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Cookie Numbers

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Abstract

This paper generalizes a party puzzle and presents an analysis that leads to the solution set, the smallest member of which is termed a cookie number. Unlike famous numbers like Avogadro number, Mach number, Prandtl number, Reynold number, etc., the cookie number has no physical significance or application. Being the general solution of a puzzle, it has entertainment value only.

Keywords: Mathematical recreations, cookie number.

1 Introduction

In gatherings of non-mathematicians, serious mathematics is rarely discussed. At a party, one may come across a puzzle or a quiz involving simple algebra. The puzzler may not even know how to solve his puzzle; he may just know the answer. Party atmospheres, with freely flowing alcohol and other distractions, are rarely conducive to solving difficult problems, much less their generalizations. Presented here is a generalization of a party quiz that leads to cookie numbers. The problem is stated as follows:

2 Formulation of the Problem

A group of n friends and a dog set out with a box of C_n cookies on an overnight train trip. Each friend woke up in turn (while the others slept), divided the cookies into n equal parts, fed the remainder (if any) to the dog, ate his share, put back the rest in the box, and went back to sleep. The following morning, all the friends sat for their breakfast, divided the cookies into n equal parts, fed the remainder (if any) to the dog, and ate their respective shares. How many cookies did they start with? Cookies are not fractional.

This is a generalized version of the actual party quiz that specified three friends and one cookie for the dog at each division. The questioner stated that the problem could not be solved by algebra. Diophantus would have shouted from his grave, "What nonsense!" and would have been annoyed at the questioner's ignorance.

3 Solution of the Problem

Let $A_1, A_2, \ldots A_r, \ldots A_{n+1}$ be the cookies fed to the dog, respectively at $n+1$ divisions, such that $0 \le A_r \le n-1$. Also, let X_n be the number of cookies eaten by each friend at the breakfast. And, let $n \ge 2$, and let $A,\ k,\ n,\ C_n,\ X_n$ all be positive integers. Then,

$$
\begin{aligned}
C_n =& \frac{n}{n-1}\left[\frac{n}{n-1}\left[\frac{n}{n-1}\left[---\left[\frac{n}{n-1}(nX_n + A_{n+1}) + A_n\right] + A_{n-1}\right]\right.\right. \\
&\left.\left. + A_{n-2}\right] + ---\right] + A_1 \\
=& \frac{n^{n+1}}{(n-1)^n}X_n + \frac{1}{(n-1)^n}\left[(n-1)^n A_1 + n(n-1)^{n-1}A_2\right. \\
&\left. + n^{2(n-1)^{n-2}}A_3 + --- + n^n(n-1)^{n-n}A_{n+1}\right] \\
=& \frac{n^{n+1}}{(n-1)^n}X_n + \frac{1}{(n-1)^n}\sum_{i=0}^{i=n} n^i(n-1)^{n-i}A_{1+i}.
\end{aligned} \tag{1}
$$

Equation (1) is a general solution and cannot be further simplified until values of A_r are specified. If $A_r = m$, a constant, then

$$
\begin{aligned}
C_n =& \frac{n^{n+1}}{(n-1)^n}X_n + \frac{m}{(n-1)^n}\left[n^n + n^{n-1}(n-1) + n^{n-2}(n-1)^2 + \right. \\
&\left. ---n(n-1)^{n-1} + (n-1)^n\right] \\
=& \frac{n^{n+1}}{(n-1)^n}X_n + \frac{mn^n}{(n-1)^n}\left[1 + \frac{n-1}{n} + \frac{(n-1)^2}{n^2} + \frac{(n-1)^3}{n^3} + \right. \\
&\left. --- + \frac{(n-1)^{n-1}}{n^{n-1}} + \frac{(n-1)^n}{n^n}\right] \\
=& \frac{n^{n+1}}{(n-1)^n}X_n + \frac{mn^n}{(n-1)^n}\left[\frac{1 - \frac{(n-1)^{n+1}}{n}}{1 - \frac{n-1}{n}}\right] \\
=& \frac{n^{n+1}}{(n-1)^n}X_n + \frac{mn^n}{(n-1)^n}\left[\frac{n^{n+1} - (n-1)^{n+1}}{n^{n+1}}(n)\right] \\
=& \frac{n^{n+1}}{(n-1)^n}X_n \frac{n^{n+1}}{(n-1)^n}X_n + \frac{m}{(n-1)^n}\left[n^{n+1} - (n-1)^{n+1}\right] \\
=& \frac{n^{n+1}}{(n-1)^n}(X_n + m) - m(n-1).
\end{aligned} \tag{2}
$$

