
The Bernstein-Gelfand-Gelfand correspondence

Author

Yuhad Olarte Ibrahim

Director

Juan Camilo Arias Uribe, Ph.D

Co-director

Luz Victoria de la Pava, Ph.D

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Mathematician



Universidad del Valle
Departamento de Matemáticas
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*A mis padres,
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Abstract

This work states and proves the Bernstein-Gelfand-Gelfand (BGG) correspondence, which describes an adjunction between the categories of complexes of graded modules over the exterior and symmetric algebras. To this end, the concepts of abelian category and category of complexes are introduced, along with the notions of quasi-isomorphism and homotopy. It is shown that the adjunction between the categories of complexes of graded modules over the exterior and symmetric algebras extends to an equivalence between their bounded derived categories.

Key words: Chain complexes, homotopy category, derived category, BGG correspondence.

Resumen

En este trabajo se enuncia y demuestra la correspondencia de Bernstein-Gelfand-Gelfand (BGG), que describe la adjunción entre las categorías de complejos de módulos graduados sobre el álgebra exterior y el álgebra simétrica. Para ello, se introducen los conceptos de categoría abeliana y categoría de complejos, así como las nociones de quasi-isomorfismo y homotopía. Se demuestra cómo la adjunción entre las categorías de complejos de módulos graduados sobre el álgebra exterior y simétrica se extiende a una equivalencia entre sus categorías derivadas acotadas

Palabras clave: Complejo de cadenas, categoría de homotopía, categoría derivada, correspondencia BGG.

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Introduction

In the early 1960s, A. Grothendieck introduced the concept of derived categories as a tool to formalize and generalize the Serre duality to algebraic varieties that admit arbitrary singularities. It was his student, J.L. Verdier, who, in his doctoral thesis "*Des Catégories Dérivées des Catégories Abéliennes*", developed the basic theory of derived categories, introducing the notion of triangulated categories and their localization theory, thus providing a formal construction of the derived category of an abelian category from its category of chain complexes. This new approach provides the correct theoretical framework for formalizing the idea of homology and derived functors, demonstrating its usefulness not only in algebraic geometry but also in representation theory.

J. Bernstein, I. Gelfand and S. Gelfand introduced in [3] what is known as the *Bernstein-Gelfand-Gelfand correspondence* or *Koszul duality*, as an intermediate step to compute the derived category of coherent sheaves over the n -dimensional projective space. The *BGG correspondence*, describes an adjunction between the category of complexes of graded modules over the exterior algebra and the category of complexes of graded modules over the symmetric algebra. This correspondence extends to an exact equivalence between the corresponding derived categories

$$\mathbf{L}: \mathbf{D}^b(E) \xrightleftharpoons{\quad} \mathbf{D}^b(S): \mathbf{R}.$$

The *BGG correspondence* has been observed in different contexts related to representation theory and algebraic geometry, and it becomes an important tool in representation theory, where it is used to get a deeper understanding of categories of representations for Lie algebras and algebraic groups.

In the present work, we will present a proof of the *BGG correspondence* and explain all the ingredients needed for this. For this purpose, we divided it into three chapters. The first chapter introduces abelian categories and their associated category of chain complexes. Regarding chain morphisms, we provide definitions of chain homotopy and quasi-isomorphism. We then define the homotopy category and describe its structure as a triangulated category, highlighting several of its fundamental properties. Finally, the derived category is constructed by formally inverting quasi-isomorphisms via the localization process, and its structure as a triangulated category is established. Moreover, we present the basics to establish well-defined functors between derived categories.

The second chapter establishes and proves the *BGG correspondence*. To this end, we introduce the

categories of graded modules over the exterior and symmetric algebra, and we present some of their basic properties. We define functors between its categories of complexes and show how to extend it to an equivalence between bounded derived categories.

In the third chapter is a sketch of how using the *BGG correspondence* we can characterize the derived category of coherent sheaves on the n -dimensional projective space as a certain category of modules over the exterior algebra.

We conclude with a brief appendix containing some definitions on the graded structure of modules and algebras.

Chapter 1

Derived categories

The main objective of this chapter is to introduce derived categories and the construction of their derived functors. To achieve this, we will start by discussing abelian categories and the category of complexes as a specific example of these, including the concepts of homology, chain homotopy, and quasi-isomorphisms. Next, we will define the homotopy category, which will enable us to construct the derived category through a process of localization. The derived category thus formed will have the structure of a triangulated category, which will be explored later. Additionally, we will introduce the concepts of projective and injective resolutions. These are crucial not only for defining derived functors but also for the calculations over these categories.

The construction of the derived category can be summarized in the following schematic steps:

$$\mathcal{A} \longrightarrow \mathbf{C}(\mathcal{A}) \longrightarrow \mathbf{K}(\mathcal{A}) \longrightarrow \mathbf{D}(\mathcal{A}).$$

- (1) Begin with an abelian category $\mathcal{A}$.
- (2) Construct the category of chain complexes $\mathbf{C}(\mathcal{A})$.
- (3) Define the homotopy category $\mathbf{K}(\mathcal{A})$ whose objects are the same as $\mathbf{C}(\mathcal{A})$ but morphisms are up to chain homotopy.
- (4) Construct the derived category $\mathbf{D}(\mathcal{A})$ by formally inverting the quasi-isomorphisms in $\mathbf{K}(\mathcal{A})$.

To keep the focus pragmatic and improve readability, some proofs will be referred or omitted. Detailed proofs will be given for those results that are directly relevant to the following chapters.

1.1 Abelian categories

Abelian categories represent the most general framework within which homological algebra can be formulated. The concept and term *abelian category* were initially introduced by MacLane [13], while the modern axiomatization and the first major applications were provided by Grothendieck in his renowned Tohoku paper [9]. For complete details about abelian categories and their properties (including the proofs) the reader may consult [17, Derived categories], [8, II] and [7].

We assume that the reader is familiar with the basics on category theory. The categories in this document are considered locally small. Functors will be assumed covariant unless stated otherwise.

Definition 1.1.1 (Preadditive category). A category $\mathcal{A}$ is called **preadditive** if each hom-set $\text{Hom}_{\mathcal{A}}(x, y)$ is endowed with the structure of an abelian group such that the compositions

$$\circ: \text{Hom}_{\mathcal{A}}(x, y) \times \text{Hom}_{\mathcal{A}}(y, z) \rightarrow \text{Hom}_{\mathcal{A}}(x, z),$$

are bilinear. A functor $F: \mathcal{A} \rightarrow \mathcal{B}$ of preadditive categories is called **additive** if the map $F: \text{Hom}_{\mathcal{A}}(x, y) \rightarrow \text{Hom}_{\mathcal{B}}(F(x), F(y))$ is a homomorphism of abelian groups for all $x, y \in \mathcal{A}$.

Let $\mathcal{A}$ be a category with a zero object. Given objects x and y in $\mathcal{A}$. Recall that the **zero morphism** between x and y is defined as the unique arrow $x \rightarrow y$ that factors through the zero object, meaning the arrow given by composition of the unique arrows $x \rightarrow 0 \rightarrow y$. This morphism is usually denoted by $x \xrightarrow{0} y$ if it does not lead to confusion. The composition of a zero morphism with any other morphism is again a zero morphism. It is easy to prove that for a category $\mathcal{A}$ with a zero object, the hom-set $\text{Hom}_{\mathcal{A}}(x, y)$ is non-empty as it contains the zero morphism as the zero element of the group. Additionally, the following property holds: $1_x = x \xrightarrow{0} x$ if and only if x is initial/terminal, that is, x is the zero object.

The following lemma is an important property of preadditive categories.

Lemma 1.1.2. Let $\mathcal{A}$ be a preadditive category. Let $x, y \in \mathcal{A}$. If the product $x \times y$ exists, then so does the coproduct $x \amalg y$. If the coproduct $x \amalg y$ exists, then so does the product $x \times y$. In this case, $x \amalg y \cong x \times y$.

Definition 1.1.3 (Direct sum). Let $\mathcal{A}$ be a preadditive category and let x, y in $\mathcal{A}$. The **direct sum** of x and y , denoted by $x \oplus y$, is the product/coproduct of x and y endowed with unique morphisms

$$x \xleftarrow[p]{i} x \oplus y \xrightarrow[q]{j} y \text{ such that } p \circ i = 1_x, q \circ j = 1_y, p \circ j = 0 \text{ and } i \circ p + j \circ q = 1_{x \oplus y}.$$

Lemma 1.1.4. Let $F: \mathcal{A} \rightarrow \mathcal{B}$ to be an additive functor between preadditive categories and let x, y be objects of $\mathcal{A}$. If $x \oplus y$ exists then $F(x \oplus y) = F(x) \oplus F(y)$. Moreover, $F(0) = 0$ as an object, and $F(0: x \rightarrow y) = 0: F(x) \rightarrow F(y)$ as a morphism.

Definition 1.1.5 (Additive category). A category $\mathcal{A}$ is called a **additive** if it is preadditive, has zero object and every finite product exists.

From now on we assume that the reader is familiar with the concepts of kernel, cokernel, image, and coimage of arbitrary maps altogether with their universal properties.

Lemma 1.1.6. Let $f: x \rightarrow y$ be a morphism in a preadditive category $\mathcal{A}$ such that the kernel, cokernel, image and coimage all exist. Then f can be factored uniquely as follows

$$\begin{array}{ccccccc} \ker f & \longrightarrow & x & \xrightarrow{f} & y & \longrightarrow & \text{coker } f \\ & & \downarrow & & \uparrow & & \\ & & \text{coim } f & \xrightarrow[h]{\quad} & \text{im } f & & \end{array}$$

where $\text{coim } f = \text{coker}(\ker f \rightarrow x)$ and $\text{im } f = \ker(y \rightarrow \text{coker } f)$.

Definition 1.1.7 (Abelian category). A category $\mathcal{A}$ is called **abelian** if it is additive, all kernels and cokernels exist, and the map $\text{coim } f \rightarrow \text{im } f$ is an isomorphism for all morphisms f of $\mathcal{A}$.

Lemma 1.1.8 (Characterization of additive functors). Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories and $F: \mathcal{A} \rightarrow \mathcal{B}$ be a functor. The following are equivalent

- (1) F is additive.
- (2) $F(x \oplus y) \rightarrow F(x) \oplus F(y)$ is an isomorphism for every $x, y \in \mathcal{A}$.

Following the Lemma 1.1.6 we know that every morphism in an abelian category $\mathcal{A}$ factors through an epimorphism and a monomorphism respectively, that is, any map f in an abelian category can be written as the composition

$$x \twoheadrightarrow \text{coim } f \cong \text{im } f \hookrightarrow y. \quad (1.1)$$

Before introducing the next definition, let us note the following. Consider $f: x \rightarrow y$ and $g: y \rightarrow z$ to be morphisms over an abelian category such that $g \circ f = 0$. The decomposition made in (1.1) of f , along with the condition $g \circ f = 0$, implies, by the universal property of the kernel of g , the existence of a unique morphism $\varphi: \text{im } f \rightarrow \ker g$. Moreover, the map φ is a monomorphism since the composition $\ker g \circ \varphi = \text{im } f$.

Definition 1.1.9 (Short exact sequences). Let $\mathcal{A}$ be an abelian category. A sequence of morphisms in $\mathcal{A}$

$$\cdots \longrightarrow x_{n-1} \xrightarrow{f_{i-1}} x_n \xrightarrow{f_i} x_{n+1} \longrightarrow \cdots,$$

is said to be **exact** if $f_i \circ f_{i-1} = 0$ and all the morphisms $\text{im } f_{i-1} \rightarrow \ker f_i$ are isomorphisms, in which case we write $\text{im } f_{i-1} = \ker f_i$. In particular, an exact sequence is called **short exact** if it is of the form: $0 \rightarrow x \rightarrow y \rightarrow z \rightarrow 0$.

Definition 1.1.10 (Exact functor). Let $F: \mathcal{A} \rightarrow \mathcal{B}$ be an additive functor between abelian categories. The functor F is said to be **left exact** if for every exact sequence $0 \rightarrow x \rightarrow y \rightarrow z$ in $\mathcal{A}$, the sequence $0 \rightarrow F(x) \rightarrow F(y) \rightarrow F(z)$ is exact in $\mathcal{B}$. Similarly, F is said to be **right exact** if for every exact sequence $x \rightarrow y \rightarrow z \rightarrow 0$ in $\mathcal{A}$, the sequence $F(x) \rightarrow F(y) \rightarrow F(z) \rightarrow 0$ is exact in $\mathcal{B}$. Finally, F is **exact** if it is both left and right exact.

Example 1.1.11. The following are examples of abelian categories.

- (1) The opposite category $\mathcal{A}^{op}$ of a abelian category $\mathcal{A}$ is abelian.
- (2) The category of abelian groups, **Ab**.
- (3) The category of k -vector spaces, **Vect**(k) over a field k .
- (4) The category of (left) R -modules, **R-Mod** over a commutative ring R .
- (5) The category of sheaves, **Sh**(X) over a topological space X .
- (6) The category of coherent sheaves, **Coh**(X) and quasi-coherent sheaves, **QCoh**(X) over a topological space X .

(7) The category of $\mathcal{O}_X$ -modules, $\mathcal{O}_X\text{-Mod}$ over a ringed space $(X, \mathcal{O}_X)$.

Example 1.1.12 (Example of exact functors). Some of the most well-known functors have their additive version in the context of abelian categories and satisfy a certain type of exactness. For example, if $\mathcal{A}$ is an abelian category and $x \in \mathcal{A}$, then the functors $\text{Hom}(x, -)$ and $\text{Hom}(-, x)$ are left exact. See [16, Theorem 2.40].

Similarly, in the category of R -modules $\mathbf{R-Mod}$, the functors $- \otimes M$ and $M \otimes -$ are right exact for any R -module M . See [16, Theorem 2.63].

Other example of a left exact functor is the global sections functor $\Gamma(X, -): \mathbf{Sh}(X) \rightarrow \mathbf{Ab}$ between the category of sheaves over a topological space X and the category of abelian groups. See [16, Lemma 6.68].

In some instances within the study of abelian categories, it suffices to state and prove results specifically within the context of categories of R -modules. Essentially, these results can be extended to any small abelian category by applying the *Freyd-Mitchell Theorem* stated below. The original proof of this theorem can be found in [7, Theorem 7.34].

Theorem 1.1.13 (Freyd-Mitchell theorem). Let $\mathcal{A}$ be a small abelian category. Then $\mathcal{A}$ is equivalent to a full abelian subcategory of $\mathbf{R-Mod}$ for a suitable ring R .

We finish this subsection stating the famous *Yoneda embedding*, which remains valid even in the setting of abelian categories, for this we set $\mathcal{A}$ an abelian category and by $[\mathcal{A}^{op}, \mathbf{Ab}]$ we will denote the functor category whose objects are functors $F: \mathcal{A}^{op} \rightarrow \mathbf{Ab}$ and maps are natural transformations.

Theorem 1.1.14 (Yoneda embedding). The Yoneda functor $\mathcal{A} \rightarrow [\mathcal{A}^{op}, \mathbf{Ab}]$ defined by $x \mapsto \text{Hom}_{\mathcal{A}}(-, x)$ is fully faithful.

Proof. See [19, Corollary 1.6.13]. □

1.1.1 The category of chain complexes. In this subsection we introduce one of the most important and basic tools in homological algebra which is the notion of chain complexes. In what follows, $\mathcal{A}$ will be an abelian category.

