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# On Hilbert schemes and Chen-Ruan cohomology

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**Abstract**

Through a geometric approach, we explain the origin of the crepant resolution conjecture of Y. Ruan. More precisely, we calculate the Chen-Ruan cohomology and the quantum corrections of Ruan for the cohomology of Hilbert schemes in the particular case of the two-fold symmetric product of $\mathbb{C}P^2$, which corresponds to the invariant part by the action of the symmetric group $\mathfrak{S}_2$ on the blow-up of $\mathbb{C}P^2 \times \mathbb{C}P^2$ along the diagonal.

**Keywords:** Chen-Ruan cohomology, Hilbert schemes, crepant resolution conjecture, symmetric product.

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## 1 Introduction

A crepant resolution of a reduced, compact, complex orbifold $Q$ of dimension $n$ (i.e. a compact Hausdorff space locally modeled on $\mathbb{C}^n/G$ where $G$ is a finite group which acts effectivily on $\mathbb{C}^n$ with a set of fixed points of codimension at least two) is a pair $(Y, \pi)$ where $Y$ is a smooth complex manifold of dimension $n$, $\pi : Y \to Q$ is a surjective map which is biholomorphic away from the singular set of $Q$, and $\pi^* K_Q = K_Y$, where $K_Q$ and $K_Y$ denote the canonical line bundles of $Q$ and $Y$, respectively.

As it was shown by A. Beauville in [1], if $Q$ is the $n$-fold symmetric product $X^{(n)}$ of a proyective surface $X$, then a crepant resolution of $X^{(n)}$ is the Hilbert scheme $X^{[n]}$ of $n$-points on $X$. In this case a conjecture of Y. Ruan (see [10]) states that the orbifold cohomology ring $H^*_{CR}(X^{(n)})$ defined by Chen and Ruan in [2] is isomorphic to the quantum corrected cohomolgy ring $H^*_\pi(X^{[n]})$ of the Hilbert scheme.

A simple case of this happens when $n = 2$, since we have a complete description of the Hilbert scheme $X^{[2]}$, namely

$$X^{[2]} = Blow_\Delta\{X \times X\}/\mathfrak{S}_2$$

where $Blow_\Delta\{X \times X\}$ is the blow-up of $X \times X$ along the diagonal $\Delta$, and $\mathfrak{S}_2$ denotes the symmetric group of order 2.

In this article, which is a summary of my master thesis carried out in the Universidad de los Andes, we will present Ruan's conjecture on the relation between the Chen-Ruan cohomology of an orbifold and the cohomology of its crepant resolution. We explain the proof of Ruan that states that the conjecture is true for the two-fold symmetric product. The proof included in this article expands the calculations of Ruan and gives the geometric proofs necesary to understand the Ruan's result. We emphasize the geometric construction and give all the details that were not included in Ruan's original proof.

## 2 Some preliminaries

### 2.1 The Chen-Ruan cohomology

In this section we define the Chen-Ruan cohomology groups for an orbifold given by the quotient of a manifold by a finite group and endow the cohomology module with a ring structure. A specific and very important case of this is the symmetric product, constructed as the orbit space of the action of the symmetric group over $n$ copies of a topological space in the natural way (by permutation of coordinates).

Let $X$ be a complex manifold on which a finite group $G$ acts holomorphically. The twisted sectors of the orbifold $Y = X/G$ are the sets

$$Y_{(\mathbf{g})} = X^{(\mathbf{g})}/C(\mathbf{g})$$

where $X^{(\mathbf{g})} = X^{g_1} \cap ... \cap X^{g_k}$ with $X^{g_i}$ the set of fixed points of the action of $g_i$ and $C(\mathbf{g}) = C(g_1) \cap ... \cap C(g_k)$ with $C(g_i)$ the centralizer of $g_i$ in $G$.

The Chen-Ruan cohomology module structure is defined as the vector space

$$H^*_{CR}(Y; \mathbb{C}) = \bigoplus_{(g) \in T_1} H^*(Y_{(g)}; \mathbb{C})$$

where $T_k$ is the set of conjugacy classes of $k$-tuples $(g_1, ...g_k)$ of elements in $G$.

We endow this $\mathbb{C}$-module with a graded ring structure. Suppose that the complex dimension of $X$ is $D$. For $g \in G$ and $y \in X^g$, let $\lambda_1, ..., \lambda_D$ be the eigenvalues of the action of $g$ on the tangent space $T_{X,y}$; note that they are roots of unity since $G$ is finite and therefore exist $k \in \mathbb{N}$ such that $g^k = 1$.

**Definition 2.1.** *If we write $\lambda_j = e^{2\pi i r_j}$, where $r_j$ is a rational number in $[0, 1[$, the age of $g$ in $y$ is the rational number $a(g, y) = \sum_{j=1}^{D} r_j$.*

**Proposition 2.2.** *The age $a(g, y)$ only depends on the conjugacy class of $g$ and the connected component $Z$ of $X^g$ in which $y$ lies. If we denote it by $a(g, Z)$, we have that*

$$a(g, Z) + a(g^{-1}, Z) = codim(Z \subset Y)$$

*where the codimension refers to complex codimension.*

*Proof.* The first part is not dificult, since the tangent spaces are isomorphic for all $x, y \in Z$, and if $g' \in (g)$ is an element in the conjugacy class of $g$, then the eigenvalues of the action of $g'$ and $g$ on the tangent space $T_{X,y}$ are the same. For the second part, we can suppose that $X^g$ is connected. Then, if $N_y$ denotes the fiber of the normal bundle of $X^g$ in $X$ over $y \in Y^g$ we have that

$$T_{y,X} = T_{y,X^g} \oplus N_y.$$

Moreover, $g$ acts as the identity on $T_{y,X^g}$. Then the action of $g$ on $T_{y,X}$ has as many eigenvalues equal to 1 as the complex dimension of $Y^g$. □

**Definition 2.3.** *The grading on the Chen-Ruan cohomology groups are defined as*

$$H^d_{CR}(Y;\mathbb{C}) = \bigoplus_{(g)\in T_1} H^{d-2a(g)}(Y_{(g)};\mathbb{C})$$

Poincaré Duality for the Chen-Ruan cohomology is as follows:
For any $0 \le d \le 2D$, the pairing

$$\langle,\rangle_{orb} : H^d_{CR}(Y;\mathbb{C}) \otimes H^{2D-d}_{CR}(Y;\mathbb{C}) \to \mathbb{C}$$

is defined by the direct sum of

$$\langle,\rangle^{(g)}_{orb} : H^{d-2a(g)}(Y_{(g)};\mathbb{C}) \otimes H^{2D-d-2a(g^{-1})}(Y_{(g^{-1})};\mathbb{C}) \to \mathbb{C}$$

where

$$\langle \alpha', \beta' \rangle^{(g)}_{orb} = \int_{Y_{(g)}} \alpha' \cup I^*(\beta'),$$

with $I : Y_{(g)} \to Y_{(g^{-1})}$ defined by $(p) \to (p)$, is nondegenerate. Note that when restricted to the non-twisted sector this is the ordinary Poincaré pairing.

The ring structure relies on the construction of an obstruction bundle over the twisted sector $Y_{(\mathbf{g})}$ where $(\mathbf{g}) = (g_1, g_2, g_3) \in T_3$ is the conjugacy class of the triple $(g_1, g_2, g_3)$ with $g_1 g_2 g_3 = 1$, and $T_3^0$ is the set of those conjugacy classes.

Let $e : Y_{(\mathbf{g})} \to Y$ be the map induced by the inclusion and $e^*TY$ the pullback of the tangent bundle over $Y_{(\mathbf{g})}$. Let $\Gamma$ the subgroup of $G$ generated by $g_1, g_2, g_3$; then $\Gamma$ acts on $e^*TY$ while fixing $Y_{(\mathbf{g})}$.

