 \in S$, $\kappa(m_0) = 0$. Using (3) we get that the sectional curvature of $\sigma = T_{m_0}S$ vanishes and therefore the metric g on $S^2 \times S^2$ does not have positive sectional curvature. ■

Remark 3.1: Indeed the proof above shows a stronger result: If (M, g) is any compact n-dimensional riemannian manifold with $n \geq 2$ and there exists a compact surface S with genus greater than zero totally geodesic immersed in M, then the manifold (M, g) does not have positive sectional curvature. We have decided to make the theorem for $M = S^2 \times S^2$ to fix

ideas and make the definitions of connection and vector field easier and also because of the relevance of this case with the Hopf's conjecture.

Theorem 3.2: *Let g be a metric on $M = S^2 \times S^2$. If the metric g is invariant under a diffeomorphism of the form*

$$\phi_\alpha(x, y) = (x_1 \cos\alpha + x_2 \sin\alpha, -x_1 \sin\alpha + x_2 \cos\alpha, x_3, y_1, y_2, y_3)$$

for some $\alpha \in (0, 2\pi)$, then the metric g does not have positive sectional curvature.

Proof: We will start proving some lemmas that are needed for this proof.

lemma 3.1: *The submanifolds $S_1 = \{(0, 0, 1, y) : y \in S^2 \subset \mathbf{R}^3\}$ and $S_2 = \{(0, 0, -1, y) : y \in S^2 \subset \mathbf{R}^3\}$ have the property that if $v \in T_{m_0}S_i$ and γ is the geodesic in M with $\gamma(0) = m_0$ and $\gamma'(0) = v$ then $\gamma(t) \in S_i$ for all t.*

Proof of the lemma: Notice that the points in $S_1 \cup S_2$ are the only points in M that are fixed under the isometrie ϕ_α. Therefore, in order to prove that certain point $m \in M$ is in either S_1 or S_2 it is enough to show that $\phi_\alpha(m) = m$. Let us assume that $m_0 \in S_1$. We will show that $\gamma(t) \in S_1$ for any t. Let us consider the curve $\beta(t) = \phi_\alpha(\gamma(t))$. Since $\phi_\alpha : M \to M$ is an isometry, then β is also a geodesic. Notice that $\beta(0) = m_0 \in S_1$ and $\beta'(0) = d\phi_\alpha(m_0)(v) = v$ because the first three components of v are zero and ϕ_α leaves invariant the last 3 components of every point in $S^2 \times S^2$. Therefore by the proposition 2.1 we have that $\beta(t) = \gamma(t)$ for all t, therefore $\gamma(t)$ is fixed by ϕ_α and $\gamma(t) \in S_1 \cup S_2$. Since $\{\beta(t) : t \in [0, \infty)\}$ is connected, $\beta(0) \in S_1$ and $S_1 \cap S_2$ is empty, we have that $\beta(t) = \gamma(t) \in S_1$ for all t. ■

Let $\eta : [0, l] \to M$ be the geodesic in M that assumes the distance between S_1 and S_2. By the proposition 2.4 we have that if $m_0 = \eta(0) \in S_1$ and $m_1 = \eta(l) \in S_2$ then $\eta'(0)$ is perpendicular to $T_{m_0}S_1$ and $\eta'(l)$ is perpendicular to $T_{m_1}S_2$. Let $P : T_{m_0}M \to T_{m_1}M$ be the linear map given by parallel transporting the vector in $T_{m_0}M$ along η. Since $P(\eta'(0)) = \eta'(l)$ because η is a geodesic and P is an isometry, then we have that $P(T_{m_0}S_1)$ is a 2 dimensional subspace of $T_{\eta(l)}$ which vectors are perpendicular to $\eta'(l)$. Counting dimensions we conclude that the subspace $T_{m_1}S_2 \cap P(T_{m_0}S_1)$ must have dimension greater than 1, therefore, we can pick a non zero vector $V_0 \in T_{m_0}S_1$ such that $P(V_0) \in T_{m_1}S_2$. Let $V(t)$ be the parallel transport of V_0 along η from 0 to t. Let $f : (-\epsilon, \epsilon) \times [0, l] \to M$ be the variation of the geodesic η defined by $f(s, t) = \exp(\eta(t), sV(t))$. By lemma 3.1 we have that $f(s, 0) \in S_1$ and $f(s, l) \in S_2$ for all $s \in (-\epsilon, \epsilon)$, therefore,

for any $s \in (-\epsilon, \epsilon)$ the curve $\beta_s : [0, l] \to M$ defined by $\beta_s(t) = f(s,t)$ connects a point in S_1 with a point in S_2, hence the length of β_s is greater than or equal to the length of $\eta = \beta_0$. Therefore the function $E(s) = \text{length}(\beta_s)$ has a minimum at $s = 0$. On the other hand we have that: **(i)** $\frac{D}{ds}\frac{\partial f}{\partial s}$ vanishes because the curves $\alpha(s) = f(s,t)$ are geodesics; **(ii)** $\frac{DV}{dt} = V'(0,t)$ vanishes by definition of parallel transport. Using **(i)**, **(ii)** and the proposition 2.3 we get that

$$E''(0) = -2\int_0^a g(R(\eta', V)\eta', V)dt \tag{4}$$

Since $E : (-\epsilon, \epsilon) \to \mathbf{R}$ has a minimum at $s = 0$ we have $E''(0) \geq 0$. By the expression given in the equation (4) for $E''(0)$ we must have that $g(R(\eta'(t_0), V(t_0))\eta'(t)_0, V(t_0)) \leq 0$ for some $t_0 \in [0, l]$. Therefore the sectional curvature in the plane spanned by the orthogonal vectors $V(t_0)$ and $\eta'(t_0)$ in $T_{\eta(t_0)}M$ must be less than or equal to zero. ■

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