Definition 1.1.15 (Chain complexes). A **chain complex**, or just a **complex** $A^\bullet = (A^\bullet, d_A^\bullet)$ over $\mathcal{A}$ is a sequence of objects and morphisms in $\mathcal{A}$ of the form

$$\dots \longrightarrow A^{i-1} \xrightarrow{d_A^{i-1}} A^i \xrightarrow{d_A^i} A^{i+1} \longrightarrow \dots,$$

such that $d_A^i \circ d_A^{i-1} = 0$ for all i . The morphisms $d_A^i: A^i \rightarrow A^{i+1}$ are called **differentials** of $A^\bullet$ and we say that A^i corresponds to the i -th degree of the complex $A^\bullet$. If there is no confusion, we will denote the complex $(A^\bullet, d^\bullet)$ simply by A .

For chain complexes A, B in $\mathcal{A}$, a **morphism between chain complexes**, or just a **chain map** $f: A \rightarrow B$ corresponds to a sequence of maps $f^i: A^i \rightarrow B^i$ on $\mathcal{A}$ such that the following diagram commutes

$$\begin{array}{ccccccc} \cdots & \longrightarrow & A^i & \xrightarrow{d_A^i} & A^{i+1} & \longrightarrow & \cdots \\ & & \downarrow f^i & & \downarrow f^{i+1} & & \\ \cdots & \longrightarrow & B^i & \xrightarrow{d_B^i} & B^{i+1} & \longrightarrow & \cdots \end{array}$$

The chain complexes over $\mathcal{A}$ and the morphisms between them form a category denoted by $\mathbf{C}(\mathcal{A})$.

Definition 1.1.16 (Concentrated complex). Let X be an object in $\mathcal{A}$. We define a **complex concentrated in n degree**, denoted by $X[n]$ as the chain complex having X in the n -th degree and zero in all other degrees. We take all the differentials in this complex to be the zero morphism.

Following the previous definition note that given a map $f: X \rightarrow Y$ in $\mathcal{A}$, we can obtain a chain complex map, which for simplicity we will continue to denote by $f: X[n] \rightarrow Y[n]$ setting $f^i := 0$ and $f^n := f$ for $i \neq n$. In other words $f: X[n] \rightarrow Y[n]$ is the morphism:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & X & \longrightarrow & 0 \longrightarrow \cdots \\ & & \downarrow & & \downarrow f & & \downarrow \\ \cdots & \longrightarrow & 0 & \longrightarrow & Y & \longrightarrow & 0 \longrightarrow \cdots \end{array} \tag{1.2}$$

Lemma 1.1.17. Let $f: A \rightarrow B$ be a chain map. f is a monomorphism (resp. epimorphism) if and only if each component $f^i: A^i \rightarrow B^i$ in $\mathcal{A}$ is a monomorphism (resp. epimorphism).

Theorem 1.1.18. The category of chain complexes $\mathbf{C}(\mathcal{A})$ is abelian. Moreover, it contains $\mathcal{A}$ as a full subcategory.

Proof. See [11, Proposition 2.5, Page 9] for the abelian structure.

For the second statement, we define the inclusion functor as follows: $I: \mathcal{A} \rightarrow \mathbf{C}(\mathcal{A})$, $X \mapsto X[0]$ and I assigns maps $f: X \rightarrow Y$ in $\mathcal{A}$ into chain morphisms $f: X[0] \rightarrow Y[0]$ putting f in degree 0 as is made in diagram (1.2). The fully faithful part follows easily. $\square$

Definition 1.1.19 (Bounded complexes). Let A be a complex. We say that A is **bounded below** if $A^i = 0$ for all $i \ll n$. Similarly, A is said to be **bounded above** if $A^i = 0$ for all $i \gg n$. Finally, A is said to be **bounded** if it is bounded both above and below. Additionally, we define $\mathbf{C}^*(\mathcal{A})$ with $*$ = +, -, b as the full subcategory of complexes that are bounded below, bounded above, and bounded, respectively. All these subcategories are abelian.

Now, we introduce the concept of **homology of a chain complex**. To this end recall that given a chain complex A the condition $d_A^i \circ d_A^{i-1} = 0$ over the differentials implies for each i , the existence of a unique monomorphism $\text{im } d_A^{i-1} \rightarrow \ker d_A^i$, these maps will be referred as the **associated morphisms** of the complex A .

Definition 1.1.20 (Homology object). Let A be a chain complex. We define i -th homology object of A to be the (corresponding object of) the cokernel of the i -th associated morphisms of A . Explicitly

$$H^i(A) := \text{coker}(\text{im } d^{i-1} \rightarrow \ker d^i).$$

As a slogan we may say that homology measures how far the chain complex is from being exact.

The following lemma is a simple application of the universal property of kernel and the decomposition made in (1.1) over the differential maps.

Lemma 1.1.21. Let $f: A \rightarrow B$ be a morphism between chain complexes. For each i , f induces unique monomorphisms $Z(f)^i: \ker d_A^i \rightarrow \ker d_B^i$ and $B(f)^i: \text{im } d_A^i \rightarrow \text{im } d_B^i$.

Proof. We prove and describe the existence of the morphism $B(f)^i: \text{im } d_A^i \rightarrow \text{im } d_B^i$ the other follows in a similar way. Consider the decomposition of the differential d_A^i as in (1.1). This induces a commutative diagram

$$\begin{array}{ccccc} & & \text{im } d_A^i & & \\ & \text{epi} \nearrow & & \searrow \text{im } d_A^i & \\ A^i & \xrightarrow{d_A^i} & B^i & & \\ f^i \downarrow & & \downarrow f^{i+1} & & \\ B^i & \xrightarrow{d_B^i} & B^{i+1} & \longrightarrow & \text{coker } d_B^i, \end{array}$$

From the condition $\text{coker } d_B^i \circ d_B^i \circ f^i = 0$ and using the commutativity we get that $\text{coker } d_B^i \circ f^{i+1} \circ \text{im } d_A^i \circ \text{epi}$ is zero, this implies that $\text{coker } d_B^i \circ f^{i+1} \circ \text{im } d_A^i$ is also zero since epi is an epimorphism. By the universal property of $\text{im } d_B^i$ we get the existence of a unique monomorphism $\text{im } d_A^i \rightarrow \text{im } d_B^i$ which we assign to $B^i(f)$. $\square$

Remark 1.1.22 (Homology functor construction). Taking the i -th homology object defines an additive functor

$$H^i(-): \mathbf{C}(\mathcal{A}) \longrightarrow \mathcal{A}, \quad A \mapsto H^i(A),$$

called the i -th homology functor. For each morphism $f: A \rightarrow B$ between chain complexes, the functorial assignation $H^i(f)$ in $\mathcal{A}$ is defined as follows. Let α, β to be the associated morphisms of the complexes A and B , respectively. These maps, as well as the morphisms $Z^i(f)$, $B^i(f)$, obtained from the Lemma 1.1.21 fit into the following commutative diagram

$$\begin{array}{ccccc} \text{im } d_A^{i-1} & \xrightarrow{\alpha} & \ker d_A^i & \longrightarrow & H^i(A) \\ \downarrow B^i(f) & & \downarrow Z^i(f) & & \downarrow H^i(f) \\ \text{im } d_B^i & \xrightarrow{\beta} & \ker d_B^i & \longrightarrow & H^i(B). \end{array}$$

It is easy to check that the composition $\text{coker } \beta \circ Z^i(f) \circ \alpha$ is zero, implying, by the universal property of the cokernel of α , the existence of a unique morphism (dashed arrow above) $H^i(A) \rightarrow H^i(B)$ which we assign to $H^i(f)$.

Note that the $H^i(f)$ morphism turns out to be an epimorphism, as it is the morphism obtained from the universal property of a cokernel. Moreover, it is not hard to convince ourselves that $H^i(-)$ is actually an additive functor as all of the universal property constructions behave nicely with composition of chain maps and they are compatible with the Hom-set addition operation.

Remark 1.1.23 (Homology over the category of **R-Mod**). Following the Freyd-Mitchell Theorem 1.1.13 it suffices to prove some results about abelian categories within the category of R -modules. For this reason, it is useful to compute homology objects and the induced homology morphisms within this category. From now on, let us fix the notation $\mathbf{C}(\mathbf{R-Mod}) := \mathbf{C}(\mathbf{R})$ over the category of complexes for R -mod.

First recall that the cokernel of an R -module homomorphism $f: A \rightarrow B$ is given by the quotient module $\text{coker } f = B/\text{im } f$. If f is additionally an injective homomorphism, then the cokernel becomes $\text{coker } f = B/A$. Now, as for each complex $A \in \mathbf{C}(\mathbf{R})$ the associated morphisms correspond to the submodule inclusions $\text{im } d^{-1} \subseteq \ker d^i$ which are injective maps, then the homology objects results to be module quotients $H^i(A) = \ker d_A^i / \text{im } d_A^{i-1}$ and the i -th homology functor corresponds to

$$H^i(-): \mathbf{C}(\mathbf{R}) \longrightarrow \mathbf{R-Mod}, \quad A \mapsto H^i(A) = \frac{\ker d_A^i}{\text{im } d_A^{i-1}}.$$

Now let $f: A \rightarrow B$ be an R -module homomorphism. From Lemma 1.1.21 we know that the restrictions $f^i: \ker d_A^i \rightarrow \ker d_B^i$ and $f^i: \text{im } d_A^{i-1} \rightarrow \text{im } d_B^{i-1}$ are well-defined injective maps, and therefore the morphism $H^i(f)$ is defined as the class morphism $H^i(f)(x + \text{im } d_A^{i-1}) = f(x) + \text{im } d_B^{i-1}$.

Definition 1.1.24 (Acyclic complex). A complex A is said to be **acyclic** if all of its homology objects $H^i(A)$ are zero. Note that this is equivalent to say that the complex A with respect to its differentials is exact.

Definition 1.1.25 (Quasi-isomorphism). A morphism $f: A \rightarrow B$ between complexes is called a **quasi-isomorphism**, if the induced map taking the homology functor $H^i(f): H^i(A) \rightarrow H^i(B)$ is an isomorphism for all i . If f is a quasi-isomorphism we will say simply that f is a qis.

Remark 1.1.26 (Quasi-isomorphisms between acyclic complexes). Let $f: A \rightarrow B$ be a morphism between acyclic complexes. Note that f acts trivially on homology; that is, each $H^i(f)$ is the zero morphism, which is an isomorphism. Therefore, any well-defined chain map between acyclic complexes is a quasi-isomorphism.

Should A and B be acyclic except for a finite amount of homological components, it suffices to verify that in these components the morphism induced by homology is a isomorphism.

The following proposition allows us to extend short exact sequences of chain complexes to a long exact sequence at the level of homology objects. The proof is carried out using the well-known **Snake Lemma** (see [16, Corollary 6.12]). These types of diagrams are a consequence of what is known as **diagram chasing**.

Proposition 1.1.27. Consider $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ an exact sequence of chain complexes. Then, there exists arrows $\delta^i: H^i(C) \rightarrow H^{i+1}(A)$ called **connecting morphisms**, such that the following is a long exact sequence

$$\cdots \longrightarrow H^i(B) \xrightarrow{H^i(g)} H^i(C) \xrightarrow{\delta^i} H^{i+1}(A) \xrightarrow{H^{i+1}(f)} H^{i+1}(B) \longrightarrow \cdots$$

Proof. See [17, Lemma 12.13.12] or [16, Theorem 6.10]. □

Definition 1.1.28 (Homotopy between chain maps). Let $f, g: A \rightarrow B$ be morphisms between chain complexes. A **homotopy** between f and g is a collection of morphisms $h^i: A^i \rightarrow B^{i-1}$ such that

$$d_B^{i-1} \circ h^i + h^{i+1} \circ d_A^i = f^i - g^i$$

for each i . If such a homotopy exists, we say that f and g are **homotopic**, and we write $f \sim g$. In particular, if f is homotopic to the zero morphism, we say that f is **nullhomotopic**. Note that $f \sim g$ if and only if $f - g \sim 0$.

Definition 1.1.29 (Homotopy equivalence). Let $f: A \rightarrow B$ be a morphism between chain complexes. If there exists a morphism $g: B \rightarrow A$ such that $f \circ g \sim 1_B$ and $g \circ f \sim 1_A$, then we say that f is a **homotopy equivalence**, and that A and B are **homotopically equivalent objects**.

Definition 1.1.30 (Homotopy category). The **homotopy category** of an abelian category $\mathcal{A}$, denoted by $\mathbf{K}(\mathcal{A})$, is defined as follows

$$\text{Ob}(\mathbf{K}(\mathcal{A})) := \text{Ob}(\mathbf{C}(\mathcal{A})), \quad \text{Hom}_{\mathbf{K}(\mathcal{A})}(A, B) := \text{Hom}_{\mathbf{C}(\mathcal{A})}(A, B) / \sim$$

We denote the homotopy classes $[f]$ in $\mathbf{K}(\mathcal{A})$ simply by $[f] := f$, if it does not lead to confusion.

Proposition 1.1.31. Let A, B, C be chain complexes, then the following properties hold:

- (1) The homotopy equivalence is an equivalence relation.
- (2) Let $f: A \rightarrow B$ and $g: B \rightarrow C$ be morphisms between complexes. If either f or g is nullhomotopic, then so is $g \circ f$.
- (3) The set of nullhomotopic morphisms $\text{Ht}(A, B)$ is a subgroup of $\text{Hom}_{\mathbf{C}(\mathcal{A})}(A, B)$.
- (4) If f and g are two homotopic morphism of chain complexes, then the homology induced morphisms $H^i(f), H^i(g)$ from $H^i(A)$ to $H^i(B)$ coincide.
- (5) Let $f: A \rightarrow B$ be a homotopy equivalence with homotopy inverse $g: B \rightarrow A$. Then f and g are quasi-isomorphism, and $H^i(f)^{-1} = H^i(g)$.

Proof. To facilitate the writing process, we omit the indices in the proof.

(1) Let $f, g, k: A \rightarrow B$ be a chain maps. We can easily see that $f \sim f$ taking $h = 0$ for the reflexivity.

For the symmetry take $f \sim g$ via an homotopy h , then we have $d_B \circ h + h \circ d_A = f - g$. Multiplying by -1 we have $d_B \circ (-h) + (-h) \circ d_A = g - f$ thus $g \sim f$ via $-h$.

For the transitivity let h and h' be homotopies such that $f \sim g$ and $g \sim k$ respectively, then we have the equations $d_B \circ h + h \circ d_A = f - g$ and $d_B \circ h + h \circ d_A = g - k$ we add this two equations to obtain $d_A \circ (h + h') + (h + h') \circ d_A = f - k$ thus $f \sim k$ as desired.

(2) Without loss of generality, suppose that f is nullhomotopic, for g being nullhomotopic follows similarly. By definition we have $d_B \circ h + h \circ d_A = f$. This implies that

$$g \circ f = g \circ d_B \circ h + g \circ h \circ d_A = d_B \circ g \circ h + g \circ h \circ d_A,$$

thus $g \circ f$ is nullhomotopic via $g \circ h$.

(3) Clearly the zero morphism is in $\text{Ht}(A, B)$. Assume that $f, g \in \text{Ht}(A, B)$. Then there exists homotopies h and k such that $f = d_B \circ h + h \circ d_A$ and $g = d_B \circ k + k \circ d_A$. This implies that

$$f + g = d_B \circ (h + k) + (h + k) \circ d_A,$$

thus $f + g$ is nullhomotopic. Therefore $\text{Ht}(A, B)$ is closed under addition. Lastly, multiplying by -1 we get the equation $-f = d_B \circ (-h) + (-h) \circ d_A$ and so $-f$ is nullmotopic. This implies that $\text{Ht}(A, B)$ is a subgroup.