For the Riemann orbifold sphere $S^2_{(x_1,x_2,x_3)}$ with only three orbifold points $x_1, x_2, x_3 \in S^2$ each one being a cone point with angle $2\pi/k_i$ where $k_i$ is the order of $g_i$(i.e. a neighborhood of $x_i$ is homeomorphic to $\mathbb{C}/\mathbb{Z}_{k_i}$), it is a result of [11] that there exist a closed Riemann surface $\Sigma$ such that $\Gamma$ acts on it holomorphically and, $\Sigma/\Gamma$ and $S^2_{(x_1,x_2,x_3)}$ are isomorphic as orbifolds. Now the group $\Gamma$ acts on both $H^1(\Sigma)$ and $e^*TY$ when considering $H^1(\Sigma)$ as a trivial bundle over $Y_{(\mathbf{g})}$.

The obstruction bundle $E_{(\mathbf{g})}$ we require is the invariant part of $H^1(\Sigma) \otimes e^*TY$ under the action of $\Gamma$. Let $c(E_{(\mathbf{g})}) = eu((H^1(\Sigma) \otimes e^*TY)^\Gamma)$ be the Euler class of $E_{(\mathbf{g})}$ and consider the maps $e_i : Y_{(\mathbf{g})} \to Y_{(g_i)}$ induced by the inclusions.

**Definition 2.4.** *For $\alpha', \beta', \gamma' \in H^*_{CR}(Y;\mathbb{C})$ a three-point function is defined by*

$$\left\langle \alpha', \beta', \gamma' \right\rangle_{orb} = \sum_{(\boldsymbol{g} \in T_3^0)} \int_{Y_{(\boldsymbol{g})}} e_1^* \alpha' . e_2^* \beta' . e_3^* \gamma' . c(E_{(\boldsymbol{g})})$$

*and let the orbifold cup product be defined by the relation*

$$\left\langle \alpha' \cup_{orb} \beta', \gamma' \right\rangle_{orb} = \left\langle \alpha', \beta', \gamma' \right\rangle_{orb}.$$

There exist an equivalent description of this ring (see [3]). Here we start with the vector space

$$H^*(X;G) = \bigoplus_{g \in G} H^*(X^g;\mathbb{C}).$$

For $g \in G$ and $\alpha'_g \in H^*(X^g;\mathbb{C})$, denote by $(\alpha'_g)$ the corresponding element in $H^*(X;G)$.

If $g, h$ are two elements of $G$, then $h(X^g) = X^{hgh^{-1}}$. Hence $G$ acts on $H^*(X;G)$ by

$$h(\alpha'_g) = (h_* \alpha')_{hgh^{-1}}$$

where $h_*$ denote the pushforward $h_* : H^*(X^g;\mathbb{C}) \to H^*(X^{hgh^{-1}};\mathbb{C})$.

**Proposition 2.5.** *The invariant subspace $H^*(X;G)^G$ under the action of $G$ is isomorphic to*

$$\bigoplus_{g \in T} H^*(X^g;\mathbb{C})^{C(g)}$$

*where $T \subset G$ is a set of representatives of the conjugacy classes of $G$.*

*Proof.* We denote an element $t \in T$ by $\bar{g}$ if $t$ is in the conjugacy class of $g$. We define the application:

$$\begin{array}{rcl} \varphi : (H^*(Y;G))^G & \longrightarrow & \bigoplus_{\bar{g} \in T} H^*(Y^{\bar{g}})^{C(\bar{g})} \\ (\alpha'_g) & \longmapsto & (\alpha'_{\bar{g}}) \end{array}$$

and, we extend this linearly . This function is well defined since if $\alpha'_g$ is invariant by the action of $G$, then for all $h \in G$, $h.(\alpha'_g) = \alpha'_{hgh^{-1}}$, in particular if $h \in C(g)$, $hgh^{-1} = g$ and we have that

$$h.(\alpha'_g) = \alpha'_{hgh^{-1}} = \alpha'_g.$$

Then $\alpha'_{\bar{g}}$ is invariant by the action of $C(g)$.

Now, we suppose that $\varphi(\alpha'_g)_{g \in G} = (\alpha'_{\bar{g}})_{\bar{g} \in T} = 0$ and let $r$ be an element in the conjugacy class of $g$. We must see that $\alpha'_r = 0$, but it is trivial since there exist $h \in G$ such that $hrh^{-1} = \bar{g}$. Therefore

$$h.(\alpha'_r) = \alpha'_{hrh^{-1}} = \alpha'_{\bar{g}} = 0.$$

Since $h^*$ is an isomorphism, then $\alpha'_r = 0$.

To see that $\varphi$ is surjective, we take $(\alpha'_g)_{g\in T} \in \bigoplus_{g\in T} H^*(Y^g)^{C(g)}$, i.e., for all $x \in C(g)$ we have that

$$x.(\alpha'_g) = (x^*)^{-1}\alpha'_g = \alpha'_g.$$

Now, since the conjugacy class of $g$ is the set $[g] = \{g, h_1 g {h_1}^{-1}, ..., h_s g {h_s}^{-1}\}$ for some $h_i \in G$, then $\varphi(\alpha'_g, h_1.(\alpha'_g), ..., h_s.(\alpha'_g), 0, ..., 0) = \alpha'_g$; because if $xgx^{-1} = ygy^{-1}$, then $x^{-1}y \in C(g)$ and therefore $y.(\alpha'_g) = x.(\alpha'_g)$. □

**Definition 2.6.** *We define a (rational) grading on $H^*(X;G)$ as follows. Let $g \in G$ and let $Z$ be a connected component of $Y^g$, and $j : Z \to Y^g$ the inclusion. Let $\alpha' \in H^i(Z;\mathbb{C})$; we assign to $j_*\alpha'_g$ the degree $i + 2a(g,Z)$.*

**Definition 2.7.** *Define a bilinear map*

$$\mu : H^*(X;G) \times H^*(X;G) \to H^*(X;G)$$

*by*

$$\mu(\alpha'_g, \beta'_h) = \gamma'_{gh}$$

*where*

$$\gamma' = i_*(f^*(\alpha')g^*(\beta')c(E_{(\boldsymbol{g})})),$$

*and $f : X^g \cap X^h \to X^g$, $g : X^g \cap X^h \to X^h$, $i : X^g \cap X^h \to X^{gh}$ are the natural inclusions, and $E_{(\boldsymbol{g})}$ is the obstruction bundle.*

**Proposition 2.8.** *The bilinear map $\mu$ sends $H^i(X;G) \otimes H^j(X;G)$ to $H^{i+j}(X;G)$. Hence, it defines a graded and associative product on $H^*(X;G)$, which extends to $H^*(X;G)^G$ and whose three point function is*

$$\left\langle \alpha'_g, \beta'_h, \gamma'_{(gh)^{-1}} \right\rangle = \int_{X^g \cap X^h} f^*(\alpha')g^*(\beta')r^*(\gamma')\pi^*(c(E_{(\boldsymbol{g})}))$$

*where $r : X^g \cap X^h \to X^{(gh)^{-1}}$ is the inclusion, and $\pi : X^g \cap X^h \to \frac{X^g \cap X^h}{C(g,h)}$ is the projection.*

A complete proof of this proposition is in [3]. Note the similarity with the definition of the orbifold cup product, in fact the two definitions are equivalent (see [12] and [3]).

Now, we will focus in the case of the symmetric product. Let $M$ be a complex connected manifold and let $X = M^n$ be the usual cartesian product. The $n$-fold symmetric product of $M$ is the space $Y = X/\mathfrak{S}_n$, where $\mathfrak{S}_n$ denotes the permutation group in $n$ elements, and the action on $X$ is the natural one. For $\sigma \in \mathfrak{S}_n$ and $(x_1, ..., x_n) \in X$

$$\sigma.(x_1, ...x_n) = (x_{\sigma(1)}, ..., x_{\sigma(n)}).$$

**Lemma 2.9.** *Let $h_1, h_2 \in \mathfrak{S}_n$ and let $H$ be a subgroup generated by $\{h_1, h_2\}$. Then, the range of the obstruction bundle $E_{(\boldsymbol{h})}$ is*

$$r(h_1, h_2) = a(h_1) + a(h_2) - a(h_1 h_2) - codim(X^H \subset X^{h_1 h_2}).$$

A complete proof of this is in [3].