Now for C_n to be an integer,

$$X_n + m = k(n-1)^n, \quad \text{where } k = 1, 2, 3, \text{ etc.} \tag{3}$$

From which,

$$C_n = kn^{n+1} - m(n-1), \tag{4}$$

and,

$$X_n = k(n-1)_n - m. \tag{5}$$

For $k = 1$ and $m = 1$,

$$C_n = n^{n+1} - (n-1), \tag{6}$$

and,

$$X_n = (n-1)^n - 1. \tag{7}$$

From (4) it can be seen that the smallest positive value of C_n results when $k = 1$ and $m = n - 1$. Then,

$$C_n = n^{n+1} - (n-1)^2, \tag{8}$$

and

$$X_n = (n-1)\left[(n-1)^{n-1} - 1\right] \tag{9}$$

4 Solution of Specific Problems

The party problem can now be solved by substituting $m = 1$ and $k = 1, 2, 3 \ldots$etc. in (4), giving $C_3 = 79$, 160, 241, 322, etc. And, although the quiz did not ask for the cookies each friend ate at the breakfast, it can be readily calculated from (5) as $X_3 = 7$, 15, 23, 31, etc. In such problems, the minimum values are usually required. But, unaware of the possible multiple solutions, the party questioner did not ask for minimums.

Some limitations on cookie numbers would be in order. For instance, m=0 implies that no one is feeding the companion dog. Cookie numbers resulting from such cruelty to animals ought to be inadmissible. Hence, $C_n = n^{n+1}$ is not a cookie number. Similarly, on humanitarian grounds, cookie numbers that leave nothing for the friends' breakfast are also inadmissible. As can be seen from (7), this situation can arise only in case of $n = 2$.

The difference, D, between the largest and the smallest cookie numbers for a given n and $m = 1$ & $m = n-1$, can be obtained by subtracting equation (8) from equation (6). This yields $D = n^2 - 3n + 2$, which represents a bandwidth of cookie numbers. This bandwidth increases with

$C_2 = 15(*)$	$W_2 = 0.375\ lb.$
$C_3 = 79$	$W_3 = 1.975\ lb.$
$C_4 = 1021$	$W_4 = 25.525\ lb.$
$C_5 = 15621$	$W_5 = 390.525\ lb.$
$C_6 = 279931$	$W_6 = 3.50\ tons$
$C_7 = 5764795$	$W_7 = 72.06\ tons$
$C_8 = 134217721$	$W_8 = 1677.7\ tons$
$C_9 = 3486784393$	$W_9 = 43584.5\ tons$
$C_{10} = 99999999991$	$W_{10} = 1250,000\ tons$

(*) It should be noted that $C_2 = 7$ obtained from (6) results in $X_2 = 0$, which is inadmissible. So, $C_2 = 15$ obtained from (3) is used. For $X_2 = 1$ and $m = 1$, (3) has a solution for $k = 2$ only.

Table 1: Cookie numbers C_n and their corresponding weights W_n

increasing values of n, but is very narrow compared to the magnitude of the corresponding cookie numbers. Because $dD/dn = 2n - 3$, the bandwidth increases linearly with n, while the cookie numbers increase exponentially.

From (6), assuming 40 cookies per pound and 2,000 pounds per ton, the first nine cookie numbers C_n and their corresponding weights W_n, are given in Table 1.

Why did the party quiz have three friends? Several reasons can be seen at once. The case of two friends is easy to solve by trial and error. And, without a general formula like (6) at hand, a problem involving four or more friends would demand too much concentration for a party atmosphere, discouraging even the math lovers. Further, the results of $n > 3$ would appear to be incredible, and hence impractical. For instance, in the case of three friends, the first friend would have to eat 26 cookies, which a good eater can manage. In the case of four friends, however, the first friend would have to eat 255 cookies! He would have to be a real cookie monster to accomplish this. And, larger groups of friends would create other serious problems in addition to the gastronomic kind: The problems of carrying their snacks! The box size, the coach size, and ultimately the train's capacity to carry the snacks would restrict the friends to a group of perhaps eight, which would call for 40 wagon loads of cookies!

The significant cookie numbers, however, are restricted by the size of the cookie we choose. There is no reason why the cookies cannot be made smaller so that larger cookie numbers can be made meaningful.

n	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8
2	1	2	4	–	–	–	–	–
3	8	12	18	27	–	–	–	–
4	81	108	144	192	256	–	–	–
5	1024	1280	1600	2000	2500	3125	–	–
6	15625	18750	22500	27000	32400	38880	46656	–
7	279936	326592	381024	444528	518616	605052	705894	823543

Table 2: Coefficients of A's

Let us then assume that the cookies are really very small... as big as atoms. Also, let us assume that the gastronomic capacities of each of the friends are truly astronomic. Then how far can we go into the world of cookie numbers? Well, C_{50} will equal about $4.4 \cdot 10^{86}$ cookies, which is over 400 times the number of atoms in the universe! Do we need bigger cookie numbers?

So far our discussion has assumed that $A_r = m$, a constant. The party quiz can be made a little more difficult, and perhaps a bit more interesting, by specifying different values of A_r. For instance, in case of three friends ($n = 3$), we can specify the values of A_1, A_2, A_3, and A_4, as 2, 1, 1, and 2 respectively. Then from (1), $C_3 = (81/8)X_3 + (1/8)(8A_1 + 18A_2 + 27A_3)$, which on substitution for the specified A's gives $C_3 = 10X_3 + 12 + (X_3+4)/8$. C_3 will be an integer only if $X_3+4 = 8$ or $X_3 = 4$, which gives $C_3 = 10X_4 + 12 + 1 = 53$.