(4) Following Remark 1.1.23 it suffices to do explicit calculations over modules. Let h be a homotopy between f and g , so $f^i - g^i = h^{i+1} \circ d_A^i + d_B^{i-1} \circ h^i$. We claim that the morphism $f - g$ maps $\ker d_A^i$ into $\text{im } d_B^{i-1}$. Indeed, this follows by taking $x \in \ker d_A^i$ and replacing in the equation before to get $(f^i - g^i)(x) = (d_B^{i-1} \circ h^i)(x)$. Then, this clearly implies that the induced homology map $H^i(f - g)$ must be the zero map, and by functoriality of $H^i(-)$, we deduce that $H^i(f) = H^i(g)$.

(5) It follows from (4) and by the functoriality of $H^i(-)$. □

Remark 1.1.32.

- From item (1), the homotopy category $\mathbf{K}(\mathcal{A})$ is well-defined.
- From item (2), the usual composition $[g] \circ [f] = [g \circ f]$ of morphisms in $\mathbf{K}(\mathcal{A})$ is well-defined.
- Since $\text{Ht}(A, B)$ is a subgroup of $\text{Hom}_{\mathbf{C}(\mathcal{A})}(A, B)$ for any complexes A and B , and since the latter is an abelian group, it follows that $\text{Ht}(A, B)$ is a normal subgroup. Consequently, we obtain an alternative definition of morphisms in the homotopy category by taking the quotient group. Explicitly,

$$\text{Hom}_{\mathbf{K}(\mathcal{A})}(A, B) = \text{Hom}_{\mathbf{C}(\mathcal{A})}(A, B) / \text{Ht}(A, B). \quad (1.3)$$

- From item (4), the concept of a quasi-isomorphism in the category of chain complexes can be naturally extended to the homotopy category.

Definition 1.1.33 (Bounded homotopy category). We define $\mathbf{K}^*(\mathcal{A})$ for $* = +, -, b$ as the full subcategory of bounded below, bounded above, and bounded chain complexes, where the morphisms are given by homotopy classes. All these subcategories are additive.

Theorem 1.1.34. The homotopy category $\mathbf{K}(\mathcal{A})$ is additive and contains $\mathcal{A}$ as a full subcategory.

Proof. For all the details in the additive structure the reader may refer to [11, Proposition 1.7].

For the second statment, we define a functor $C: \mathbf{C}(\mathcal{A}) \rightarrow \mathbf{K}(\mathcal{A})$ which acts as the identity on objects but sends each chain map f to its corresponding homotopy class $[f]$. This construction together with the inclusion functor $I: \mathcal{A} \rightarrow \mathbf{C}(\mathcal{A})$ makes the functorial composition $\mathcal{A} \rightarrow \mathbf{C}(\mathcal{A}) \rightarrow \mathbf{K}(\mathcal{A})$ a fully faithful functor.

Indeed, it is suffices to prove that the subgroup $\text{Ht}(A[0], B[0])$ is zero, for that take any null homotopic map g via an homotopy h . In components, we must have that $h^i: A^i \rightarrow B^{i-1}$ but $A^i = B^i = 0$ wherever $i \neq 0$ therefore $h^i = 0$ for every i , and then $g = 0$. Now, by the equation (1.3) and by Theorem 1.1.18 we have

$$\text{Hom}_{\mathcal{A}}(A, B) = \text{Hom}_{\mathbf{C}(\mathcal{A})}(A[0], B[0]) \cong \text{Hom}_{\mathbf{K}(\mathcal{A})}(A[0], B[0]),$$

and thus the composition $\mathcal{A} \rightarrow \mathbf{C}(\mathcal{A}) \rightarrow \mathbf{K}(\mathcal{A})$ is a fully faithful functor as desired. $\square$

Definition 1.1.35 (Double complex). A **double complex**, commonly denoted as $A^{\bullet, \bullet}$ is a sequence of objects $A^{i,j}$ and morphisms $d_1^{i,j}: A^{i,j} \rightarrow A^{i+1,j}$, $d_2^{i,j}: A^{i,j} \rightarrow A^{i,j+1}$ in $\mathcal{A}$ such that for every i, j we have the relations

$$d_1^{i+1,j} \circ d_1^{i,j} = 0, \quad d_2^{i,j+1} \circ d_2^{i,j} = 0 \quad \text{and} \quad d_1^{i,j+1} \circ d_2^{i,j} = d_2^{i+1,j} \circ d_1^{i,j}.$$

This is equivalent to say that each $A^{i, \bullet}$ is a complex and that each $d_1^{i, \bullet}$ is a morphism of complexes $A^{i, \bullet} \rightarrow A^{i+1, \bullet}$ such that $d_1^{i+1, \bullet} \circ d_1^{i, \bullet} = 0$. A double complex can be seen as a complex of complexes. So we get a commutative diagram

$$\begin{array}{ccccc} & & \vdots & & \vdots \\ & & \uparrow & & \uparrow \\ \dots & \longrightarrow & A^{i,j+1} & \xrightarrow{d_1^{i+1,j+1}} & A^{i+1,j+1} & \longrightarrow \dots \\ & & \uparrow d_2^{i,j+1} & & \uparrow d_2^{i+1,j+1} \\ \dots & \longrightarrow & A^{i,j} & \xrightarrow{d_1^{i,j}} & A^{i+1,j} & \longrightarrow \dots \\ & & \uparrow & & \uparrow \\ & & \vdots & & \vdots \end{array}$$

Remark 1.1.36 (Double complex warning). In the literature one encounters a different definition where a double complex has the property that the squares in the diagram anti-commute. In this document we always assume that the double complex commutes.

Definition 1.1.37 (Total complex). Let $A^{\bullet, \bullet}$ to be a double complex. The **total complex** associated to $A^{\bullet, \bullet}$, denoted as $\text{Tot}(A)$ is given by

$$\text{Tot}(A)^n := \bigoplus_{i+j=n} A^{i,j} \quad \text{with differentials} \quad d_{\text{Tot}(A)}^n := \bigoplus_{i+j=n} d_1^{i,j} + (-1)^i d_2^{i,j}.$$

The total complexes of a double complex provide a way to represent double complexes in an abelian category $\mathcal{A}$ using chain complexes within $\mathcal{A}$. Additionally, there are sufficient conditions on double complexes that ensure the exactness of their associated total complexes in $\mathcal{A}$. Specifically, if a double complex is bounded below or above, the exactness of its row or column complexes ensures that its total complexes are also exact. This result is also known as the **acyclic assembly lemma**. See [19, Lemma 2.7.3].

Proposition 1.1.38. *The total complex $\text{Tot}(A)$ is acyclic whenever A is a bounded double complex with exact rows (or exact columns).*

Proof. Let $A^{\bullet,\bullet}$ be a double complex. By interchanging rows and columns, it suffices to prove the case when all the rows are exact. We check that $\text{Tot}(A)$ is exact at homological degree 0. Up to some complex shifting (see definition 1.2.10) we may assume that $A^{i,-i} = 0$ for $i < 0$ and for $i > n$ for some n . In the following we omit the differentials indices.

Let $a = (a_0, \dots, a_n) \in \text{Tot}^0(A)$ with $a_i \in A_{i,-i}$ such that $a \in \ker d_{\text{Tot}(A)}^0$ and so

$$d_1(a_i) + (-1)^{i+1}d_2(a_{i+1}) = 0 \quad \text{for } i = 0, \dots, n-1. \quad (\star)$$

We want to find an element $b = (b_0, \dots, b_n) \in \text{Tot}^{-1}(A)$ such that $d_{\text{Tot}(A)}^{-1}(b) = a$. This is equivalent to $d_1(b_{i-1}) + (-1)^i d_2(b_i) = a_i$ for $i = 0, \dots, n$ with $b_j \in A^{j,-j-1}$.

Noticing that the double complex ends up with $A^{n,-n}$ for $\text{Tot}^0(A)$, we have trivially that $d_1(a_n) = 0$. Since the rows are exact we can find $b_{n-1} \in A^{n-1,-n}$ such that $d_1(b_{n-1}) = a_n$, then by taking $b_n = 0$ we have

$$d_1(b_{n-1}) + (-1)^n d_2(b_n) = a_n.$$

We construct b_i inductively, for that suppose there exists $b_j, \dots, b_{n-1}$ such that

$$d_1(b_{i-1}) + (-1)^i d_2(b_i) = a_i \quad \text{for } j+1 \leq i \leq n-1. \quad (\star\star)$$

$$\begin{aligned} d_1(a_j - (-1)^j d_2(b_j)) &= d_1(a_j) + (-1)^{j+1} d_1(d_2(b_j)) \\ &= d_1(a_j) + (-1)^{j+1} d_2(d_1(b_j)). \quad (\text{By the commutativity of the differential}) \\ &= d_1(a_j) + (-1)^{j+1} d_2(a_{j+1} - (-1)^{j+1} d_2(b_{j+1})). \quad (\text{Replacing } (\star\star)) \\ &= d_1(a_j) + (-1)^{j+1} d_2(a_{j+1}) = 0. \quad (\text{Since } d_2^2 = 0 \text{ and replacing } (\star)) \end{aligned}$$

As the rows are exact there exists $b_{j-1} \in A^{j,-j-1}$ such that $d_1(b_{j-1}) = a_j - (-1)^j d_2(b_j)$. Continuing with this process, note that in the end, we can take $d_1(b_0) = 0$ so the induction must finish. $\square$

Definition 1.1.39 (Direct sum, tensor, and Hom complexes). *Let $A, B \in \mathbf{C}(\mathcal{A})$.*

- (1) *The **Direct sum complex** is defined as $(A \oplus B)^i = A^i \oplus B^i$ with differential maps $d^i(a, b) = (d_A^i(a), d_B^i(b))$.*

(2) The **Tensor complex** is defined as the total complex of the double Tensor complex with objects $(A \otimes B)^{i,j} = A^i \otimes B^j$ and differentials $d_1^{i,j}(a \otimes b) = d_A(a) \otimes b$ and $d_2^{i,j} = a \otimes d_B(b)$. This definition is taken over the category of R -modules.

(3) The **Hom complex** is defined as the total complex of the double Hom complex with $\text{Hom}(A, B)^{i,j} = \text{Hom}_A(A^{-i}, B^j)$ and differentials $d_1^{i,j}(f) = f \circ d_A$ and $d_2^{i,j}(f) = -(-1)^{j+1} d_B \circ f$.

We end this subsection introducing the **cone complex**, before that we recall the matrix notation of maps between finite direct sums.

Remark 1.1.40 (Matrix notation over finite direct sums). Let $\mathcal{C}$ to be an abelian category and let x_r and y_s be objects in $\mathcal{C}$ for $0 \leq r \leq n$ and $0 \leq s \leq m$. Let $f_{rs}: x_r \rightarrow y_s$ be maps in $\mathcal{C}$ and recall the Definition 1.1.3 with give us canonical maps $i_r: x_r \rightarrow x_1 \oplus \cdots \oplus x_n$ and $q_s: y_1 \oplus \cdots \oplus y_m \rightarrow y_s$. Then, there exists a unique map $F: x_1 \oplus \cdots \oplus x_n \rightarrow y_1 \oplus \cdots \oplus y_m$ such that $q_s \circ F \circ i_r = f_{rs}$ for every $0 \leq i \leq n$ and $0 \leq s \leq m$.

We write this morphism F as the $m \times n$ matrix (f_{sr}) with $1 \leq s \leq m$ and $1 \leq r \leq n$ where the matrix multiplication coincides by the composition making the calculation easier.

Definition 1.1.41 (Cone complex). Let $f: A \rightarrow B$ be a morphism between chain complexes. We define the **cone of f** denoted by $\text{Cone}(f)$, as the chain complex with objects and differential given as follows

$$\text{Cone}(f)^i := A^{i+1} \oplus B^i, \quad d_{\text{Cone}(f)}^i := \begin{pmatrix} -d_A^{i+1} & 0 \\ f^{i+1} & d_B^i \end{pmatrix}.$$

Remark 1.1.42 (Short exact sequence induced by cone complex). Let $f: A \rightarrow B$ be a chain map. Associated with $\text{Cone}(f)^i$, we have two canonical morphisms: the projection $\beta(f)^i: \text{Cone}(f)^i \rightarrow A^{i+1}$ and the inclusion $\alpha(f)^i: B^i \rightarrow \text{Cone}(f)^i$ which are chain maps, these induce the following short exact sequence of complexes

$$0 \longrightarrow B \xrightarrow{\alpha(f)} \text{Cone}(f) \xrightarrow{\beta(f)} A[1] \longrightarrow 0. \quad (1.4)$$

Proposition 1.1.43. Let $f: A \rightarrow B$ to be a morphism between chain complexes. Then f is a quasi-isomorphism if and only if the $\text{Cone}(f)$ complex is acyclic.

Proof. See [11, Proposition 7.7]. □

Lemma 1.1.44. Let $f: A \rightarrow B$ be a morphism between chain complexes and $\alpha(1_A)$ be the canonical inclusion associated to $\text{Cone}(1_A)$. Then f can be factorized through $\text{Cone}(f)$ via $\alpha(1_A)$ if and only if f is nullhomotopic.

Proof. Assume that f is nullhomotopic. This means there exists a chain homotopy $h^i: A^i \rightarrow B^{i-1}$ such that $f^i = d_B^{i-1} \circ h^i + h^{i+1} \circ d_A^i$. We can then define the map $g: \text{Cone}(1_A) \rightarrow B$ by components as

$g^i(a, a') := f^i(a) + h^{i+1}(a')$. We note that each $g^i := (h^{i+1} \ f^i)$ commutes with differentials. Indeed

$$\begin{aligned} \begin{pmatrix} h^{i+2} & f^{i+1} \end{pmatrix} \begin{pmatrix} -d_A^{i+1} & 0 \\ 1_{A^{i+1}} & d_A^i \end{pmatrix} &= \begin{pmatrix} -h^{i+2} \circ d_A^{i+1} + f^{i+1} & f^{i+1} \circ d_A^i \end{pmatrix} \\ &= \begin{pmatrix} d_B^i \circ h^{i+1} & d_B^i \circ f^i \end{pmatrix} = d_B^i \circ g^i. \end{aligned}$$

Now the equality $(g \circ \alpha(1_A))(a) = g(a, 0) = f(a)$ proves that f indeed factors through $\text{Cone}(1_A)$ complex as desired.

For the converse. Let $g: \text{Cone}(1_A) \rightarrow B$ be a map of chain complexes such that $g \circ \alpha(1_A) = f$. We claim that the map $\alpha(1_A)$ is nullhomotopic. Let $h^i: A^i \rightarrow A^i \oplus B^{i-1}$ be the natural injective inclusion map $h^i := \begin{pmatrix} 1_{A^i} \\ 0 \end{pmatrix}$. We check the equality $h \circ d_A + d_{\text{Cone}(1_A)} \circ h = \alpha(1_A)$. Indeed

$$\begin{pmatrix} 1_{A^{i+1}} \\ 0 \end{pmatrix} d_A^i + \begin{pmatrix} -d_A^i & 0 \\ 1_{A^i} & d_A^{i-1} \end{pmatrix} \begin{pmatrix} 1_{A^i} \\ 0 \end{pmatrix} = \begin{pmatrix} d_A^i \\ 0 \end{pmatrix} + \begin{pmatrix} -d_A^i \\ 1_{A^i} \end{pmatrix} = \begin{pmatrix} 0 \\ 1_{A^i} \end{pmatrix},$$

and then we get that $\alpha(1_A)$ is nullhomotopic. Now, by the item (2) in the Proposition 1.1.31 we conclude that $f = g \circ \alpha(1_A)$ is a nullhomotopic map as desired. $\square$

Later, we provide an example (Example 1.2.16) showing that the homotopy category $\mathbf{K}(\mathcal{A})$ is generally not abelian. Consequently, we cannot discuss the concepts of kernels, images, or exact sequences in this context. Instead, this category possesses a triangulated structure which we will introduce in the next section. In such categories, the concept of exact sequences is replaced by *distinguished triangles*.