Now, if $\sigma_1, \sigma_2 \in \mathfrak{S}_n$ are diferent transpositions, we have by proposition 2.2 that

$$a(\sigma_1) = a(\sigma_2) = \frac{n}{2}$$

and then,

$$r(\sigma_1, \sigma_1) = n - a(1) - codim(X^{\sigma_1} \subset X) = n - 0 - n = 0. \tag{1}$$

As the definition of the obstruction bundle is symmetric with respect to the elements in $\mathfrak{S}_n$, then

$$2r(\sigma_1, \sigma_2) = 2n - (a(\sigma_1\sigma_2) + a(\sigma_2\sigma_1)) - 2codim(Y^{\sigma_1,\sigma_2} \subset Y^{\sigma_1\sigma_2}),$$

but since $X^{\sigma_1,\sigma_2} = X^{\sigma_1\sigma_2}$, because $\sigma_1\sigma_2$ is a minimal descomposition of $\sigma_1\sigma_2$ as product of transpositions, and the $codim(Y^{\sigma_1\sigma_2} \subset Y) = 2n$, we obtain that

$$2r(\sigma_1, \sigma_2) = 2n - 2n - 0 = 0. \tag{2}$$

Note that for the last calculations $c(E_{(\sigma_1,\sigma_2)}) = 1$. Now, consider the 4-fold symmetric product of $M$. Let $\{h_i\}$ be a basis for $H^*(M;\mathbb{C})$ as a $\mathbb{C}$-module. By Kunneth formula $\{h_{i_1} \otimes ... \otimes h_{i_k}\}$ is a basis for $H^*(M^k;\mathbb{C})$ as a $\mathbb{C}$-module. We have that

$$\begin{aligned}
X^{(1,2)} &\cong \{(x_{12}, x_3, x_4) \in M^3\} = M^3\\
X^{(3,4)} &\cong \{(x_1, x_2, x_{34}) \in M^3\} = M^3\\
X^{(12)(34)} &\cong \{(x_{12}, x_{34}) \in M^2\} = M^2.
\end{aligned}$$

Let $\{h_i \otimes h_j \otimes h_k\}$, $\{\tilde{h_i} \otimes \tilde{h_j} \otimes \tilde{h_k}\}$ and $\{\bar{h_i} \otimes \bar{h_j}\}$ be basis for $H^*(X^{(1,2)};\mathbb{C})$, $H^*(X^{(3,4)};\mathbb{C})$ and $H^*(X^{(1,2)(3,4)};\mathbb{C})$ as $\mathbb{C}$-modules respectively.

If $f : X^{(1,2)(3,4)} \to X^{(1,2)}$ and $g : X^{(1,2)(3,4)} \to X^{(3,4)}$ are the natural inclusions, it is easy to see that

$$f^*(h_i \otimes h_j \otimes h_k) = \bar{h_i} \otimes \bar{h_j}\bar{h_k}$$

and

$$g^*(\tilde{h_i} \otimes \tilde{h_j} \otimes \tilde{h_k}) = \bar{h_i}\bar{h_j} \otimes \bar{h_k}.$$

And the product

$$\begin{aligned}
\mu(h_i \otimes h_j \otimes h_k, \tilde{h_i} \otimes \tilde{h_j} \otimes \tilde{h_k}) &= f^*(h_i \otimes h_j \otimes h_k) g^*(\tilde{h_i} \otimes \tilde{h_j} \otimes \tilde{h_k})\\
&= [\bar{h_i} \otimes \bar{h_j}\bar{h_k}][\bar{h_i}\bar{h_j} \otimes \bar{h_k}]\\
&= \bar{h_i}\bar{h_i}\bar{h_j} \otimes \bar{h_j}\bar{h_k}\bar{h_k}.
\end{aligned}$$

Note that it is true in general for $\sigma_1, \sigma_2$ disjoint transpositions. If $\sigma_1$ and $\sigma_2$ are diferent transpositions which intersect, the situation can be modeled by the choice of $\sigma_1 = (1,2)$ and $\sigma_2 = (2,3)$; in this case

$$\begin{aligned} X^{(1,2)} &\cong \{(x_{12}, x_3, x_4) \in M^3\} = M^3 \\ X^{(2,3)} &\cong \{(x_1, x_{23}, x_4) \in M^3\} = M^3 \\ X^{(1,2,3)} &\cong \{(x_{123}, x_4) \in M^2\} = M^2 \end{aligned}$$

and for $f : X^{(1,2,3)} \to X^{(1,2)}$ and $g : X^{(1,2,3)} \to X^{(2,3)}$ the inclusions, and $\{h_i \otimes h_j\}$, $\{\tilde{h_i} \otimes \tilde{h_j} \otimes \tilde{h_k}\}$, $\{\bar{h_i} \otimes \bar{h_j} \otimes \bar{h_k}\}$ basis as a $\mathbb{C}$-module for $H^*(X^{(1,2,3)}; \mathbb{C})$, $H^*(X^{(1,2)}; \mathbb{C})$, $H^*(X^{(2,3)}; \mathbb{C})$ respectively, we have that

$$\begin{aligned} f^*(\tilde{h_i} \otimes \tilde{h_j} \otimes \tilde{h_k}) &= h_i h_j \otimes h_k \\ g^*(\bar{h_i} \otimes \bar{h_j} \otimes \bar{h_k}) &= h_i h_j \otimes h_k. \end{aligned}$$

And the product is as before.

This shows that the Chen-Ruan cohomology ring may be generated by the twisted sectors corresponding to the transpositions, more precisely

**Proposition 2.10.** *If $\tau = \sigma_1 ... \sigma_k$ is a minimal descomposition of $\tau \in \mathfrak{S}_n$ ($\tau \neq (1)$) as product of transpositions and $\alpha'_\tau \in H^*(Y^\tau; \mathbb{C})$, then there exist elements $\alpha'_{\sigma_i} \in H^*(X^{\sigma_i}; \mathbb{C})$ such that*

$$\alpha'_\tau = \alpha'_{\sigma_1} ... \alpha'_{\sigma_k}$$

*in the ring $H^*(X; \mathfrak{S}_n)$.*

*Proof.* It is suficient to prove the proposition for the case which $\tau = \sigma_1\sigma_2\sigma_3$ with $\sigma_i \neq \sigma_j$ for $i \neq j$. Then a inductive argument works in general.

Note that the range of the obstruction bundle is

$$\begin{aligned} r(\sigma_1\sigma_2, \sigma_3) &= a(\sigma_1\sigma_2) + a(\sigma_3) - a(\sigma_1\sigma_2\sigma_3) \\ &= \frac{n}{2} + 2\frac{n}{2} - 3\frac{n}{2} = 0. \end{aligned}$$

Therefore $c(E_\sigma) = 1$ and, since $X^{\sigma_1\sigma_2} \cap X^{\sigma_3} = X^{\sigma_1\sigma_2\sigma_3}$ and the pullbacks $f^* : H^*(X^{\sigma_3}; \mathbb{C}) \to H^*(X^{\sigma_1\sigma_2\sigma_3}; \mathbb{C})$ and $g^* : H^*(X^{\sigma_1\sigma_2}; \mathbb{C}) \to H^*(X^{\sigma_1\sigma_2\sigma_3}; \mathbb{C})$ induced by the inclusions are surjective, we have the result. $\square$

### 2.2 The homology of Hilbert schemes of points on surfaces

Let us consider the two-fold symmetric product of a projective surface $X^{(2)}$. Since the singular set of $X^{(2)}$ is the set $\Delta = \{(x,x) \mid x \in X\}$, then a resolution of singularities is the quotient by the extended action of $\mathfrak{S}_2$ of the blow-up of $X^2$ along the diagonal $\Delta$

$$Blow_\Delta(X \times X)/\mathfrak{S}_2 \xrightarrow{\pi} X^{(2)}.$$

This resolution is in fact a crepant resolution of $X^{(2)}$. In order to study crepant resolutions for the $n$-fold symmetric product $X^{(n)}$, one can ask: what space should take the place of the Blow-up?

$$? \xrightarrow{\pi} X^{(n)}.$$

The answer is *The Hilbert scheme of $n$ points on $X$*, which corresponds in the case $n = 2$ to the blow-up mentioned above.

To explain what is the Hilbert scheme of points, we will start saying how are its elements in the special case of the Hilbert scheme $(\mathbb{C}^2)^{[n]}$ of $n$ points on $\mathbb{C}^2$.