This shows how the diophatine analysis should be used when the dog feeding is non-uniform. It also shows that cases of non-uniform dog feeding may lead to smaller cookie numbers than the cases of uniform dog feeding, as given by (8). In fact, for $n = 3$, the largest $C_3 = 95$, corresponding to $(A_1,\ A_2,\ A_3,\ A_4) = (2,\ 2,\ 1,\ 2)$, and the smallest $C_3 = 15$, corresponding to $(A_1,\ A_2,\ A_3,\ A_4) = (0,\ 1,\ 0,\ 1)$. For $n = 2$ the expression for cookie numbers turns out to be simpler: $C_2 = 8X_2 + (A_1 + 2A_2 + 4A_3)$, the largest and the smallest values of which are 15 and 9, corresponding to values of $(A_1,\ A_2,\ A_3) = (1,\ 1,\ 1)$ and $(1,\ 0,\ 0)$, respectively. Within these ranges, all integers are cookie numbers. Again, numbers of the form n^{n+1} are not cookie numbers. Thus, 8, 81, 1024, etc. are not cookie numbers. It is obvious that for every n, there are $n^{n+1} - 1$ valid cookie numbers.

As an aid to solving the general equation (1), Table 2 provides the coefficients of A's for $n = 2$ to $n = 7$.

As an example of the use of this table, consider the case of 5 friends who feed one cookie to dog at each division:

For $n = 5$; $n^{n+1} = 5^6 = 15625$; $(n-1)^n = 4^5 = 1024$; and $A_1 = A_2 = \cdots A_5 = 1$,

$\sum 1024A_1 + 1280A_2 + 1600A_3 + 2000A_4 + 2500A_5 + 3125A_6 = 11529$. Then, $C_5 = \frac{15625}{1024}X_5 + \frac{11529}{1024} = \frac{15625}{1024}X_5 + \frac{15625}{1024} - \frac{4096}{1024} = (15625/1024)(X_5 + 1) - 4$.

For C_5 to be an integer, $X_5 + 1 = 1024$, or $X_5 = 1023$, and $C_5 = 15625 - 4 = 15621$, which checks with the result obtained by formula (6), as it should. Also, formula (7) yields $X_5 = 1023$.

In case of $n = 4$, $A_1 = A_2 = A_3 = 0$, $A_4 = 2$ and $A_5 = 3$, $\sum 192X_2 + 256X_3 = 1152$, $n^{n+1} = 1024$, $(n-1)^n = 81$, and

$$\begin{aligned} C_4 &= (1024/81)X_4 + 1152/81 \\ &= (1024/81)X_4 - (1152X_8/81) + (1152X_9/81) \\ &= (1024/81)(X_4 - 9) + 128. \end{aligned}$$

For C_4 to be an integer, $X_4 - 9 = 81$, or $X_4 = 90$, and $C_4 = (1024 + 128) = 1152$, which the reader may check if he doubts. In passing, it may be mentioned that Table 2 contains help to solve some 296,669 cookie problems, no more, no less!

And that's the way the cookie crumbles...

5 Extension of The Problem

The problem can be further extended to add complications and complexities without adding flavor. For instance, each friend can get up more than once to munch his share, feeding the dog as he pleases. Or a general expression can be derived to find how many cookies each friend ended up eating. Or one may even try to find an expression for the cookie number that ensures all friends eating equal number of cookies. This, however, leads the party puzzle, far from being a mere brain teaser, and is not attempted here.

6 Concluding Remarks

Cookie numbers, unlike most famous numbers, are integers and shall cover almost entire field of positive whole numbers, unless, as is done here, restricted by some arbitrary conditions. Even the negative integers provide perfectly legitimate, if not practical solutions. For instance, in case of the party problem, substituting $n = 3$, $m = 1$ and $k = 0$ in equation (4) C_n turns out to be negative two ($C_n = -2$). In the world of negative beings, this is not only a legitimate solution, but also a beautiful one. For it ensures equal number of cookies eaten by each friend. Each friend eats -2 cookies and the good dog (the unworldly one!) eats $+4$ cookies. This would make the day of the animal lovers (of the real world!). In all fairness, one may feed the dog negative cookies too. Substituting $n = 3$, $m = -1$ and $k = 0$ in equation (4) one would get $C_n = 2$ and each friend will then end up eating two cookies and the poor quadruped will have to be content with negative four cookies. Without general expressions, who would have thought of such solutions!

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References

[1] D.O. Shklarsky, N.N. Chentzov and I.M. Yaglom, *The USSR Olympiad problem* problem book, Dover publ., New York(1994)

[2] P.S. Herwitz, *The theory of numbers*, Scientific American, July(1951).

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