1.2 Triangulated categories

The triangulated structure on a category is an extra piece of data, consisting of a translation functor and a set of distinguished triangles chosen suitably to satisfy certain axioms, these categories provide a natural framework for studying objects up to homotopy equivalence, and their structure generalizes many classical concepts from abelian categories. The references for this section are [17, Derived categories], [11] and [8].

First, recall that given a category $\mathcal{C}$ (not necessary abelian), an **automorphism** over $\mathcal{C}$ is a functor $T: \mathcal{C} \rightarrow \mathcal{C}$ with an inverse $T^{-1}: \mathcal{C} \rightarrow \mathcal{C}$, that satisfies the equality $T \circ T^{-1} = 1_{\mathcal{C}} = T^{-1} \circ T$ as functors. Clearly this notion is stronger than a categorical equivalence.

Definition 1.2.1 (Triangles). *Let $\mathcal{C}$ be an additive category and let $[1]: \mathcal{C} \rightarrow \mathcal{C}$ be a translation functor. A **triangle** in $\mathcal{C}$ is a diagram of the form*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1],$$

commonly denoted as (X, Y, Z, f, g, h) . A **morphism between triangles** is a triple (f, g, h) of morphisms in $\mathcal{C}$ such that the following diagram commutes

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\ \downarrow f & & \downarrow g & & \downarrow h & & \downarrow f[1] \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X'[1]. \end{array}$$

We say that two triangles are isomorphic if there exists a map of triangles (f, g, h) where f, g, h are isomorphisms in $\mathcal{C}$.

Definition 1.2.2 (Triangulated category). A **triangulated category** is an additive category $\mathcal{C}$ equipped with a translation functor $T := [1]: \mathcal{C} \rightarrow \mathcal{C}$ along with a family of **distinguished triangles** that satisfy the following axioms:

(TR1a) For every object X , the triangle $X \xrightarrow{1_X} X \longrightarrow 0 \longrightarrow X[1]$ is distinguished.

(TR1b) Every triangle isomorphic to a distinguished triangle is distinguished.

(TR1c) Every morphism $f: X \rightarrow Y$ can be extended to a distinguished triangle

$$X \xrightarrow{f} Y \longrightarrow Z \longrightarrow X[1].$$

(TR2) (Rotation Axiom) A triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1],$$

is distinguished if and only if the triangle

$$Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1].$$

is distinguished. The latter triangle is called the **rotated triangle**.

(TR3) Suppose there is a commutative diagram between distinguished triangles with vertical morphisms f and g . Then, the diagram

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\ \downarrow f & & \downarrow g & & \downarrow h & & \downarrow f[1] \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X'[1], \end{array}$$

can be completed with a morphism, not necessarily unique, $h: Z \rightarrow Z'$ such that the whole diagram commutes.

(TR4) (Octahedral Axiom) Given distinguished triangles

$$X \xrightarrow{f} Y \longrightarrow Z' \longrightarrow X[1], \quad Y \xrightarrow{g} X \longrightarrow X' \longrightarrow Y[1], \quad X \xrightarrow{g \circ f} Z \longrightarrow Y' \longrightarrow X[1].$$

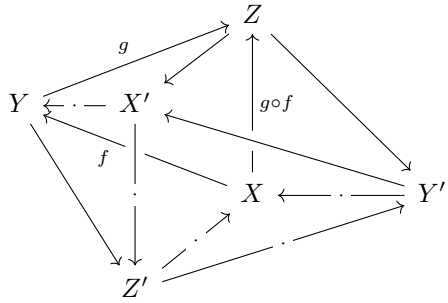
there exists a distinguished triangle

$$Z' \longrightarrow Y' \longrightarrow X' \longrightarrow Z'[1],$$

such that the following diagram commutes

$$\begin{array}{ccccccc}
 X & \xrightarrow{f} & Y & \longrightarrow & Z' & \longrightarrow & X[1] \\
 \downarrow 1_X & & \downarrow g & & \downarrow & & \downarrow 1_{X[1]} \\
 X & \xrightarrow{g \circ f} & Z & \longrightarrow & Y' & \longrightarrow & X[1] \\
 \downarrow f & & \downarrow 1_Z & & \downarrow & & \downarrow f[1] \\
 Y & \xrightarrow{g} & Z & \longrightarrow & X' & \longrightarrow & Y[1] \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 Z' & \longrightarrow & Y' & \longrightarrow & X' & \longrightarrow & Z'[1].
 \end{array}$$

This axiom is called **octahedral axiom** because we can draw it as follows:



where any distinguished triangle is represented by $\begin{array}{ccc} & Z & \\ \swarrow & & \nwarrow \\ X & \longrightarrow & Y \end{array}$ and the morphism

$Z \dashrightarrow X$ stands for $Z \rightarrow X[1]$.

Let $\mathcal{C}, \mathcal{D}$ be triangulated categories with translation functors $T_{\mathcal{C}}$ and $T_{\mathcal{D}}$ respectively and consider $F: \mathcal{C} \rightarrow \mathcal{D}$ to be an additive functor. The functor F is called **exact** or **triangulated** if the following two conditions are satisfied

- (1) F **commutes with translation**, that is, there exists a natural isomorphism $F \circ T_{\mathcal{C}} \xrightarrow{\sim} T_{\mathcal{D}} \circ F$, explicitly $F(X[1]) \cong (F(X))[1]$ for very $X \in \mathcal{C}$.
- (2) F **takes distinguished triangles in $\mathcal{C}$ to distinguished triangles in $\mathcal{D}$** , that is, any distinguished triangle $X \rightarrow Y \rightarrow Z \rightarrow X[1]$ in $\mathcal{C}$ is mapped to a distinguished triangle $F(X) \rightarrow F(Y) \rightarrow F(Z) \rightarrow F(X)[1]$ in $\mathcal{D}$, where $F(X[1])$ is identified $F(X)[1]$ via item (1).

Before introducing the next subsection, we present several properties and remarks about the triangulated structure.

Proposition 1.2.3. *Let $\mathcal{C}$ be a triangulated category, then the following properties hold:*

- (1) If (X, Y, Z, f, g, h) is a distinguished triangle in $\mathcal{C}$, then $g \circ f = 0$, $h \circ g = 0$ and $f[1] \circ h = 0$.
- (2) Any change in the sign of any two morphisms of a distinguished triangle results again in a distinguished triangle.
- (3) If (X, Y, Z, f, g, h) is a distinguished triangle in $\mathcal{C}$. Then f is an isomorphism in $\mathcal{C}$ if and only if $Z = 0$.
- (4) Given the distinguished triangles (X, Y, Z, f, h, g) and (X', Y', Z', f', g', h') then the triangle formed by the direct sum of each component

$$(X \oplus X', Y \oplus Y', Z \oplus Z', f \oplus f', g \oplus g', h \oplus h'),$$

is distinguished. The converse also holds.

Proof. (1) By TR1 we know that $(X, X, 0, 1, 0, 0)$ is a distinguished triangle. We can now apply TR3 to complete the diagram

$$\begin{array}{ccccccc} X & \longrightarrow & X & \longrightarrow & 0 & \longrightarrow & X[1] \\ \downarrow 1_X & & \downarrow f & & \downarrow 0 & & \downarrow 1[1] \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & 0 & \xrightarrow{h} & X[1]. \end{array}$$

Of course the dotted arrow is the zero map. Hence the commutativity of the diagram implies that $g \circ f = 0$. For the other cases just rotate the original triangle.

(2) Let (X, Y, Z, f, g, h) be a distinguished triangle and considering the change of sign on the maps f and g , that is, the triangle $(X, Y, Z, -f, -g, h)$. The result follows directly from TR1b by considering the isomorphism $(1_X, -1_Y, 1_Z): (X, Y, Z, f, g, h) \rightarrow (X, Y, Z, -f, -g, h)$.

(3) In this item we will assume the Triangulated five lemma, Lemma 1.2.6. By item (1) we already know that $g \circ f = 0$, from these we can obtain the following map between distinguished triangles.

$$\begin{array}{ccccccc} X & \longrightarrow & X & \longrightarrow & 0 & \longrightarrow & X[1] \\ \downarrow 1_X & & \downarrow f & & \downarrow & & \downarrow 1[1] \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & X[1]. \end{array}$$

If $Z = 0$ then the third column is an isomorphism which implies that f is also by Lemma 1.2.6. Now if f is an isomorphism again by Lemma 1.2.6 then $0 \rightarrow Z$ is an isomorphism.

(4) See [17, Derived categories, Lemma 4.10.]. □

Definition 1.2.4 (Homological functor). *Let $\mathcal{A}$ be an abelian category. We say that an additive functor $H: \mathcal{C} \rightarrow \mathcal{A}$ is **homological** if for every distinguished triangle (X, Y, Z, f, g, h) the sequence*

$$H(X) \xrightarrow{H(f)} H(Y) \xrightarrow{H(g)} H(Z),$$

*is exact in $\mathcal{A}$. Additionally, we say that a functor $H: \mathcal{C}^{op} \rightarrow \mathcal{A}$ is **cohomological** if the corresponding functor $\mathcal{C} \rightarrow \mathcal{A}^{op}$ is homological.*

Proposition 1.2.5. *Let W any object in $\mathcal{C}$. The functor $\mathrm{Hom}_{\mathcal{C}}(W, -): \mathcal{C} \rightarrow \mathbf{Ab}$ is homological and $\mathrm{Hom}_{\mathcal{C}}(-, W): \mathcal{C} \rightarrow \mathbf{Ab}$ is cohomological.*

Proof. We will show the statement for the functor $\mathrm{Hom}_{\mathcal{D}}(W, -)$. Let (X, Y, Z, f, g, h) be a distinguished triangle. Using the Rotation axiom it suffices to prove that the following induced sequence

$$\mathrm{Hom}(W, X) \xrightarrow{f_*} \mathrm{Hom}(W, Y) \xrightarrow{g_*} \mathrm{Hom}(W, Z),$$

is exact, where $f_* = f \circ -$ and $g_* = g \circ -$. Let $a: W \rightarrow Y$ be a map such that $g \circ a = 0$, this is, $a \in \ker g_*$. This induces the following diagram

$$\begin{array}{ccccccc} W & \longrightarrow & W & \longrightarrow & 0 & \longrightarrow & W[1] \\ \downarrow b & & \downarrow a & & \downarrow 0 & & \downarrow b[1] \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \longrightarrow & X[1]. \end{array}$$

The top row is distinguished by TR1. The dashed arrow can be obtained first rotating using Rotation axiom, then apply TR3, and then rotating back. From the commutativity we get that $f \circ b = a$, equivalently $a \in \mathrm{im} f_*$ as desired. $\square$

Lemma 1.2.6 (Triangulated five lemma). *Let $\mathcal{C}$ be a triangulated category. Let*

$$(f, g, h): (X, Y, Z, a, b, c) \longrightarrow (X', Y', Z', a', b', c'),$$

be a morphism of distinguished triangles. If any of two f, g, h are isomorphisms then is so the third one.

Proof. Because of the Rotation axiom we may assume that f and h are isomorphisms. Consider the following diagram

$$\begin{array}{ccccccccc} \mathrm{Hom}_{\mathcal{C}}(W, Z[-1]) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(W, X) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(W, Y) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(W, Z) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(W, X[1]) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathcal{C}}(W, Z'[-1]) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(W, X') & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(W, Y') & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(W, Z') & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(W, X'[1]). \end{array}$$

This diagram is clearly commutative because $\mathrm{Hom}_{\mathcal{C}}(W, -)$ is a functor. The exactness of the rows follows by Proposition 1.2.5. By assumption the right two and left two vertical arrows are isomorphisms. Now by the Five Lemma over $\mathbf{Ab}$ we have that the middle vertical arrow $g_*: \mathrm{Hom}_{\mathcal{C}}(W, Y) \rightarrow \mathrm{Hom}_{\mathcal{C}}(W, Y')$ is an isomorphism. By Yoneda Lemma, Lemma 1.1.14 we have that g is an isomorphism as desired. $\square$

Lemma 1.2.7. *Let $\mathcal{C}$ be a triangulated category and let (X, Z, Y, f, g, h) be a distinguished triangle. If $h = 0$, then the previous triangle is isomorphic to the distinguished triangle $(X, X \oplus Y, Y, i, p, h = 0)$. Moreover, $Z \cong X \oplus Y$.*

Proof. By TR3 there exist a map $h: Z \rightarrow X \oplus Y$ such that the triple $(1_{Z[-1]}, 1_X, h)$ defines a morphism between the rotated triangles $(Y[-1], X, Z, 0, f, g)$ and $(Y[-1], X, X \oplus Y, 0, i, p)$. Now we can apply the Lemma 1.2.6 to see that h is an isomorphism. Rotating back we get the desired isomorphism. $\square$

Remark 1.2.8. The previous lemma follows, using the rotation axiom, even if one of the maps f or g are zero obtaining a list of useful equalities: in the distinguished triangle (X, Y, Z, f, g, h)

- if $f = 0$ then $Z[-1] = Y[-1] \oplus X$. Equivalently, $Z = Y \oplus X[1]$.
- if $g = 0$ then $X = Z[-1] \oplus Y$.
- if $h = 0$ then $Y = X \oplus Z$.

Remark 1.2.9 (Z object in TR1c axiom is unique). Consider $f: X \rightarrow Y$ to be a morphism over a triangulated category $\mathcal{C}$. The object Z obtained in TR1c axiom is unique up to non-canonical isomorphism as we describe. Let (X, Y, Z, f, g, h) and (X, Y, Z', f, g', h') be completions of f given by TR1c axiom. Clearly by TR3 axiom, there exists a map $h: Z \rightarrow Z'$ such that the triple

$$(1_X, 1_Y, h): (X, Y, Z, f, g, h) \longrightarrow (X, Y, Z', f, g', h'),$$

defines a morphism between the previous distinguished triangles, now applying Lemma 1.2.6 we get that h is an isomorphism.

1.2.1 The homotopy category is triangulated. We conclude this section by defining the triangulated structure of the homotopy category. Additionally, we present an example that illustrates why this category is not generally abelian. We essentially follow the first chapter of [11]. For the following let $\mathcal{A}$ be an abelian category. Lets start by defining a translation functor over the category of complexes

Definition 1.2.10 (Translation functor on complexes). The **translation functor on complexes** or **shift functor** is the functor $T: \mathbf{C}(\mathcal{A}) \rightarrow \mathbf{C}(\mathcal{A})$, defined on objects by

$$T: \mathbf{C}(\mathcal{A}) \longrightarrow \mathbf{C}(\mathcal{A}), \quad (A^i, d_A^i) \mapsto T(A) := (A^{i+1}, -d_A^{i+1}),$$

and on morphism by $f: A \rightarrow B$ in $\mathbf{C}(\mathcal{A})$ is mapped to the morphism $T(f): T(A) \rightarrow T(B)$ in $\mathbf{C}(\mathcal{A})$ defined by $T(f)^i := f^{i+1}$. It is straightforward to prove that the translation functor defines an automorphism on $\mathbf{C}(\mathcal{A})$. We denote the n -th composition as $T^n(A) := A[n]$, where it holds that $[n] \circ [m] = [n + m]$ and $[0] = 1_{\mathbf{C}(\mathcal{A})}$ as functor equality.