An element in $(\mathbb{C}^2)^{[n]}$ is a closed 0-dimensional subscheme of length $n$, i.e. such element can be represented by an ideal $I \subset \mathbb{C}[x, y]$ such that the set

$$V(I) = \{(x, y) \in \mathbb{C}^2 \mid f(x, y) = 0, \text{ for all } f \in I\}$$

is finite and $\mathbb{C}[x, y]/I$ has dimension $n$ as a complex vector space.

The multiplicity of a point $P = (p_1, p_2) \in V(I)$ is the length of the Artin local ring $\big(\mathbb{C}[x, y]/I\big)_{\langle x-p_1, y-p_2\rangle}$. It is clear that the multiplicities of all points in $V(I)$ sum $n$, giving rise to a 0-dimensional algebraic cycle $\sum_i m_i P_i$ of weight $\sum_i m_i = n$. We may view this cycle as an unordered tuple $[P_1, ..., P_n] \in (\mathbb{C}^2)^{(n)}$, in which each point is repeated accordingly to its multiplicity.

The *Hilbert-Chow morphism*

$$\pi : (\mathbb{C}^2)^{[n]} \to (\mathbb{C}^2)^{(n)}$$

is the morphism mapping each $I \in (\mathbb{C}^2)^{[n]}$ to the corresponding algebraic cycle $\pi(I) = [P_1, ..., P_n]$.

As an example, we can study the Hilbert scheme $(\mathbb{C}^2)^{[2]}$. If $V(I)$ contains two points, then $I$ correspond to an element in $(\mathbb{C}^2)^{(2)}$. Now, let us suppose that $V(I) = \{(0, 0)\}$ and consider the projection

$$\mathbb{C}[x, y] \to \mathbb{C}[x, y]/I$$

since the ring $(\mathbb{C}[x, y]/I)_{\langle x,y\rangle}$ has length two, then the kernel of this morphism should contain the ideal $m^2 = \langle x^2, xy, y^2\rangle$. It is clear that $\mathbb{C}[x, y]/m^2$ has dimension three over $\mathbb{C}$, it follows that the kernel will contain a nonzero homogeneus linear form $\alpha' x + \beta' y$.

The subscheme

$$X_{\alpha',\beta'} = Spec\mathbb{C}[x, y]/\langle x^2, xy, y^2, \alpha' x + \beta' y\rangle$$

can be characterized as the subscheme of $\mathbb{C}^2$ associated to the ideal of functions $f \in C[x, y]$ that vanish at the origin and have partial derivatives satisfying

$$\beta' \frac{\partial f}{\partial x} - \alpha' \frac{\partial f}{\partial y}$$

(since this implies that $f = c(\alpha' x + \beta' y)+$ higher-order terms). In this way, the Hilbert scheme of two points on $\mathbb{C}^2$ is the blow-up of $\mathbb{C}^2$ along the diagonal.

More theory is necesary in order to make a formal definition.

We define the *functor of points* of a scheme $X$; that is, the functor

$$h_X : (schemes)^\circ \to (sets),$$

where $(schemes)^\circ$ and $(sets)$ represent the category of schemes with the arrows reversed and the category of sets, respectively; $h_X$ sends each scheme $Y$ to the set

$$h_X(Y) = Mor(Y, X)$$

and each morphism $f : Y \to Z$ to the map of sets

$$h_X(Z) \to h_X(Y)$$

defined by sending an element $g \in h_X(Z) = Mor(Z, X)$ to the composition $g \circ f \in Mor(Y, X)$. We say that the functor $h_X$ is a *representable functor* and that it is representable by the scheme $X$.

**Example 2.11.** *The classical geometric categories of topological spaces and manifolds have a unique (up to isomorphism) one-point object $x$ which has the property that the morphism $f : x \to X$ correspond to the points of $X$, giving rise to a natural bijection $h_X(x) = \{f : x \to X\} \leftrightarrow \{x \in X\}$ hence the terminology "functor of points".*

**Theorem 2.12** (Existence of the Hilbert scheme)**.** *Fix a projective $S$-scheme $X$ and a polynomial $P(n)$. Then the Hilbert functor $\mathscr{H}_{X,P(n)}$ defined by:*

$$\mathscr{H}_{X,P(n)}(T) = \{\textit{closed subschemes } Z \subset X \times T \textit{ that are flat over } T \textit{ with Hilbert polynomial } P(n)\}$$

$$\mathscr{H}_{X,P(n)}(f) = \textit{base extension by } f$$

*is represented by a projective $S$-scheme, denoted by $Hilb_X^{P(n)}$ and called the Hilbert scheme of $X$ with polynomial $P(n)$.*

A proof of this is in [5].

**Definition 2.13.** *For each $n$ let $P$ be the constant polynomial given by $P(m) = n$, for all $m \in \mathbb{Z}$. We denote by $X^{[n]} = Hilb_X^P$ the corresponding Hilbert scheme and call it the Hilbert scheme of $n$ points in $X$.*

Is clear from the definition that an element in $X^{[n]}$ is represented by a length-$n$ zero-dimensional closed subscheme $\xi$ of $X$. Now, let $x_1, ..., x_n \in X$ be $n$ distinct points and consider $Z = \{x_1, ..., x_n\} \subset X$ as a closed subscheme. Then

$$\text{length}(Z) = \sum_{i=1}^{n} \text{length}(x_i) = n,$$

and $Z$ can be considered as a element in $X^{[n]}$. This is the reason $X^{[n]}$ is called the Hilbert scheme of $n$ points in $X$.

**Example 2.14.** *Suppose $X$ nonsingular and consider $X^{[2]}$. As mentioned above, $\{x_1, x_2\}$ can be considered as a point in $X^{[2]}$ if $x_1$ and $x_2$ are distinct points. What happens when $x_1 = x_2$? For each point $x \in X$, a vector $v \neq 0 \in T_xX$ defines an ideal $\mathscr{J} \subset \mathscr{O}_X$ given by*

$$\mathscr{J} = \{f \in \mathscr{O}_X \mid f(x) = 0,\ df_x(v) = 0\}.$$

*Then $\mathscr{J}$ has colength 2, and hence $\mathscr{O}_X/\mathscr{J}$ defines a zero-dimensional subscheme $Z \in X^{[2]}$. Actually, this gives a complet description of $X^{[2]}$. Namely,*

$$X^{[2]} = Blow_\Delta(X \times X)/\mathfrak{S}_2,$$

*where $Blow_\Delta(X \times X)$ is the blowup of $X \times X$ along the diagonal $\Delta$.*

**Theorem 2.15.** *Suppose $X$ is a nonsingular projective surface. Then the Hilbert-Chow map*

$$\pi : X^{[n]} \to X^n/\mathfrak{S}_n$$

*sending an element in $X^{[n]}$ to its support in $X^n/\mathfrak{S}_n$ is a resolution of singularities. In fact, the pair $(X^{[n]}, \pi)$ is a crepant resolution of the symmetric product.*

See details in [9].

Let $X$ be a simply-connected smooth projective surface, and $X^{[n]}$ be the Hilbert scheme of points in $X$. For a subset $Y \subset X$, we define the subset

$$M_n(Y) = \{\xi \in X^{[n]} \mid |\xi| \text{is a point in Y}\} \subset X^{[n]}.$$