Remark 1.2.11.

- Although the notation may be the same, the notation of the translation functor on $\mathbf{C}(\mathcal{A})$, and the definition of chain complexes concentrated in a given degree should not be confused.
- The definition of the translation functor on $\mathbf{C}(\mathcal{A})$ is compatible with homotopic morphisms this is, f is homotopic to g via h , if and only if, $f[1]$ is homotopic to $g[1]$ via $-h$. This follows by the

following calculation

$$\begin{aligned}
 f - g = d_B \circ h + h \circ d_A &\iff T(f - g) = T(d_B \circ h + h \circ d_A). \\
 &\iff T(f) - T(g) = T(d_B) \circ T(h) + T(h) \circ T(d_A). \\
 &\iff f - g = -d_B \circ h + h \circ -d_A.
 \end{aligned}$$

This induces a well-defined translation functor $T = [1]: \mathbf{K}(\mathcal{A}) \rightarrow \mathbf{K}(\mathcal{A})$ on the homotopic category which is also an automorphism.

- As the sign does not affect either the kernels and images then taking homology functor commutes with translation.

Definition 1.2.12 (Standard triangles). Let $f: A \rightarrow B$ a morphism between complexes. The $\text{Cone}(f)$ complex altogether with the canonical arrows $\alpha(f)$ and $\beta(f)$ induces the sequence

$$A \xrightarrow{f} B \xrightarrow{\alpha(f)} \text{Cone}(f) \xrightarrow{\beta(f)} A[1]. \quad (1.5)$$

We will refer to these triangles as the **standard triangle associated to f** .

Definition 1.2.13 (Triangles on $\mathbf{K}(\mathcal{A})$). A triangle in the homotopy category $\mathbf{K}(\mathcal{A})$ is said to be **distinguished** if it is isomorphic in the homotopy category to the image of a standard triangle (1.5) through the functor

$$\mathbf{C}(\mathcal{A}) \longrightarrow \mathbf{K}(\mathcal{A}),$$

defined in Theorem 1.1.34. That is, the distinguished triangles in $\mathbf{K}(\mathcal{A})$ are standard triangles up to homotopy.

Theorem 1.2.14. The functor $[1]: \mathbf{K}(\mathcal{A}) \rightarrow \mathbf{K}(\mathcal{A})$ together with the Definition 1.2.13 induces a triangulated structure in the homotopy category $\mathbf{K}(\mathcal{A})$.

Proof. See [11, Theorem 6.7] for the details in the triangulated structure. $\square$

Lemma 1.2.15. Let (X, Y, Z, f, g, h) be a distinguished triangle in $\mathbf{K}(\mathcal{A})$. Then f is a quasi-isomorphism if and only if Z is acyclic.

Proof. We already know by Proposition 1.1.43 that f is a quasi-isomorphism if and only if $\text{Cone}(f)$ is acyclic. Since (X, Y, Z, f, g, h) is a distinguished triangle in $\mathbf{K}(\mathcal{A})$ then by definition, Z is isomorphic to $\text{Cone}(f)$ in $\mathbf{K}(\mathcal{A})$, so Z is acyclic if and only if f is a quasi-isomorphism. $\square$

Example 1.2.16 (The homotopy category is not abelian). In this example we will see that in general, the homotopy category $\mathbf{K}(\mathcal{A})$ is non abelian. To this end, consider $\mathcal{A} = \mathbf{Ab}$ and assume by way of contradiction that $\mathbf{K}(\mathbf{Ab})$ is abelian. Take the abelian groups $\mathbb{Z}, \mathbb{Z}/2\mathbb{Z} \in \mathbf{Ab}$ with the canonical projection

$\pi: \mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z}$. This homomorphism induces the following chain map $\pi: \mathbb{Z}[0] \rightarrow \mathbb{Z}/2\mathbb{Z}[0]$ between concentrated complexes

$$\begin{array}{ccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \longrightarrow \cdots \\ & & \downarrow & & \downarrow \pi & & \downarrow \\ \cdots & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & 0 \longrightarrow \cdots \end{array}$$

Let $k: \ker \pi \rightarrow \mathbb{Z}[0]$ to be the kernel of π , which exists by hypothesis, since $\mathbf{K}(\mathbf{Ab})$ is abelian. Note that since $K = \ker \pi$ is the complex obtained by taking kernels in the columns maps in the above diagram, thus it is concentrated in degree 0. By TR1c axiom, there exist $W \in \mathbf{C}(\mathbf{Ab})$ such that k can be extended to a distinguished triangle

$$W[-1] \xrightarrow{w} K \xrightarrow{k} \mathbb{Z}[0] \longrightarrow W.$$

Formally, this distinguished triangle is rotated one degree clockwise. Note that $w = 0$ since $k \circ w = 0$ and k is a monomorphism, which by the Remark 1.2.8 implies that $\mathbb{Z}[0] = K \oplus W$ as complexes. Now as $\mathbb{Z}[0]$ and K are concentrated in 0 degree we must have the equality $\mathbb{Z} = \ker \pi \oplus W^0$, where $W^0 \in \mathbf{Ab}$, but this can not be possible since $\mathbb{Z}$ is indecomposable group.

Remark 1.2.17. Recall, by Proposition 1.1.31, that any two homotopic maps induce the same homology morphism. This extends the definition of i -th homology functor to the homotopy category

$$H^i(-): \mathbf{K}(\mathcal{A}) \rightarrow \mathcal{A}.$$

Proposition 1.2.18. The i -th homology functor $H^i(-): \mathbf{K}(\mathcal{A}) \rightarrow \mathcal{A}$ is homological.

Proof. By Remark 1.2.11 it suffices to show that $H^0(-): \mathbf{K}^b(\mathcal{A}) \rightarrow \mathcal{A}$ is homological, that means, given any distinguished triangle $A \xrightarrow{f} B \rightarrow \text{Cone}(f) \rightarrow A[1]$ their image through $H^0(-)$ is exact. For that recall that the cone complex induces a short exact sequence $0 \rightarrow B \rightarrow \text{Cone}(f) \rightarrow A[1] \rightarrow 0$ this induces a long exact sequence in homology which includes

$$H^{-1}(A[1]) \xrightarrow{\partial} H^0(B) \longrightarrow H^0(\text{Cone}(f)).$$

But $H^{-1}(X[1]) = H^0(X)$, and furthermore we can trace through the construction of the connecting homomorphism ∂ and note that coincides with $H^0(f)$ thus, the sequence $H^0(A) \rightarrow H^0(B) \rightarrow H^0(\text{Cone}(f))$ is exact. $\square$

1.3 Derived categories

The central concept behind the definition of the derived category is that *quasi-isomorphisms of complexes should be treated as isomorphisms*. This category is obtained by a formal process called *localization of categories*, which roughly speaking adds inverses to a specific class of morphisms in a category.

There are several comprehensive references on derived categories. While Verdier's original work [18] remains fundamental, more detailed and structured expositions can be found in [17, Derived Categories] and [8]. Additionally, [12] provides introductory chapters that outline the main concepts and their applications to algebraic geometry

1.3.1 Construction of the derived category. We now proceed with the construction of the derived category. To this end, we introduce the category of left fractions also known as left roofs, and the morphisms between them.

Definition 1.3.1 (Left roofs). *Let A, B be chain complexes. A **left roof** or **left fraction** between A and B is a diagram*

$$\begin{array}{ccc} & L & \\ s \swarrow & & \searrow f \\ A & & B, \\ & \text{qis} & \end{array}$$

where the maps $L \xrightarrow{s} A$ and $L \xrightarrow{f} B$ are morphisms in $\mathbf{K}(\mathcal{A})$, and the map $L \xrightarrow{s} A$ is a quasi-isomorphism in $\mathbf{K}(\mathcal{A})$ (note the shorthand **qis** in the above diagram). We shall denote $(f, s): A \rightarrow B$ for such left roofs.

Definition 1.3.2 (Equivalent roofs). *Let A, B be chain complexes. Two left roofs $(f, s), (g, t): M \rightarrow N$ are **equivalent** if there exists maps $p: H \rightarrow L$ and $q: H \rightarrow K$ in $\mathbf{K}(\mathcal{A})$ such that the compositions $s \circ p, t \circ q$ are quasi-isomorphisms in $\mathbf{K}(\mathcal{A})$ and making the following diagram commute*

$$\begin{array}{ccccc} & & H & & \\ & \text{qis} \swarrow & \downarrow p & \searrow q & \\ & & L & & K \\ & \text{qis} \swarrow & \downarrow s & \searrow f & \downarrow g \\ A & & & & B. \end{array}$$

Definition 1.3.3 (Roof composition). *Let A, B, C be chain complexes. Consider the left roofs $(f, s): A \rightarrow B$ and $(g, t): C \rightarrow D$ as the diagrams*

$$\begin{array}{ccc} & L & \\ s \swarrow & & \searrow f \\ A & & B, \end{array} \quad \begin{array}{ccc} & K & \\ t \swarrow & & \searrow g \\ B & & C. \end{array}$$

We define their **roof composition** as the roof obtained in the outer part (dashed maps) of the following commutative diagram over $\mathbf{K}(\mathcal{A})$

$$\begin{array}{ccccc} & & U & & \\ & \text{qis} \swarrow & \downarrow u & \searrow k & \\ & & L & & K \\ & \text{qis} \swarrow & \downarrow s & \searrow f & \downarrow t \\ A & & & & B. \end{array}$$

This above definition presents two issues: first, guaranteeing the existence of such a diagram, and second, ensuring the well-defined composition by equivalence classes of left roof morphisms. To deal with these issues, we introduce a short proposition.

Proposition 1.3.4. *Let $f: A \rightarrow B$ and $g: C \rightarrow B$ be chain maps with f a quasi-isomorphism. There there is a complex C_0 together with a quasi-isomorphism $C_0 \rightarrow C$ and a morphism $C_0 \rightarrow A$ such that the following diagram commutes in the homotopy category $\mathbf{K}(\mathcal{A})$.*

$$\begin{array}{ccc} C_0 & \xrightarrow{qis} & C \\ \downarrow & & \downarrow g \\ A & \xrightarrow[qis]{f} & B. \end{array} \quad (1.6)$$

Proof. We start by extending $A \xrightarrow{f} B$ to a distinguished triangle $A \xrightarrow{f} B \xrightarrow{\alpha} U \xrightarrow{\beta} A[1]$ where U corresponds to the cone of f . Rotating back we obtain the distinguished triangle

$$B \xrightarrow{\alpha} \text{Cone}(f) \xrightarrow{\beta} A[1] \xrightarrow{-f[1]} B[1].$$

On the other hand we can complete the composition map $C \xrightarrow{g} B \xrightarrow{\alpha}$ to a distinguished triangle $C \xrightarrow{g \circ \alpha} \text{Cone}(f) \xrightarrow{r} V \xrightarrow{u} C[1]$ where V corresponds to the cone of $g \circ \alpha$. These extensions to distinguished triangles can be seen in the following diagram

$$\begin{array}{ccccccc} C & \xrightarrow{g \circ \alpha} & \text{Cone}(f) & \xrightarrow{r} & V & \xrightarrow{u} & C[1] \\ g \downarrow & & 1 \downarrow & & v \downarrow & & \downarrow g[1] \\ B & \xrightarrow{\alpha} & \text{Cone}(f) & \xrightarrow{\beta} & A[1] & \xrightarrow{f[1]} & B[1]. \end{array}$$

where the left square is clearly commutative. Then, there exists a map $v: V \rightarrow A[1]$ (dashed arrow) such that the entire diagram commutes in the homotopy category. Rotating this diagram

$$\begin{array}{ccccccc} V[-1] & \xrightarrow{-u[-1]} & C & \xrightarrow{g \circ \alpha} & \text{Cone}(f) & \xrightarrow{r} & V \\ v[-1] \downarrow & & g \downarrow & & 1 \downarrow & & \downarrow v \\ A & \xrightarrow{f} & B & \xrightarrow{\alpha} & \text{Cone}(f) & \xrightarrow{\beta} & A[1]. \end{array}$$

We conclude by taking $C_0 = V[-1]$ and noticing that $-u[-1]: C_0 \rightarrow C$ is a quasi-isomorphism since $\text{Cone}(f)$ is acyclic. $\square$

Lemma 1.3.5. *Here are some basic properties between left roofs.*

- (1) *The relation defined between left roofs is an equivalence relation.*
- (2) *The composition rule given above is well defined on equivalence classes*
- (3) *Composition is associative.*

Proof. See [17, Categories, Lemma 27.2] □

For the following definition, consider $\mathcal{S}$ be the **class of quasi-isomorphisms** over the homotopy category $\mathbf{K}(\mathcal{A})$ and $\simeq$ the equivalence relation between left roofs.

Definition 1.3.6 (Derived category). *The **Derived category of $\mathcal{A}$** , denoted as $\mathbf{D}(\mathcal{A})$, is defined as the category which shares the same objects as $\mathbf{K}(\mathcal{A})$, and whose morphisms are given by equivalence classes of left roofs. Explicitly*

$$\text{Ob}(\mathbf{D}(\mathcal{A})) := \text{Ob}(\mathbf{K}(\mathcal{A})), \text{Hom}_{\mathbf{D}(\mathcal{A})}(A, B) := \left\{ \begin{array}{c} \begin{array}{ccc} & L & \\ s \swarrow & & \searrow f \\ A & & B \end{array} : s, f \in \text{Hom}_{\mathbf{K}(\mathcal{A})}(A, B), s \in \mathcal{S} \end{array} \right\} / \simeq.$$

In the above definition note that the identity map in $\text{Hom}(A, A)_{\mathbf{D}(\mathcal{A})}$ is given by the left roof $(1_A, 1_A): A \rightarrow A$, which is well-defined since 1_A is a quasi-isomorphism.

We now define a functor called **localization functor** $Q: \mathbf{K}(\mathcal{A}) \rightarrow \mathbf{D}(\mathcal{A})$, which acts as the identity on objects. For each morphism $f: A \rightarrow B$ in $\mathbf{K}(\mathcal{A})$, the image $Q(f)$ is the equivalence class of the left roof $(f, 1_A): A \rightarrow B$. This assignment ensures that $Q(f) = (f, 1_A)$ has a well-defined inverse $(1_A, f): B \rightarrow A$ whenever f is a quasi-isomorphism, so Q sends quasi-isomorphisms of $\mathbf{K}(\mathcal{A})$ into isomorphisms in $\mathbf{D}(\mathcal{A})$. Furthermore, any map $\psi: A \rightarrow B$ in $\mathbf{D}(\mathcal{A})$ represented by the left roof

$$\begin{array}{ccc} & U & \\ s \swarrow & & \searrow f \\ A & & B, \end{array}$$

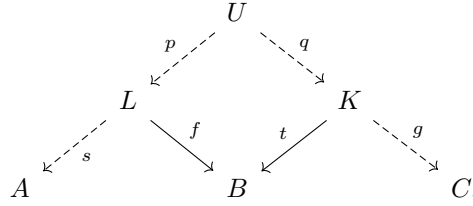
can be expressed as $\psi = Q(f) \circ Q(s)^{-1}$. The derived category together with the localization functor satisfy the following universal property.

Proposition 1.3.7 (Universal property of derived category). *Let $\mathbf{D}(\mathcal{A})$ be the derived category of $\mathcal{A}$. If given another functor $G: \mathbf{K}(\mathcal{A}) \rightarrow \mathcal{D}$ such that for every $s \in \mathcal{S}$ the image $G(s)$ is an isomorphism in $\mathcal{D}$, then there exists a unique functor $\tilde{G}: \mathbf{D}(\mathcal{A}) \rightarrow \mathcal{D}$ such that $\tilde{G} \circ Q = G$. That is, the following diagram commutes.*

$$\begin{array}{ccc} \mathbf{K}(\mathcal{A}) & \xrightarrow{Q} & \mathbf{D}(\mathcal{A}) \\ & \searrow G & \swarrow \tilde{G} \\ & \mathcal{D} & \end{array}$$

Proof. Let $(f, s): A \rightarrow B$ be a left roof map in $\mathbf{D}(\mathcal{A})$. Define $\tilde{G}$ as the identity on objects and $\tilde{G}(f, s) = G(f) \circ G(s)^{-1}$ on maps. Clearly $\tilde{G} \circ Q = G$ and $\tilde{G}$ maps the identity morphism to the identity morphism. It remains to check that G preserves composition and equivalent left roofs. Let $(g, t): B \rightarrow C$ be a left roof map in $\mathbf{D}(\mathcal{A})$. The composition $(g, t) \circ (f, s): A \rightarrow C$ is given by the

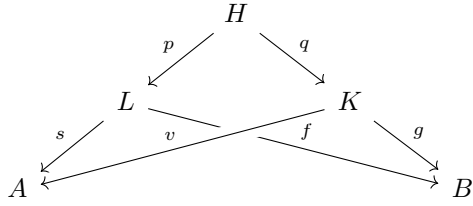
following commutative diagram



where s and p are quasi-isomorphisms. Then we have

$$\begin{aligned}\tilde{G}((g, t) \circ (f, s)) &= G(g \circ q) \circ G(s \circ p)^{-1} \\ &= G(g) \circ G(q) \circ G(p)^{-1} \circ G(s)^{-1} \\ &= G(g) \circ G(t)^{-1} \circ G(f) \circ G(s)^{-1} = \tilde{G}(g, t) \circ \tilde{G}(f, s).\end{aligned}$$

Let $(h, v): A \rightarrow B$ be left roof map in $\mathbf{D}(\mathcal{A})$ equivalent to $(f, s): A \rightarrow B$. Then there exists an commutative diagram



where s and p are quasi-isomorphisms. Then we have

$$\begin{aligned}\tilde{G}(g, v) &= G(g) \circ G(v)^{-1} \\ &= G(g) \circ G(q) \circ G(p)^{-1} \circ G(v)^{-1} \\ &= G(f) \circ G(s)^{-1} = \tilde{G}(f, s).\end{aligned}$$

The uniqueness can be easily checked from the fact that every map $\psi = (f, s): A \rightarrow B$ can be written as $\psi = Q(f) \circ Q(s)^{-1}$. $\square$

Remark 1.3.8 (Localization of categories, general case). *The derived category, in which morphisms are constructed using left roofs, is an instance of what is more generally known as the **localization of categories**. More precisely, given a category $\mathcal{C}$ and a **multiplicative system** S , we can define the localized category, denoted by $S^{-1}\mathcal{C}$, along with a functor $Q: \mathcal{C} \rightarrow S^{-1}\mathcal{C}$, such that every element of S becomes invertible in $S^{-1}\mathcal{C}$. For further details see [17, Localization in categories].*

Now we establish a triangulated structure on the derived category, and for doing so, we first define a translation functor. We start by considering the translation functor $T = [1]: \mathbf{K}(\mathcal{A}) \rightarrow \mathbf{K}(\mathcal{A})$ over the homotopy category and set $F := Q \circ T: \mathbf{K}(\mathcal{A}) \rightarrow \mathbf{D}(\mathcal{A})$ defined as $A \mapsto Q(A[1])$. Clearly for every quasi-isomorphism $s \in S$ we know $s[1] \in S$, so $F(s) := Q(s[1])$ is an isomorphism for every $s \in S$. Thus by the universal property of the derived category, there exists a unique functor $\mathbf{D}(\mathcal{A}) \rightarrow \mathbf{D}(\mathcal{A})$

which we label as $T_{\mathbf{D}(\mathcal{A})}$ such that $T_{\mathbf{D}(\mathcal{A})} \circ Q = F$.

$$\begin{array}{ccc} \mathbf{K}(\mathcal{A}) & \xrightarrow{T=[1]} & \mathbf{K}(\mathcal{A}) \\ Q \downarrow & & \downarrow Q \\ \mathbf{D}(\mathcal{A}) & \xrightarrow{T_{\mathbf{D}(\mathcal{A})}} & \mathbf{D}(\mathcal{A}). \end{array}$$

Similarly, we can consider the composition $Q \circ T^{-1}: \mathbf{K}(\mathcal{A}) \rightarrow \mathbf{D}(\mathcal{A})$ and apply the same universal property to get a functor which is inverse to $T_{\mathbf{D}(\mathcal{A})}$, then $T_{\mathbf{D}(\mathcal{A})}$ is an automorphism over the derived category which we will call **induced translator functor** and denoted again by $[1]: \mathbf{D}(\mathcal{A}) \rightarrow \mathbf{D}(\mathcal{A})$.

Definition 1.3.9 (Distinguished triangles on derived category). *We define a triangle in $\mathbf{D}(\mathcal{A})$ to be distinguished if it is isomorphic to the image of a distinguished triangle in $\mathbf{K}(\mathcal{A})$.*

Theorem 1.3.10. *The induced translator functor $T_{\mathbf{D}(\mathcal{A})}: \mathbf{D}(\mathcal{A}) \rightarrow \mathbf{D}(\mathcal{A})$, along with the definition of distinguished triangle induces a triangulated structure in $\mathbf{D}(\mathcal{A})$. Under this triangulated structure, the localization functor $Q: \mathbf{K}(\mathcal{A}) \rightarrow \mathbf{D}(\mathcal{A})$ is exact.*

Proof. See [17, Derived categories, Proposition 5.6] □

The following proposition allows us to extend functors defined on the homotopy category $\mathbf{K}(\mathcal{A})$ to the derived category $\mathbf{D}(\mathcal{A})$.

Proposition 1.3.11. *Let $\mathcal{D}$ be a triangulated category and $\mathcal{B}$ be an abelian category. Let $Q: \mathbf{K}(\mathcal{A}) \rightarrow \mathbf{D}(\mathcal{A})$ be the localization functor.*

- (1) *If $F: \mathbf{K}(\mathcal{A}) \rightarrow \mathcal{D}$ is an exact functor such that $F(s)$ is an isomorphism in $\mathcal{D}$ for every quasi-isomorphism $s \in \mathcal{S}$. Then there exists a unique exact functor $\tilde{F}: \mathbf{D}(\mathcal{A}) \rightarrow \mathcal{D}$ such that $\tilde{F} \circ Q = F$.*
- (2) *If $G: \mathcal{C} \rightarrow \mathcal{B}$ is a homological functor such that $G(s)$ is an isomorphism for every quasi-isomorphism $s \in \mathcal{S}$. Then there exists a unique homological functor $\tilde{G}: \mathbf{D}(\mathcal{A}) \rightarrow \mathcal{B}$ such that $\tilde{G} \circ Q = G$.*

Recall the i -th homological functor $H^i(-): \mathbf{K}(\mathcal{A}) \rightarrow \mathcal{A}$ defined in Remark 1.2.17 which clearly sends quasi-isomorphisms into isomorphisms in $\mathcal{A}$ then by the item (2) of the above proposition there exists a functor

$$H^i(-): \mathbf{D}(\mathcal{A}) \rightarrow \mathcal{A}$$

which coincides with $H^i(-): \mathbf{K}(\mathcal{A}) \rightarrow \mathcal{A}$ on the image of Q . To keep everything organized, the following diagram shows the categories and the homology functors defined so far.

$$\begin{array}{ccccccc} \mathcal{A} & \xrightarrow{I} & \mathbf{C}(\mathcal{A}) & \xrightarrow{C} & \mathbf{K}(\mathcal{A}) & \xrightarrow{Q} & \mathbf{D}(\mathcal{A}) \\ & & & \searrow & \searrow & \searrow & \downarrow H^i \\ & & & & & & \mathcal{A}. \end{array}$$

Theorem 1.3.12. *The derived category $\mathbf{D}(\mathcal{A})$ contains $\mathcal{A}$ as full subcategory.*

Recall that we define for $* = +, -, b$ the subcategories $\mathbf{C}^*(\mathcal{A})$ of chain complexes which induce the subcategories $\mathbf{K}^*(\mathcal{A})$ over the homotopy category. To each subcategory $\mathbf{K}^*(\mathcal{A})$ we can define the class S^* of quasi-isomorphism between bounded below, bounded above and bounded respectively. Define the **bounded derived categories**, denoted as $\mathbf{D}^*(\mathcal{A})$, by replacing $\mathbf{K}(\mathcal{A})$ and S , for $\mathbf{K}^*(\mathcal{A})$ and S^* respectively. These subcategories have equipped a localization functor $Q: \mathbf{K}^*(\mathcal{A}) \rightarrow \mathbf{D}^*(\mathcal{A})$. Together the subcategories $\mathbf{D}^*(\mathcal{A})$ with the localization functor satisfy the following universal property. There is a natural forgetful functor

$$\mathbf{D}^*(\mathcal{A}) \longrightarrow \mathbf{D}(\mathcal{A}), \quad (1.7)$$

which just forgets boundedness condition.

Remark 1.3.13 (Universal property of bounded derived categories). *The full subcategories $\mathbf{D}^*(\mathcal{A})$ with $* = +, -, b$ come with an equipped functor $Q: \mathbf{K}^*(\mathcal{A}) \rightarrow \mathbf{D}^*(\mathcal{A})$ such that for every $s \in S^*$ the image $Q(s)$ is an isomorphism in $\mathbf{D}^*(\mathcal{A})$ and if a functor $F: \mathbf{K}^*(\mathcal{A}) \rightarrow \mathcal{D}$ sends quasi-isomorphisms in S^* to isomorphisms, then there is a unique functor $\tilde{F}: \mathbf{D}^*(\mathcal{A}) \rightarrow \mathcal{D}$ such that $\tilde{F} \circ Q = F$, that is, the following diagram commutes*

$$\begin{array}{ccc} \mathbf{K}^*(\mathcal{A}) & \xrightarrow{Q} & \mathbf{D}^*(\mathcal{A}) \\ & \searrow F & \swarrow \tilde{F} \\ & \mathcal{D} & \end{array}$$

Lemma 1.3.14. *Let K be a complex.*

1. *If $H^i(K) = 0$ for all $i \ll 0$, then there exists a quasi-isomorphism $A \rightarrow L$ with L bounded below.*
2. *If $H^i(K) = 0$ for all $i \gg 0$, then there exists a quasi-isomorphism $M \rightarrow K$ with M bounded above.*
3. *If $H^i(K) = 0$ for all $|i| \gg 0$, then there exists a commutative diagram of complexes*

$$\begin{array}{ccc} K & \longrightarrow & L \\ \uparrow & & \uparrow \\ M & \longrightarrow & N, \end{array}$$

where all the arrows are quasi-isomorphisms, L bounded below, M bounded above, and N a bounded complex.

Proof. See [17, Lemma 13.11.5]. □

Proposition 1.3.15. *The natural forgetful functor $\mathbf{D}^*(\mathcal{A}) \rightarrow \mathbf{D}(\mathcal{A})$ with $* = +, -, b$, gives an equivalence of $\mathbf{D}^*(\mathcal{A})$ with the full triangulated subcategories of all complexes $A \in \mathbf{D}(\mathcal{A})$ with $H^i(A) = 0$ for $i \ll 0$, for $i \gg 0$, or for $|i| \gg 0$, respectively.*

Proof. Let A and B be bounded above complexes such that $H^i(A) = H^i(B) = 0$ for all $i \gg 0$. Let $(f, s): A \rightarrow B$ be a map in $\mathbf{D}(\mathcal{A})$ with $f: A \rightarrow B$ and a quasi-isomorphism $s: B \rightarrow A$. This implies

that L has bounded above homology. By the Lemma 1.3.14 we can choose a quasi-isomorphism $s': C \rightarrow L$ with C bounded above. The pair $(f \circ s', s \circ s'): A \rightarrow B$ define a map in $\mathbf{D}^-(\mathcal{A})$. Hence the functor is full.

For the faithfulness let A and B be bounded above complexes and $(f, s): A \rightarrow B$ be a map in $\mathbf{D}^*(\mathcal{A})$ which is zero in $\mathbf{D}(\mathcal{A})$, with $f: L \rightarrow B$ and a quasi-isomorphism $s: L \rightarrow A$. This means that there exists a quasi-isomorphism $s': L' \rightarrow L$ such that $f \circ s' = 0$. By Lemma 1.3.14 we can choose a quasi-isomorphism $s'': L'' \rightarrow L'$ with L'' bounded above. Easily we can check that the maps (f, s) and $(f \circ s' \circ s'', s \circ s' \circ s'')$ are equivalent, now since $f \circ s' \circ s'' = 0$ we have that (f, s) is zero in $\mathbf{D}^-(\mathcal{A})$. Hence the functor is faithful. $\square$

Remark 1.3.16. *Characterizing the derived category $\mathbf{D}(\mathcal{A})$ of an abelian category $\mathcal{A}$ is generally a difficult and non-trivial task. However, when $\mathcal{A}$ is **semisimple**, we can provide a complete description of it as $\mathbf{D}(\mathcal{A}) = \prod_{\mathbb{Z}} \mathcal{A}$. For further details see [8, III, 3].*

1.3.2 Verdier Quotient. Given a triangulated category and a triangulated subcategory we can construct another triangulated category by taking the quotient, by means of localization. We start by defining the concept of kernel of a functor.

Definition 1.3.17 (Thick categories). *Let $\mathcal{D}$ be a triangulated category. We say a full triangulated subcategory $\mathcal{D}'$ of $\mathcal{D}$ is **thick** whenever $X \oplus Y$ is isomorphic to an object in $\mathcal{D}'$ then both X and Y are isomorphic to objects of $\mathcal{D}'$.*

Definition 1.3.18 (Kernel of Functors). *Let $\mathcal{D}$ and $\mathcal{D}'$ be triangulated categories and $\mathcal{A}$ be an abelian category.*

(1) *Let $F: \mathcal{D} \rightarrow \mathcal{D}'$ be an exact functor. The subcategory, denoted by $\ker F$, consists of objects*

$$\text{Ob}(\ker F) = \{X \in \mathcal{D} \mid F(X) = 0\}.$$

*This subcategory is called the **kernel of F** .*

(2) *Let $H: \mathcal{D} \rightarrow \mathcal{A}$ be a homological functor. The subcategory, denoted by $\ker H$, consists of objects*

$$\text{Ob}(\ker(H)) = \{X \in \mathcal{D} \mid H(X[i]) = 0 \text{ for all } i\}.$$

*This subcategory is called the **kernel of H** .*

Lemma 1.3.19.