By the results in [9] about the homology groups of the Hilbert scheme, we have that the space

$$\mathbb{H} = \bigoplus_{n=0}^{\infty}\bigoplus_{k=0}^{\infty} H_k(X^{[n]}; \mathbb{C})$$

is an irreducible highest weight representation of the Heisenberg algebra generated by elements $\mathfrak{a}_{-n}(\alpha')$, $n \in \mathbb{N}$, $\alpha' \in H_*(X; \mathbb{C})$. Moreover, $|0\rangle = 1 \in H_0(X^{[0]}, \mathbb{C})$ is a highest weight vector and the space $\mathbb{H}$ is a linear span of elements of the form $\mathfrak{a}_{-n_1}(\alpha'_1)...\mathfrak{a}_{-n_k}(\alpha'_k)|0\rangle$ where $k \geq 0$, $n_1, ..., n_k > 0$, and $\alpha'_1, ..., \alpha'_k \in H_*(X; \mathbb{C})$. The geometric interpretation of $\mathfrak{a}_{-n_1}(\alpha'_1)...\mathfrak{a}_{-n_k}(\alpha'_k)|0\rangle$ for homogeneous classes $\alpha'_1, ..., \alpha'_k \in H_*(X; \mathbb{C})$ can be understood as follows. For $i = 1, ..., k$, let $\alpha'_i \in H_{|\alpha'_i|}(X; \mathbb{C})$ and $A_i$ a submanifold of $X$ which represent the class $\alpha'_i$, such that $A_1, ..., A_k$ are in general position. Then,

$$\mathfrak{a}_{-n_1}(\alpha'_1)...\mathfrak{a}_{-n_k}(\alpha'_k)|0\rangle \in H_m(X^{[n]}; \mathbb{C})$$

where $n = \sum_{i=1}^{k} n_i$ and $m = \sum_{i=1}^{k}(2n_i - 2 + |\alpha'_i|)$. In addition, up to scalar the element $\mathfrak{a}_{-n_1}(\alpha'_1)...\mathfrak{a}_{-n_k}(\alpha'_k)|0\rangle$ is represented by the closure of the real-$m$-dimensional subset

$$\{\xi_1 + ... + \xi_k \in X^{[n]} \mid \xi_i \in M_{n_i}(A_i),\ |\xi_i| \cap |\xi_j| = \ for\ i \neq j\}.$$

**Definition 2.16.** *Let $x \in X$, and $C$, $\bar{C}$ be real-2-dimensional submanifolds of $X$. Then, we define the following homology classes*

$$\begin{aligned}
\beta'_C &= \mathfrak{a}_{-1}(C)\mathfrak{a}_{-1}(x)^{n-1}|0\rangle & \beta'_n &= \mathfrak{a}_{-2}(x)\mathfrak{a}_{-1}^{n-2}|0\rangle \\
\mathfrak{s}_{n,1} &= \mathfrak{a}_{-1}(X)\mathfrak{a}_{-1}(x)^{n-1}|0\rangle & \mathfrak{s}_{n,2} &= \mathfrak{a}_{-2}(x)\mathfrak{a}_{-2}(x)\mathfrak{a}_{-1}(x)^{n-4}|0\rangle \\
\mathfrak{s}_{n,3} &= \mathfrak{a}_{-3}(x)\mathfrak{a}_{-1}(x)^{n-3}|0\rangle & \mathfrak{s}_{C,1} &= \mathfrak{a}_{-1}(C)\mathfrak{a}_{-2}(x)\mathfrak{a}_{-1}(x)^{n-3}|0\rangle \\
\mathfrak{s}_{C,2} &= \mathfrak{a}_{-2}(C)\mathfrak{a}_{-1}(x)^{n-2}|0\rangle & \mathfrak{s}_{C,\bar{C}} &= \mathfrak{a}_{-1}(C)\mathfrak{a}_{-1}(\widetilde{C})\mathfrak{a}_{-1}(x)^{n-2}|0\rangle.
\end{aligned}$$

**Lemma 2.17.** *Assume that $n \geq 2$ and that $X$ is a simply-connected complex surface. Let $\{\alpha'_1, ..., \alpha'_s\}$ be a basis of $H_2(X;\mathbb{C})$ represented by real surfaces $\{C_1, ..., C_s\}$ respectively. Then,*

*(i) a basis of $H_2(X^{[n]};\mathbb{C})$ consists of the homology classes $\beta'_n, \beta'_{C_1}, ..., \beta'_{C_s}$;*

*(ii) a basis of $H_4(X^{[n]};\mathbb{C})$ consists of the homology classes $\mathfrak{s}_{n,1}, \mathfrak{s}_{n,2}, \mathfrak{s}_{n,3}, \mathfrak{s}_{C_i,1}$ $(i = 1, ..., s)$, $\mathfrak{s}_{C_i,2}$ $(i = 1, ..., s)$, and $\mathfrak{s}_{C_i,C_j}$ $(i, j = 1, ..., s)$.*

*Proof.* We only prove $(i)$, since a similar argument works for $(ii)$. Fix a point $x \in X$. Expand the basis $\{\alpha'_0 = x, \alpha'_1, ..., \alpha'_s, \alpha'_{s+1} = X\}$ of $H_*(X;\mathbb{C}) = H_0(X;\mathbb{C}) \oplus H_2(X;\mathbb{C}) \oplus H_4(X;\mathbb{C})$. A basis of $H_2(X_{[n]};\mathbb{C})$ consist of

$$\mathfrak{a}_{-n_1}(\alpha'_{m_1})...\mathfrak{a}_{-n_k}(\alpha'_{m_k})|0\rangle$$

satisfying $n_i \geq 1$, $\sum_{i=1}^{k} n_i = n$, and $\sum_{i=1}^{k}(2n_i - 2 + |\alpha'_{m_i}|) = 2$. Since $X$ is simply-connected $|\alpha'_{m_i}| \in \{0, 2, 4\}$. There is two posibilities, $|\alpha'_{m_i}| = 0$ for all $i$ and then $k = n - 1$, and the generator is $\mathfrak{a}_{-2}(x)\mathfrak{a}_{-1}(x)^{n-2}|0\rangle = \beta'_n$, or $|\alpha'_{m_i}| = 2$ for some $i$ and then $n = k$ and the generator is $\mathfrak{a}_{-1}(C_i)\mathfrak{a}_{-1}(x)^{n-1}|0\rangle = \beta'_{C_i}$. □

**Example 2.18.** *Let $M$ be a simply-connected complex surface, $\{h_i\}$ a basis for $H^2(M;\mathbb{C})$ and $\{C_i\}$ complex curves in $M$ representing their Poincaré duals, and $H$ the Poncaré dual to a point. Consider the Hilbert scheme $M^{[2]}$, $E = \pi^{-1}(\Delta)$ the exceptional divisor of the Hilbert-Chow map $M^{[2]} \xrightarrow{\pi} M^{(2)}$ and $C$ the fiber over a point in the diagonal $\Delta \subset M^2$. A set of generators for $H_2(M^{[2]})$ are (by lemma 2.17) the complex curves $\beta'_{C_i}$ and $\beta'_2$, where $\beta'_2$ corresponds to the closure of the set*

$$\{\xi \in M^{[2]} \mid |\xi| = x \in M\}.$$

*This is the set of points in $X^{[2]}$, which are maped in a fixed point on the diagonal, i.e. $\beta_2'$ corresponds to the fiber $C$. In the same way $\beta_{C_i}'$ corresponds to the closure of the set*

$$\{\xi_1+\xi_2 \in M^{[2]} \mid |\xi_1| \in C_i,\ |\xi_2| = x \neq |\xi_1|\}$$

*which can be identified with the Poincaré dual to the pullback $\pi^*(h_i \otimes H + H \otimes h_i)$.*

*A set of generators for $H_4(M^{[2]};\mathbb{C})$ are the complex surfaces $\mathfrak{s}_{n,1}$, $\mathfrak{s}_{C_i,2}$ and $\mathfrak{s}_{C_i,C_j}$ corresponding to the closures of the sets*

$$\{\xi_1+\xi_2 \in M^{[2]} \mid |\xi_1| \in X,\ |\xi_1| \neq |\xi_2|\} \qquad \{\xi \in M^{[2]} \mid |\xi| \in C_i\}$$

$$\{\xi_1+\xi_2 \in M^{[2]} \mid |\xi_1| \in C_i,\ |\xi_2| \in C_j,\ \textit{and}\ |\xi_1| \neq |\xi_2|\}$$

*respectively. Hence, $\mathfrak{s}_{n,1}$ can be identified with the Poincaré dual to the pullback $\pi^*(1 \otimes H + H \otimes 1)$; $\mathfrak{s}_{C_i,2}$ can be identified with the Poincaré dual to $p^{-1}(C_i)$, where $p : E \to M$ is the projection; and $\mathfrak{s}_{C_i,C_j}$ can be identified with the Poincaré dual to the pullback $\pi^*(h_i \otimes h_j + h_j \otimes h_i)$.*

*In this case, we can describe the generators for $H_6(M^{[2]};\mathbb{C})$. In the same way which we did in the proof of the lemma 2.17, we can prove that the generators of $H_6(M^{[2]};\mathbb{C})$ are the class $\mathfrak{a}_{-2}(M)|0\rangle$ corresponds to the closure of the set*

$$\{\xi \in M^{[2]} \mid |\xi| \in M\}$$

*which can be identified with the exceptional divisor $E = \pi^{-1}(\Delta)$, and the classes $\mathfrak{a}_{-1}(C_i)\mathfrak{a}_{-1}(M)|0\rangle$ that correspond to the closure of the sets*

$$\{\xi_1+\xi_2 \in M^{[2]} \mid |\xi_1| \in C_i,\ |\xi_2| \notin C_i\}$$

*which can be identified with the Poincaré dual to the pull-back $\pi^*(h_i \otimes 1 + 1 \otimes h_i)$.*

## 3 A geometric approach

In this section we present a geometric motivation to study the Crepant resolution conjecture of Ruan. Let $M$ be a symply-connected complex smooth surface and consider the Hilbert scheme $M^{[2]}$. As we showed in the previous section, this can be identifyied with the quotient under $\mathfrak{S}_2$ on the blow-up of $M^2$ along the diagonal $\Delta \subset M^2$. Moreover, the homology of $M^{[2]}$ is generated by the preimages of the generators of the homology of $M^2$ by the Hilbert-Chow map $M^{[2]} \xrightarrow{\pi} M^{(2)}$ and the generators corresponding to the exceptional divisor $E = \pi^{-1}(\Delta)$ and the fiber $C = \pi^{-1}(x,x)$. Since the Hilbert-Chow map is surjective and an isomorphism away of $E$, all the intersection products, except the corresponding to $E$, are the same as those in $M^{(2)}$. The natural question is: How different is the product in $M^{[2]}$ with respect to the product in $M^{(2)}$?

To answer the question above, we have to identify the directions on which, the homology class corresponding to $E$ should be perturbed when performing the transversal intersection. To see this, we need to characterize the normal bundle to $E$.

We start with a definition. For $E \to X$ a complex vector bundle of rank $r$ and $\mathbb{P}(E) \xrightarrow{\pi} X$ its associated projective bundle, we define the tautological line bundle $T \to \mathbb{P}(E)$ to be the subbundle of the pullback bundle $\pi^* E \to \mathbb{P}(E)$ whose fiber at any point $(p, v) \in \mathbb{P}(E)$ is the line in $E_p$ represented by $v$. Now, consider the blow-up $\pi : \widetilde{\mathbb{C}^n_V} \to \mathbb{C}^n$ of $\mathbb{C}^n$ along the subspace $V \cong \mathbb{C}^k$. Let $(p, v) \in E$, where $E = \pi^{-1}(V)$ denote the exceptional divisor. Then the map

$$\pi_* : T_{(p,v)} \widetilde{\mathbb{C}^n_V} \to T_p \mathbb{C}^n$$

between the tangent spaces, induces a map

$$\bar{\pi}_* : N_{E/\mathbb{C}^n,(p,v)} \to N_{V/\mathbb{C}^n,p}$$

between the normal spaces. Moreover, the image of $\bar{\pi}_*$ is just the line $v$ in $N_{V/\mathbb{C}^n,p}$. So the fiber over $(p, v) \in E$ of the normal vector bundle is just the line which $v$ represents. The same argument works for the blow-up $\pi : \tilde{M} \to M$ of $M$ along the submanifold $X$. Then, the normal bundle to $E$ in $\tilde{M}$ is just the tautological bundle $T$ on $E \cong \mathbb{P}(N_{X/M})$. Now, if $\alpha'$ is the Poincaré dual to $E$, we have that

$$\int_\sigma \alpha'|_E = \int_{\tilde{M}} PD_{\tilde{M}}(\sigma)\alpha' = \int_E PD_{\tilde{M}}(\sigma)|_E$$

for any cycle $\sigma \subset E$ with Poincaré dual $PD_{\tilde{M}}(\sigma)$. We know that $E$ has a neighborhood diffeomorphic to the divisor $[T]$ which represents the normal vector bundle $T$. Moreover, the Poincaré dual to $[T]$ is precisely the first Chern class $c_1(T)$ of $T$ (see [4], chapter 1,141). Then, for all $\sigma \subset E$

$$\int_\sigma c_1(T)|_\sigma = \int_{\tilde{M}} PD(\sigma)c_1(T) = \int_E PD(\sigma)$$

and therefore

$$\alpha'|_E = c_1(T). \tag{3}$$

In the other hand, for $K \to X$ a complex vector bundle of rank $r$ and $\mathbb{P}(K) \xrightarrow{\pi} X$ its associated projective bundle. Let $S$ be the quotient of the pullback $\pi^* K$ by the tautological line bundle $T$. Therefore the following sequence of vector bundles over $\mathbb{P}(K)$ is exact

$$0 \to T \to \pi^*(K) \to S \to 0,$$

we have by the multiplicative property of Chern classes that

$$c(T)c(S) = c(\pi^*(K)) = \pi^*(c(K)).$$

Set $\eta_i = c_i(S)$ and $\xi = c_1(T)$. Then,

$$(1 + \xi)(1 + \eta_1 + ... + \eta_{r-1}) = \pi^*(c(K))$$

and solving succesively, we have

$$\begin{aligned}
\eta_1 &= \pi^*(c_1(K)) - \xi \\
\eta_2 &= \pi^*(c_2(K)) - \xi\pi^*(c_1(K)) + \xi^2 \\
&\vdots \\
\eta_{r-1} &= \pi^*(c_{r-1}(K)) - \xi\pi^*(c_{r-2}(K)) + ... + (-1)^{r-1}\xi^{r-1} \\
\pi^*(c_r(K)) &= \xi\eta_{r-1}.
\end{aligned}$$

and finally

$$\xi^r - \pi^*(c_1(K))\xi^{r-1} + ... + (-1)^{r-1}\pi^*(c_{r-1}(K))\xi + (-1)^r\pi^*(c_r(K)) = 0.$$

If $\xi$ denotes again the first Chern class of the tautological line bundle $T$ over the exceptional divisor $E$, it is clear from the last observation by taking $K = N_{\Delta(M)/M^2}$ that

$$\xi^2 = \pi^*(c_1(N_{\Delta(M)/M^2}))\xi + \pi^*(c_2(N_{\Delta(M)/M^2})) \tag{4}$$

$$\xi^2 = \pi^*(c_1(M))\xi + eu(M) \tag{5}$$

where $eu(M)$ is the Euler class of $M$.

Since the directions on which we have perturbed the class of $E$ relates to curves on the normal bundle to $E$, and this can be identified with the tautological line bundle over $E$, we can see (of the last equation) that if $M$ has trivial canonical divisor ($c_1(M) = 0$), then the ordinary cohomology of $M^{[2]}$ is equal to the Chen-Ruan cohomology of $M^{(2)}$, because the square of $\xi$ becomes the dual of the diagonal with itself. This result is true in general for the Hilbert-Chow map $M^{[n]} \xrightarrow{\pi} M^{(n)}$ and it was shown by Uribe in [12] and Fantechi-Göttsche in [3].

Now, if we want to change the product in $H^*(M^{[2]};\mathbb{C})$ so that it matches the Chen-Ruan cohomology ring $H^*_{CR}(M^{(2)};\mathbb{C})$ , we have to remove the directions on which the class of $E$ was perturbed. This suggests the introduction of the Gromov-Witten invariants.

### 3.3 The Gromov-Witten invariants

The following is a brief introduction to Gromov-Witten invariants. More details are in [8].

Let $M$ be a $n$-dimensional complex manifold and for any $A \in H_2(M)$, $\mathscr{M}^*_{0,k}(A)$ the set of equivalence clases of tuples $(u, z_1, ..., z_k)$, where $u : S^2 \to M$ is a simple holomorphic sphere representing the class $A$ and the $z_i$ are the pairwise distinct points on $S^2$. The equivalence relation is given by the obvious actions of the reparametrization group. This space is a smooth oriented manifold of dimension

$$\mu(A, k) = 2n + 2c_1(A) + 2k - 6.$$

Now suppose that each class $a_i \in H^*(M)$ is dual to a an oriented submanifold $X_i \subset M$ in general position. Then the Gromov-Witten invariant $GW^M_{A,k}(a_1, ..., a_k)$ is the number of holomorphic spheres in the class $A$ passing through the submanifolds $X_i$ and counted with appropiate sings

$$GW^M_{A,k}(a_1, ..., a_k) = \#\{[u, z_1, ..., z_k] \in \mathscr{M}^*_{0,k}(A) \mid u(z_i) \in X_i\}$$

if

$$\sum_{i=1}^{k} \deg(a_i) = \mu(A, k).$$

In all other cases the invariant is zero by definition.