(1) *Let $F: \mathcal{D} \rightarrow \mathcal{D}'$ be an exact functor. Then the subcategory $\ker F$ is a thick.*

(2) *Let $H: \mathcal{D} \rightarrow \mathcal{A}$ be an homological functor. Then the subcategory $\ker H$ is a thick.*

Proof. See [17, Lemma 13.6.2] and [17, Lemma 13.6.3]. $\square$

For any exact functor $F: \mathcal{D} \rightarrow \mathcal{D}'$ we want to define the **quotient category** $\mathcal{D}/\ker F$. We start by defining $\mathcal{D}/\mathcal{B}$ to any thick subcategory $\mathcal{B} \subseteq \mathcal{D}$.

As mentioned earlier, the derived category is constructed through *localization of categories*, which formalizes the process of defining inverses for some special class of maps. The derived category was constructed as the left roof category with respect to quasi-isomorphisms chain maps. In this case, the quotient category will be defined analogously, as the category of left roofs, but with the respect to the set $\tilde{S}$ defined above. In the general framework of localization of triangulated categories (Remark 1.3.8), the so-called *compatible* maps, referred to as multiplicative systems (see [17, Definition 4.27.1]), must satisfy certain specific axioms. Here, we omit the details.

Lemma 1.3.20. *Let $\mathcal{D}$ be a triangulated category and $\mathcal{B} \subseteq \mathcal{D}$ be a thick triangulated subcategory. The induced set of morphisms*

$$\tilde{S} = \left\{ \begin{array}{l} f \in \text{Hom}_{\mathcal{D}}(X, Y) \text{ such that there exists a distinguished triangle} \\ (X, Y, Z, f, g, h) \text{ of } \mathcal{D} \text{ with } Z \text{ isomorphic to an object of } \mathcal{B} \end{array} \right\},$$

are 'compatible' in the sense of Definition [17, Definition 4.27.1].

Definition 1.3.21 (Quotient category). *Let $\mathcal{D}$ be a triangulated category. Let $\mathcal{B}$ be a full triangulated subcategory. We define the **quotient category**, denoted as $\mathcal{D}/\mathcal{B}$, as the localization of $\mathcal{D}$ with respect to the class of maps $\tilde{S}$ associated to $\mathcal{B}$ via the Lemma 1.3.20.*

The localization functor $Q: \mathcal{D} \rightarrow \mathcal{D}/\mathcal{B}$ is called the **quotient functor** in this case. The quotient category satisfies the following universal property: If $F: \mathcal{D} \rightarrow \mathcal{D}'$ is an exact functor such that $\mathcal{B} \subseteq \ker(F)$, then there exists a unique factorization $\tilde{F}: \mathcal{D}/\mathcal{B} \rightarrow \mathcal{D}'$ such that $F = \tilde{F} \circ Q$, that is, the following diagram commutes

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{Q} & \mathcal{D}/\mathcal{B} \\ & \searrow F & \swarrow \tilde{F} \\ & \mathcal{D}' & \end{array}$$

Remark 1.3.22 (Calculating quotients by the kernel). *In this remark we will write down the compatible systems obtained from a exact and homological functors.*

- Let $F: \mathcal{D} \rightarrow \mathcal{D}'$ to be an exact functor. We can take $\mathcal{B} = \ker F$ and define the quotient category $\mathcal{D}/\mathcal{B}$ as the localization of $\mathcal{D}$ with respect to

$$\tilde{S} = \left\{ \begin{array}{l} f \in \text{Mor}(\mathcal{D}) \text{ such that there exists a distinguished triangle} \\ (X, Y, Z, f, g, h) \text{ of } \mathcal{D} \text{ with } F(Z) = 0 \end{array} \right\}.$$

- Let $H: \mathcal{D} \rightarrow \mathcal{A}$ a homological functor. We can take $\mathcal{B} = \ker H$ and define the quotient category $\mathcal{D}/\mathcal{B}$ as the left roof category with respect to

$$\tilde{S} = \left\{ \begin{array}{l} f \in \text{Mor}(\mathcal{D}) \text{ such that there exists a distinguished triangle} \\ (X, Y, Z, f, g, h) \text{ of } \mathcal{D} \text{ with } H(Z[i]) = 0 \text{ for all } i \end{array} \right\}.$$

An special case of item 2 in the preceded Remark is to take the homology functor $H^0: \mathbf{K}(\mathcal{A}) \rightarrow \mathcal{B}$ and then define the category $\mathbf{K}(\mathcal{A})/\ker H^0$ where $\ker H^0(-) = \mathbf{Ac}(\mathcal{A})$ is the full subcategory consisting of acyclic complexes of $\mathbf{K}(\mathcal{A})$, with the compatible system

$$\tilde{\mathcal{S}} = \left\{ \begin{array}{l} f \in \text{Mor}(\mathcal{D}) \text{ such that there exists a distinguished triangle} \\ (X, Y, Z, f, g, h) \text{ of } \mathcal{D} \text{ with } H^i(Z) = 0 \text{ for all } i \end{array} \right\},$$

as Z correspond to $\text{Cone}(f)$ complex we conclude that the compatible maps $\tilde{\mathcal{S}}$ is exactly the class of quasi-isomorphisms $\mathcal{S}$. This provides an alternative definition of the derived category as a quotient category

$$\mathbf{D}(\mathcal{A}) = \mathbf{K}(\mathcal{A})/\mathbf{Ac}(\mathcal{A}).$$

1.3.3 Derived Functors. The aim of this subsection is to construct derived functors obtained from additive functors $\mathcal{A} \rightarrow \mathcal{B}$ between abelian categories. For reference we mainly follow [17, Derived categories] and [12].

We start with the most basic case which is to consider an additive functor $F: \mathcal{A} \rightarrow \mathcal{B}$ between additive categories. We can now apply F term-by-term to obtain an additive functor $\mathbf{C}(F): \mathbf{C}(\mathcal{A}) \rightarrow \mathbf{C}(\mathcal{B})$ between categories of chain complexes. This functor can be extended to a well-defined exact functor between homotopic categories since

- $\mathbf{C}(F)$ takes homotopic maps in $\mathbf{C}(\mathcal{A})$ into homotopic maps in $\mathbf{C}(\mathcal{B})$: Given homotopic chain morphisms $f, g: A \rightarrow B$ in $\mathbf{C}(\mathcal{A})$ via h we get that $F(f)$ is homotopic to $F(g)$ via $F(h)$ providing the desired and well-defined functor $\mathbf{K}(F): \mathbf{K}(\mathcal{A}) \rightarrow \mathbf{K}(\mathcal{B})$.
- The functor $\mathbf{K}(F)$ takes distinguished triangles in $\mathbf{K}(\mathcal{A})$ into distinguished triangles in $\mathbf{K}(\mathcal{B})$: This follows easily since $F(\text{Cone}(f)) = \text{Cone}(F(f))$.
- To conclude that $\mathbf{K}(F)$ is exact, it remains to show that it commutes with translation functors, which is straightforward.

Summarizing; taking an additive functor F between additive categories, give us an exact functor of triangulated categories $\mathbf{K}(F)$. Now we want to see if we can pass to the derived category.

Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. Consider $Q_{\mathcal{A}}: \mathbf{K}(\mathcal{A}) \rightarrow \mathbf{D}(\mathcal{B})$ and $Q_{\mathcal{B}}: \mathbf{K}(\mathcal{B}) \rightarrow \mathbf{D}(\mathcal{B})$ be the localization functors associated to the definitions of $\mathbf{D}(\mathcal{A})$ and $\mathbf{D}(\mathcal{B})$ respectively. One natural possible strategy to construct a functor between $\mathbf{D}(\mathcal{A})$ and $\mathbf{D}(\mathcal{B})$ is given as follows. If the functor $\mathbf{K}(F): \mathbf{K}(\mathcal{A}) \rightarrow \mathbf{K}(\mathcal{B})$ takes quasi-isomorphisms to quasi-isomorphisms (equivalently acyclic complexes to acyclic complexes), then the composition $Q_{\mathcal{B}} \circ \mathbf{K}(F): \mathbf{K}(\mathcal{A}) \rightarrow \mathbf{D}(\mathcal{B})$ sends quasi-isomorphisms in $\mathbf{K}(\mathcal{A})$ to isomorphisms in $\mathbf{D}(\mathcal{B})$, we can now apply the universal property of $\mathbf{D}(\mathcal{A})$ to get a functor (dashed arrow) $\mathbf{D}(\mathcal{A}) \rightarrow \mathbf{D}(\mathcal{B})$ between derived categories making the following

diagram commute

$$\begin{array}{ccc}
 \mathbf{K}(\mathcal{A}) & \xrightarrow{\mathbf{K}(F)} & \mathbf{K}(\mathcal{B}) \\
 Q_{\mathcal{A}} \downarrow & & \downarrow Q_{\mathcal{B}} \\
 \mathbf{D}(\mathcal{A}) & \dashrightarrow & \mathbf{D}(\mathcal{B}).
 \end{array} \tag{1.8}$$

A perhaps trivial case where we can get a derived functor via the universal property is when F is exact, as stated in the next proposition.

Proposition 1.3.23. *Let $F: \mathcal{A} \rightarrow \mathcal{B}$ be an exact functor of abelian categories. Then for any quasi-isomorphism s the image $\mathbf{K}(F)(s)$ is a quasi-isomorphism in $\mathbf{K}(\mathcal{B})$. Moreover, the functor F induces a functor $\mathbf{D}(\mathcal{A}) \rightarrow \mathbf{D}(\mathcal{B})$ as in diagram (1.8).*

Remark 1.3.24 (Induced bounded derived functor). *Using the universal property of the bounded derived categories of Remark 1.3.13, we can get derived functors $\mathbf{D}^*(\mathcal{A}) \rightarrow \mathbf{D}^*(\mathcal{B})$ with $*$ = +, −, b where $\mathbf{F}(\mathcal{A}): \mathbf{C}^*(\mathcal{A}) \rightarrow \mathbf{C}^*(\mathcal{B})$ and $\mathbf{K}(F): \mathbf{K}^*(\mathcal{A}) \rightarrow \mathbf{K}^*(\mathcal{B})$ are defined by applying F to each complex degree as before.*

So far, we have discussed the possible construction of a derived functor, that is, a functor between derived categories arising from an additive (potentially exact) functor between abelian categories. Now, we will focus on the case where an additive functor $F: \mathcal{A} \rightarrow \mathcal{B}$ between categories of complexes is left exact to construct *the right derived functor*

$$RF: \mathbf{D}^+(\mathcal{A}) \longrightarrow \mathbf{D}^+(\mathcal{B}).$$

To achieve this, we will need to define injective objects along with their respective resolutions and some results from these. For the following let $\mathcal{A}$ be an abelian category.

Definition 1.3.25 (Injective objects, resolutions and enough injectives). *Let A and I be objects in $\mathcal{A}$.*

(1) *The object I is called **injective** if one of the following, equivalent statements is satisfied.*

- *For every monomorphism $X \xrightarrow{f} Y$ and every morphism $X \xrightarrow{g} I$ there exists a map $Y \xrightarrow{h} I$ making the following diagram commute*

$$\begin{array}{ccccc}
 0 & \longrightarrow & X & \xrightarrow{f} & Y \\
 & & g \downarrow & \nearrow h & \\
 & & I & &
 \end{array}$$

- *The Hom functor $\text{Hom}(-, I)$ is exact.*
- *Any short exact sequence of the form $0 \rightarrow I \rightarrow A \rightarrow B \rightarrow 0$ splits.*

(2) *We say that $\mathcal{A}$ has **enough injectives** if for any object X , there is an injective object I and a monomorphism $X \rightarrow I$.*

(3) *An **injective resolution** of A is an exact sequence commonly denoted by $0 \rightarrow A \rightarrow I^\bullet$, given by*

$$0 \longrightarrow A \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots,$$

where all I^i objects are injective. Clearly, $I^\bullet$ is a cyclic complex such that $I^i = 0$ for all $0 < i$.

Remark 1.3.26 (Injective resolutions as quasi-isomorphisms). Let $0 \rightarrow A \xrightarrow{\psi} I^\bullet$ be an injective resolution of A . Considering $A[0]$ as a complex concentrated in degree 0, the above injective resolution induces the chain map

$$\begin{array}{ccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & A & \longrightarrow & 0 \longrightarrow \cdots \\ & & \downarrow & & \downarrow \psi & & \downarrow \\ \cdots & \longrightarrow & 0 & \longrightarrow & I^0 & \longrightarrow & I^1 \longrightarrow \cdots \end{array}$$

We claim that this chain map is a quasi-isomorphism. It is straightforward to verify that $H^0(A[0]) = H^0(I) = A$ and $H^i(A[0]) = H^i(I) = 0$ for all $i \neq 0$. By definition, ψ is a monomorphism. Since the resolution is exact, we have $\text{im } \psi = \ker d_I^0$, meaning that ψ is an isomorphism from A onto $\ker d_I^0$. Therefore, ψ is a quasi-isomorphism.

Conversely, the argument also works in reverse, starting from a quasi-isomorphism $A[0] \rightarrow I$, we can construct an injective resolution of A .

The above Remark allows us to generalize the concept of injective resolution of objects in $\mathcal{A}$ to complexes in $\mathbf{C}(\mathcal{A})$

Definition 1.3.27 (Injective resolution of a complex). Let $A \in \mathbf{C}(\mathcal{A})$ to be a complex. An **injective resolution of the complex A** is a complex I together with a chain map $\alpha: A \rightarrow I$ such that I is bounded below, each I^i is an injective object in $\mathcal{A}$ and the map α is a quasi-isomorphism. In other words, we have the diagram

$$\begin{array}{ccccccc} \cdots & \longrightarrow & A^{i-1} & \longrightarrow & A^i & \longrightarrow & A^{i+1} \longrightarrow \cdots \\ & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \longrightarrow & 0 & \longrightarrow & I^i & \longrightarrow & I^{i+1} \longrightarrow \cdots \end{array}$$

We summarize several important results about injective objects and maps between them in the following theorem.

Theorem 1.3.28 (Tools over injective objects). Let $\mathcal{A}$ be an abelian category with enough injectives and let $I^\bullet$ be a bounded below complex such that I^i is an injective object for each i .

- (1) Let K be a bounded below acyclic complex. Then any morphism $f: K \rightarrow I$ is nullhomotopic.
- (2) Let $f: A \rightarrow B$ be a quasi-isomorphism of chain complexes. Then the induced map obtained by precomposing with f is bijective. Therefore, we obtain the equation

$$\text{Hom}_{\mathbf{K}(\mathcal{A})}(B, I) \cong \text{Hom}_{\mathbf{K}(\mathcal{A})}(A, I).$$

- (3) Any two injective resolutions of a complexes are homotopically equivalent.

(4) Let $K \in \mathbf{K}(\mathcal{A})$. The induced Hom-map $\mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(L, I) \rightarrow \mathrm{Hom}_{\mathbf{D}(\mathcal{A})}(L, I)$ via the localization functor $Q: \mathbf{K}(\mathcal{A}) \rightarrow \mathbf{D}(\mathcal{A})$ is a bijection. Therefore, we obtain the equation

$$\mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(K, I) \cong \mathrm{Hom}_{\mathbf{D}(\mathcal{A})}(K, I).$$

Let $\mathcal{I}$ be the full **subcategory of injective objects** in $\mathcal{A}$. It can be shown that $\mathcal{I}$ is additive, though not necessarily abelian. However, recall that the construction of the homotopy category applies to any additive category. Thus, $\mathbf{K}(\mathcal{I})$ and $\mathbf{K}^*(\mathcal{I})$ with $*$ = +, −, b are well-defined as a triangulated categories. Now, note that the inclusion functor $\mathcal{I} \rightarrow \mathcal{A}$ induces a natural exact functor $\mathbf{K}^*(\mathcal{I}) \rightarrow \mathbf{K}^*(\mathcal{A})$, and by composing it with the localization functor, which is also exact we obtain an exact functor $\iota: \mathbf{K}^+(\mathcal{I}) \rightarrow \mathbf{D}^+(\mathcal{A})$. Moreover this functor induces an equivalence of triangulated categories as stated below.