### 3.4 The conjecture

Let $\pi : Y \to X$ be a crepant resolution of a orbifold $X$. Then the homology classes of rational curves $\pi$-contracted are generated by so called extremal rays. For example, the Hilbert-Chow map $\pi : M^{[n]} \to M^{(n)}$ satisfies that the extremal rays are generated by the rational curve $\beta'_n$ (see [7]). In general, if $\pi$ is non-degenerate (i.e. there exist a finite integral base of extremal rays linearly indepent), then the homology class of any effective curve $\pi$-contracted can be written as $A = \sum_i a_i A_i$ for $a_i \geq 0$. For each $A_i$, we assign a formal variable $q_i$, such that $A$ corresponds to $q_1^{a_1}...q_k^{a_k}$. In [10], Ruan defines a 3-point function

$$\left\langle \alpha', \beta', \gamma' \right\rangle_{qc} (q_1, ..., q_k) = \sum_{a_1,...,a_k} GW^M_A(\alpha', \beta', \gamma') q_1^{a_1}...q_k^{a_k}$$

We view $\langle \alpha', \beta', \gamma' \rangle_{qc} (q_1, ..., q_k)$ as an analytic function of $q_1, ..., q_k$ and set $q_i = -1$, and let

$$\left\langle \alpha', \beta', \gamma' \right\rangle_{qc} = \left\langle \alpha', \beta', \gamma' \right\rangle_{qc} (-1, ..., -1).$$

In this way, Ruan defines the quantum corrected triple intersection by the formula

$$\left\langle \alpha', \beta', \gamma' \right\rangle_{\pi} = \left\langle \alpha', \beta', \gamma' \right\rangle + \left\langle \alpha', \beta', \gamma' \right\rangle_{qc}$$

where $\langle \alpha', \beta', \gamma' \rangle = \int_Y \alpha' \cup \beta' \cup \gamma'$ is the ordinary triple intersection. Then the quantum corrected cup product $\alpha' \cup_\pi \beta'$ is given by the equation

$$\left\langle \alpha' \cup_\pi \beta', \gamma' \right\rangle = \left\langle \alpha', \beta', \gamma' \right\rangle_{\pi}$$

for arbitrary $\gamma'$. We denote the new quantum corrected cohomology ring as $H^*_\pi(Y, \mathbb{C})$.

Now we can state the

**Conjecture 3.1** (Cohomological Crepant Resolution Conjecture [2])**.** . *Suppose that $\pi$ is non-degenerate and hence $H^*_\pi(Y, \mathbb{C})$ is well-defined. Then, $H^*_\pi(Y, \mathbb{C})$ is isomorphic to orbifold cohomology ring $H^*_{CR}(X, \mathbb{C})$.*

### 3.5 The case of two copies

From now on, we will focus on verifying the Cohomological Crepant Resolution Conjecture for the case of the symmetric product $X = M^{(2)}$, where $M$ is a simply-connected complex surface.

We start by calculating the orbifold cohomology $H^*_{CR}(X, \mathbb{C})$. Since $M$ is simply connected, $H_1(M, \mathbb{C}) = H_3(M, \mathbb{C}) = 0$. Let $h_i \in H^2(M, \mathbb{C})$ be element of a basis and $H \in H^4(M, \mathbb{C})$ the Poincaré dual to a point. Then by the Kunneth formula, the cohomology of the nontwisted sectors are generated by $1, 1 \otimes h_i + h_i \otimes 1, 1 \otimes H + H \otimes 1, h_i \otimes h_j + h_j \otimes h_i, h_i \otimes H + H \otimes h_i, H \otimes H$. The twisted sector $M^{(1,2)} = \Delta(M) \subset M \times M$ is diffeomorphic to $M$ with degree shifting number 1 and generators $\bar{1}, \bar{h_i}, \bar{H}$ with degrees 2, 4 and 6. By definition of the product,

$$\begin{aligned} \langle \textit{twisted sector}, \textit{nontwisted sector}, \textit{nontwisted sector} \rangle &= 0 \\ \langle \textit{twisted sector}, \textit{twisted sector}, \textit{twisted sector} \rangle &= 0. \end{aligned}$$

Since the obstruction bundle in this case has range 0 (see equations 1 and 2), the nonzero triple intersections are:

$$\begin{aligned} \langle \bar{1}, \bar{1}, 1 \otimes H + H \otimes 1 \rangle &= \int_M 1 \cup 1 \cup \Delta^*(1 \otimes H + H \otimes 1) = \int_M 2H = 2 \\ \langle \bar{1}, \bar{1}, h_i \otimes h_j + h_j \otimes h_i \rangle &= \int_M 1 \cup 1 \cup \Delta^*(h_i \otimes h_j + h_j \otimes h_i) = \int_M 2h_i \cup h_j \\ &= 2 \langle h_i, h_j \rangle \\ \left\langle \bar{1}, \bar{h_i}, 1 \otimes h_j + h_j \otimes 1 \right\rangle &= \int_M 1 \cup h_i \cup \Delta^*(1 \otimes h_j + h_j \otimes 1) = \int_M 2h_i \cup h_j \\ &= 2 \langle h_i, h_j \rangle . \end{aligned}$$

Next, we will calculate the quantum corrected triple intersection of the Hilbert scheme $Y = M^{[2]}$. Remember that $Y$ is the quotient of the blow-up of $M^2$ along the diagonal $\Delta(M)$ by the extended action of $\mathfrak{S}_2$. As it is shown in the example 2.18 the generators of the cohomology of $Y$ are the pull-backs of the nontwisted sectors by the Hilbert-Chow map $\pi : Y \to X$ and the classes $\alpha', \tilde{h_i}, \tilde{H}$ correspond to the Poincaré dual to the exceptional divisor $E = \pi^{-1}(\Delta)$, $p^{-1}(PD(h_i))$ and the fiber $[C]$, where $p : E \to M$ is the projection.

Then the triple intersection products are

$$\begin{aligned} \left\langle \alpha', \alpha', \tilde{h_i} \right\rangle &= \int_Y \alpha' \alpha' \tilde{h_i} \\ &= \int_{p^{-1}(PD(h_i))} (\alpha'|_E)^2 \quad \text{by definition of the Poincaré dual} \\ &= \int_{p^{-1}(PD(h_i))} \xi^2 \quad \text{by 3} \\ &= \int_{p^{-1}(PD(h_i))} \pi^*(c_1(N_{\Delta(M)/M^2}))\xi + \pi^*(c_2(N_{\Delta(M)/M^2})) \text{ by 4.} \end{aligned}$$

But, since the Poincaré dual to $\frac{1}{\chi(M)}\pi^*(c_2(N_{\Delta(M)/M^2})) = \frac{1}{\chi(M)}\pi^*(eu(M))$ corresponds to the fiber $[C]$, it is possible to perturb the class $p^{-1}(PD(h_i))$ such that

$$\int_{p^{-1}(PD(h_i))} \pi^*(c_2(N_{\Delta(M)/M^2})) = 0.$$

Then

$$\begin{aligned}\left\langle \alpha', \alpha', \tilde{h}_i \right\rangle &= \int_{p^{-1}(PD(h_i))} \pi^*(c_1(T_{\Delta(M)}))\xi = \xi(C)\int_{PD(h_i)} c_1(M) \\ &= -\left\langle c_1(M), h_i\right\rangle . \end{aligned} \tag{6}$$

This is the main difference with the Chen-Ruan cohomology. The other triple intersection are the same as in the Chen-Ruan cohomology. Namely

$$\begin{aligned}
\left\langle \alpha', \alpha', 1\otimes H + H\otimes 1\right\rangle &= \int_Y \alpha'\alpha'(1\otimes H + H\otimes 1) \\
&= \int_E \xi(1\otimes H + H\otimes 1) \quad \text{by Poincaré duality} \\
&= \xi(C)\int_M 2H = -2 \\
\left\langle \alpha', \alpha', h_i\otimes h_j + h_j\otimes h_i\right\rangle &= \int_Y \alpha'\alpha'(h_i\otimes h_j + h_j\otimes h_i) \\
&= \int_E \xi(h_i\otimes h_j + h_j\otimes h_i) \quad \text{by PD and 3} \\
&= \xi(C)\int_M 2h_ih_j = -2\left\langle h_i, h_j\right\rangle \\
\left\langle \alpha', \tilde{h}_i, 1\otimes h_j + h_j\otimes 1\right\rangle &= \int_{p^{-1}(PD(h_i))} \xi(1\otimes h_j + h_j\otimes 1)|_{p^{-1}(PD(h_i))} \quad \text{by PD} \\
&= \xi(C)\int_{PD(h_i)} (2h_j) = -2\left\langle h_i, h_j\right\rangle
\end{aligned}$$

The rest are zero.

For the quantum corrections we use the results obtained by Li-Qin about the 1-point Gromov-Witten invariants. More precisely, we have the following proposition

**Proposition 3.2** ([6]). *Let $d \geq 1$ be an integer, and $C_1$, $C_2$ be smooth real surfaces in $M$.*

1. *If $\alpha'$ is the Poincaré dual of $\mathfrak{s}_{n,1}$, $\mathfrak{s}_{C_1,C_2}$, or $\mathfrak{s}_{C_1,1}$ the 1-point Gromov-Witten invariants $GW_{0,d\beta'_n}(\alpha')$ are zero.*
2. *If $\alpha'$ is the Poincaré dual of $\mathfrak{s}_{C_1,2}$, then $GW_{0,d\beta'_n}(\alpha') = -2\left\langle c_1(M), C_1\right\rangle / d^2$.*

The relation between the 1-point Gromov-Witten invariants and the 3-point function of Ruan can be seen in the following proposition:

**Proposition 3.3** ([8]). *Let $M$ be a compact complex manifold, $A \in H_2(M;\mathbb{Z})$, $k \geq 1$. If $(A,k) \neq (0,0)$ and deg$(a_k)=2$, then*

$$GW^M_{A,k}(a_1,...,a_k) = GW^M_{A,k-1}(a_1,...,a_{k-1}) \int_A a_k.$$

In this way, the only nonzero terms are

$$\begin{aligned}\left\langle \alpha', \alpha', \tilde{h_i} \right\rangle_{qc}(q) &= \sum_d GW^Y_{d[C]}(\alpha',\alpha',\tilde{h_i})q^d = \sum_d \alpha'(d[C])\alpha'(d[C])GW_{0,d[C]}\langle \tilde{h_i}\rangle q^d \\ &= \sum_d d^2 \frac{(-2\langle c_1(M)), h_i\rangle)}{d^2} q^d = -2\langle c_1(M), h_i\rangle \frac{q}{1-q}.\end{aligned}$$

Therefore

$$\left\langle \alpha', \alpha', \tilde{h_i} \right\rangle_{qc} = \langle c_1(M), h_i\rangle$$

cancels with $\left\langle \alpha', \alpha', \tilde{h_i} \right\rangle$. Hence, the triple intersection products in $H^*_{CR}(M^{(2)};\mathbb{C})$ and $H^*_\pi(M^{[2]},\mathbb{C})$ are equal, and this rings are isomorphic.

Actually, we have shown the following theorem, which appeared first in Ruan's article.

**Theorem 3.4.** *Let $M$ be a simply-connected complex surface, then the quantum corrected cohomology ring of the Hilbert scheme of two points on $M$ is isomorphic to the Chen-Ruan cohomology ring of the two-fold symmetric product of $M$.*

### 3.6 An example

To finish, we will calculate these rings for $M = \mathbb{C}P^2$ the complex projective space. Note that we only need to know the ordinary cohomology ring of $M$, in this case

$$H^*(\Delta;\mathbb{C}) = H^*(\mathbb{C}P^2;\mathbb{C}) = \mathbb{C}[\delta]/\langle\delta^3\rangle,$$

and

$$H^*(\mathbb{C}P^2 \times \mathbb{C}P^2;\mathbb{C}) = \mathbb{C}[x,y]/\langle x^3, y^3\rangle.$$

Let $\epsilon \in H^0(\Delta;\mathbb{C})$ be the unity. The $\mathfrak{S}_2$-invariant part of $H^*(\mathbb{C}P^2 \times \mathbb{C}P^2;\mathfrak{S}_2)$ is generated by $\alpha' = x+y$ and $\epsilon$ in degree 2, $\beta' = xy$, $x^2+y^2$ and $\delta$ in degree 4, and $\delta^2$, $x^2y+y^2x$ in degree 6. It is easy to verify that

$$\begin{aligned}\epsilon\epsilon &= x^2+xy+y^2 \qquad & \epsilon\beta' &= \delta^2, \qquad & \epsilon\alpha' &= 2\delta, \qquad & \alpha'^3 &= 3\alpha'\beta', \\ &= \alpha'^2 - \beta', & \beta'^3 &= 0, & \alpha'^5 &= 0.\end{aligned}$$

Actually, the Chen-Ruan cohomology ring of the symmetric product is the graded polynomial algebra

$$H^*_{CR}((\mathbb{C}P^2)^{(2)};\mathbb{C}) = \mathbb{C}[\alpha', \epsilon, \beta']/\langle \alpha'^3 - 3\alpha'\beta', \beta'^3, \epsilon^2 - \alpha'^2 + \beta', \alpha'^5\rangle.$$

Now, we will calculate the ring $H^*((\mathbb{C}P^2)^{[2]};\mathbb{C})$. We know from the example 2.18 that a set of generators of $H^*((\mathbb{C}P^2)^{[2]};\mathbb{C})$ as a $\mathbb{C}-$module are the elements

$$\begin{array}{rcl} \tilde{\alpha'} & = & \pi^*(x+y) = \pi^*(\alpha'), \\ \tilde{\beta'} & = & \pi^*(xy) = \pi^*(\beta'), \\ \tilde{\epsilon} & = & PD(E), \text{ the Poincaré dual to exceptional divisor } E, \\ \tilde{\delta} & & \text{the Poincaré dual to } \pi^{-1}(PD_\delta(\delta)), \text{ and} \\ C & = & PD(\pi^{-1}(x,x)) \text{ corresponding to the pull-back } \pi^*(\delta^2). \end{array}$$

It is clear that $\tilde{\alpha'}^3 = 3\tilde{\alpha'}\tilde{\beta'}$, $\tilde{\beta'}^3 = 0$ and $\tilde{\alpha'}^5 = 0$ by the properties of the pull-back. It is also easy to verify that

$$\tilde{\epsilon}\tilde{\alpha'} = -2\tilde{\delta}, \qquad \tilde{\epsilon}\tilde{\beta'} = -C,$$

and by 6 we have that

$$\tilde{\epsilon}^2 = -(\tilde{\alpha'}^2 - \tilde{\beta'}) - \tilde{\delta} = -(\tilde{\alpha'}^2 - \tilde{\beta'}) + \frac{\tilde{\epsilon}\tilde{\alpha'}}{2}.$$

The quantum corrections remove the term $\frac{\tilde{\epsilon}\tilde{\alpha'}}{2}$ of the last equality. Therefore, the ring $H^*_\pi((\mathbb{C}P^2)^{[2]};\mathbb{C})$ is the graded polynomial algebra

$$\mathbb{C}[\tilde{\alpha'}, \tilde{\epsilon}, \tilde{\beta'}]/\langle \tilde{\alpha'}^3 - 3\tilde{\alpha'}\tilde{\beta'}, \tilde{\beta'}^3, \tilde{\epsilon}^2 + \tilde{\alpha'}^2 - \tilde{\beta'}, \tilde{\alpha'}^5\rangle$$

and the function $\varphi : H^*_{CR}((\mathbb{C}P^2)^{(2)};\mathbb{C}) \to H^*_\pi((\mathbb{C}P^2)^{[2]};\mathbb{C})$ defined by

$$\varphi(\alpha') = i\tilde{\alpha'} \qquad \varphi(\beta') = -\tilde{\beta'} \qquad \varphi(\epsilon) = \tilde{\epsilon}$$

is clearly an isomorphism.

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