Theorem 1.3.29. *The functor $Q: \mathbf{K}^+(\mathcal{A}) \rightarrow \mathbf{D}^+(\mathcal{A})$ induces an equivalence of categories $\iota: \mathbf{K}^+(\mathcal{I}) \rightarrow \mathbf{D}^+(\mathcal{A})$. The inverse ι^{-1} is defined by choosing a complex of injective objects quasi-isomorphic to a given bounded below complex in $\mathbf{D}^+(\mathcal{A})$.*

Proof. See [12, Proposition 2.40]. □

We can summarize all the functors described so far in this subsection to get the following diagram

$$\begin{array}{ccccc} \mathbf{K}^+(\mathcal{I}) & \xleftarrow{\sim} & \mathbf{K}^+(\mathcal{A}) & \xrightarrow{\mathbf{K}(F)} & \mathbf{K}^+(\mathcal{B}) \\ & \searrow \iota & \downarrow Q_{\mathcal{A}} & & \downarrow Q_{\mathcal{B}} \\ & \swarrow \iota^{-1} & \mathbf{D}^+(\mathcal{A}) & \dashrightarrow & \mathbf{D}^+(\mathcal{B}). \end{array} \quad (1.9)$$

Definition 1.3.30 (Right derived functor $\mathbf{R}$). *The **right derived functor** of a left exact functor $F: \mathcal{A} \rightarrow \mathcal{B}$ is defined as the functor*

$$RF := Q_{\mathcal{B}} \circ \mathbf{K}(F) \circ \iota^{-1}: \mathbf{D}^+(\mathcal{A}) \longrightarrow \mathbf{D}^+(\mathcal{B}).$$

Definition 1.3.31 (Higher right derived functor). *Let $RF: \mathbf{D}^+(\mathcal{A}) \rightarrow \mathbf{D}^+(\mathcal{B})$ be the right derived functor of an additive left exact functor $F: \mathcal{A} \rightarrow \mathcal{B}$ between abelian categories, where $\mathcal{A}$ is assumed to have enough injectives. For any object $X \in \mathcal{A}$, we define the **higher right derived functor** $R^i F: \mathcal{A} \rightarrow \mathcal{B}$ as $R^i F(X) := H^i(RF(X[0]))$.*

Remark 1.3.32 (Classical right derived functor). *In this Remark we explain the classical description of derived functors. Start with an additive left exact functor $F: \mathcal{A} \rightarrow \mathcal{B}$ between abelian categories. Let X be an object in $\mathcal{A}$. Consider the complex $X[0] \in \mathbf{D}^+(\mathcal{A})$ concentrated in 0 degree. In order to calculate $R^i F$ we follow the diagram (1.9). Since ι is an equivalence, we can get a bounded below complex I quasi-isomorphic to $X[0]$ and chasing through $Q_{\mathcal{B}} \circ \mathbf{K}(F)$ who acts like the identity over the complexes. This gives us $RF(X[0]) = F(I)$. Now it remains to take homology and we are done. Explicitly, $R^i F(X) = H^i(RF(X[0])) \cong H^i(F(I))$.*

Using Remark 1.3.26, we can equivalently choose an injective resolution $0 \rightarrow X \rightarrow I$ of X then apply F term-by-term, and then compute homology. This approach aligns with the classical definition of a derived functor.

Remark 1.3.33 (Left derived functors). We finish this subsection stating the construction of left derived functors. We start with an additive right exact functor $G: \mathcal{A} \rightarrow \mathcal{B}$, and then:

(1) *Projective objects and projective resolutions.*

(a) An object P is called **projective** if one of the following equivalent statements is satisfied.

- For every epimorphism $X \xrightarrow{f} Y$ and every morphism $P \xrightarrow{g} B$ there exists a map $P \xrightarrow{h} X$ such that the diagram commutes

$$\begin{array}{ccccc} & & P & & \\ & \swarrow h & \downarrow g & & \\ X & \xrightarrow{f} & Y & \longrightarrow & 0. \end{array}$$

- The Hom functor $\text{Hom}(P, -)$ is exact.
- Any short exact sequence of the form $0 \rightarrow A \rightarrow B \rightarrow P \rightarrow 0$ splits.

(b) $\mathcal{A}$ has **enough projectives** if for any object X , there is projective object P and a epimorphism $P \rightarrow X$.

(c) A **projective resolution** of A is an exact sequence commonly denoted by $P \rightarrow B \rightarrow 0$

$$\dots \longrightarrow P^{-1} \longrightarrow P^0 \longrightarrow B \longrightarrow 0.$$

Analogously to the Remark 1.3.26, a projective resolution $P \rightarrow B \rightarrow 0$ is equivalent to have a quasi-isomorphism $P \xrightarrow{\varepsilon} B[0]$.

(2) We state a dual statement of Theorem 1.3.28 to get the tools over projective objects. Consider $\mathcal{A}$ to have enough projectives and P to be a bounded above complex such that P^i is a projective object, for all $i < 0$.

- (a) Let K to be a bounded above acyclic complex. Then any morphism $g: P \rightarrow K$ is nullhomotopic.
- (b) $\text{Hom}_{\mathbf{K}(\mathcal{A})}(P, A) = \text{Hom}_{\mathbf{K}(\mathcal{A})}(P, B)$.
- (c) $\text{Hom}_{\mathbf{K}(\mathcal{A})}(P, K) = \text{Hom}_{\mathbf{D}(\mathcal{A})}(P, K)$.

(3) Define the **full subcategory of projective objects** $\mathcal{P}$ and their corresponding (bounded) homotopy categories $\mathbf{K}(\mathcal{P})$ and $\mathbf{K}^*(\mathcal{P})$ with $*$ = +, −, b. The composition of the functors

$$j: \mathbf{K}^-(\mathcal{P}) \rightarrow \mathbf{K}^-(\mathcal{A}) \rightarrow \mathbf{D}^-(\mathcal{A}),$$

induces an equivalence of triangulated categories. Which the inverse j^{-1} is defined by choosing a complex of projective objects quasi-isomorphic to a given above complex in $\mathbf{D}^-(\mathcal{A})$.

(4) We define the **left derived functor** as the functor composition

$$LG := Q_{\mathcal{B}} \circ \mathbf{K}(G) \circ j^{-1}: \mathbf{D}^-(\mathcal{A}) \rightarrow \mathbf{D}^-(\mathcal{B}).$$

(5) We get the classical definition of left derived functor from the **higher left derived functor** $L_i G: \mathcal{A} \rightarrow \mathcal{B}$ defined as $L_i F(X) := H^{-i}(LF(X[0]))$. Note that the negative sign taking homology is because the projective complex is bounded above.

The following theorem includes properties of right (resp. left) functors and one of the most important results about the right (resp. left) derived functors; they fix the lack of exactness of the functor we are deriving, by inducing a long exact sequence in homology from any given short exact sequence.

Theorem 1.3.34. *Let $F: \mathcal{A} \rightarrow \mathcal{B}$ be an additive functor with $\mathcal{A}$ having enough injectives.*

- (1) $R^i F = 0$ for $i < 0$.
- (2) $R^0 F$ is left exact.
- (3) There exists a natural transformation $F \Rightarrow R^0 F$. Moreover, this natural transformation is an isomorphism of categories, if and only if, F is left exact.
- (4) For any short exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ in $\mathcal{A}$ there is a long exact sequence

$$0 \longrightarrow R^0 F(A) \longrightarrow R^0 F(B) \longrightarrow R^0 F(C) \longrightarrow R^1 F(A) \longrightarrow R^1 F(B) \longrightarrow \cdots.$$

Analogously, let $G: \mathcal{A} \rightarrow \mathcal{B}$ be an additive functor with $\mathcal{A}$ having enough projectives.

- (1) $L_i G = 0$ for $i > 0$.
- (2) $L_0 G$ is right exact.
- (3) There exists a natural transformation $G \Rightarrow L_0 G$. Moreover, this natural transformation is an isomorphism of categories, if and only if, G is right exact.
- (4) For any short exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ in $\mathcal{A}$ there is a long exact sequence

$$\cdots \longrightarrow L_1 G(B) \longrightarrow L_1 G(C) \longrightarrow L_0 G(A) \longrightarrow L_0 G(B) \longrightarrow L_1 G(C) \longrightarrow 0.$$

An important observation is that contravariant derived functors can be computed using a technical adjustment. Specifically, for any contravariant left (resp. right) additive exact functor $F: \mathcal{A} \rightarrow \mathcal{B}$ between abelian categories, we can consider F as a covariant functor $F: \mathcal{A}^{op} \rightarrow \mathcal{B}$. This allows us to directly apply the theory described above.

We end this subsection introducing two examples of derived functors.

Example 1.3.35 (Ext functors). *For any $A \in \mathcal{A}$ and consider the covariant left exact functor*

$$\mathrm{Hom}_{\mathcal{A}}(A, -): \mathcal{A} \rightarrow \mathbf{Ab}.$$

*This functor has a right derived functor $R\mathrm{Hom}_{\mathcal{A}}(A, -): \mathbf{D}^+(\mathcal{A}) \rightarrow \mathbf{D}^+(\mathbf{Ab})$. From these we define the **Ext functor** as*

$$\mathrm{Ext}^i(A, -) := R^i \mathrm{Hom}_{\mathcal{A}}(A, -).$$

Example 1.3.36 (Tor functors). *Let R to be a commutative ring and let $\mathbf{R}\text{-Mod}$ to be the category of R -modules. The category $\mathbf{R}\text{-Mod}$ has enough injectives and projective objectives. Fix a R -module A and consider the covariant right exact functor*

$$A \otimes_R -: \mathbf{R}\text{-Mod} \rightarrow \mathbf{R}\text{-Mod}.$$

*This functor has a left derived functor $A \otimes_R -: \mathbf{D}^-(\mathbf{R}) \rightarrow \mathbf{D}^-(\mathbf{R})$. From these we define the **Tor functor** as*

$$\mathrm{Tor}_i(A, -) := L_i(A \otimes_R -).$$

Chapter 2

The BGG Correspondence

In this chapter, we provide a detailed proof of the *BGG correspondence*. We begin by describing the categories of graded modules over exterior and symmetric algebras, highlighting their key properties. Next, we introduce the complexes over these algebras and define the necessary functors to establish the adjointness between these categories. We include the necessary results to extend the correspondence to an equivalence of derived categories. For references, we primarily follow Appendix A of [14], which supplements the original work [3], alongside with [6].

2.1 Modules and complexes over the exterior and symmetric algebras

Let k be an algebraically closed field of characteristic zero. For our purposes, we may take k be the field of complex numbers. Let V be an $(n + 1)$ -dimensional vector space over k , and let $W := V^*$ denote its dual space. We assign a grading to V and W by specifying that elements of V have degree -1 , while elements of W have degree 1 . Additionally let $\{e_i\}_{i=0}^n$ and $\{w_i\}_{i=0}^n$ be basis for V and W respectively.

Define $E = E(V)$ and $S = S(W)$ be the exterior and symmetric k -algebras of V and W , respectively. To simplify notation, we omit the wedge symbol $\wedge$ when writing the product of elements in E , unless it is necessary for clarity. For precise definitions and further details on these algebras, we refer the reader to Appendix A. Since the elements of V have degree -1 , the exterior algebra E inherits a negative grading: $E = \bigoplus_{i=0}^{n+1} E_{-i}$. In contrast, the symmetric algebra S has the standard positive grading: $S = \bigoplus_{i=0}^{\infty} S_i$.

We denote by **E-grMod** the category of graded modules over E whose maps are degree preserving homomorphisms. Additionally we set **E-grmod** to denote the full subcategory consisting of finitely generated graded E -modules. Similarly, we denote by **S-grMod** the category of graded modules over S , and by **S-grmod** the full subcategory of finitely generated graded S -modules. Given a graded module $V = \bigoplus_{j \in \mathbb{Z}} V_j$ over E or S , we refer to V_j as the j -th homogeneous component of V . Since k is contained of both E and S , every graded module over E and S can be viewed as a k -vector space

by restriction of scalars. Moreover, the algebras E and S itself are k -vector spaces.

Due to the properties of E , we may characterize the category **E-grmod**, for that, recall that each homogeneous component E_{-i} , as k -vector space has dimension $\dim_k E_{-i} = \binom{n+1}{i}$ and thus the total dimension of E is $\dim_k E = 2^{n+1}$. Therefore, E is a finite dimensional k -vector space and the finitely generated E -modules coincide with finite dimensional modules (taken as k -vector spaces) as we prove in the following lemma.

Lemma 2.1.1. *Let P be a graded E -module. Then P is finitely generated as E -module if and only if P is finite dimensional as a k -vector space.*

Proof. Assume that P is a finite dimensional k -vector space, then there exists a basis $\{p_0, \dots, p_r\} \subseteq P$. Thus every element $p \in P$ can be uniquely expressed in the form $p = \sum_{i=1}^r \alpha_i p_i$. This shows that P is generated by $\{p_0, \dots, p_r\}$ over E , and hence P is finitely generated as E -module.

For the reverse direction, assume that P is finitely generated as E -module. Then there exists a finite set of generators $\{p_0, \dots, p_r\} \subseteq P$ such that every element $p \in P$ can be written as $p = \sum_{i=1}^r \tilde{e}_i p_i$ with $\tilde{e}_i \in E$. A basis for E as k -vector space is given by all wedge products of the basis elements of V , specifically

$$\{1, e_i, e_i \wedge e_j, e_i \wedge e_j \wedge e_k, \dots, e_0 \wedge e_1 \wedge \dots \wedge e_n\},$$

where $0 \leq i < j < k \dots \leq n$. Now each $\tilde{e}_i$ can be uniquely expressed as

$$\tilde{e}_j = \sum_{S \subseteq \{0,1,\dots,n\}} \alpha_S \left(\bigwedge_{j \in S} e_j \right), \quad \alpha_S \in k.$$

Replacing this into the expression for p , we get that p is k -linear combination of the set

$$\left\{ \left(\bigwedge_{j \in S} e_j \right) p_i : 0 \leq i \leq r, S \subseteq \{0, 1, \dots, n\} \right\}.$$

Since this set is finite and the product $\bigwedge_{j \in S} e_j$ depends either on p_i or on p , then P is finite dimensional. $\square$

This characterization applies only to k -algebras that are finitely generated as k -vector spaces. Since the symmetric algebra $S = \bigoplus_{i=0}^{\infty} S_i$ has an infinitely many graded components, we cannot describe the category **S-grmod** in the same way.

2.1.1 Properties of E-grMod. In this subsection we will introduce some basic definitions, constructions and properties over the category of graded E -modules.

Definition 2.1.2 (Basic definitions and operations of **E-grMod**).

- (a) Let $\mathbb{1}$ be the **trivial one dimensional module**, that is, $\dim_k(\mathbb{1}_0) = 1$ as k -vector space and $\mathbb{1}_i = \{0\}$ for $i \neq 0$.

- (b) For any integer r , we define the **shifted graded module** $P(r)$ by setting $P(r)_j := P_{j+r}$ for all j . It is straightforward to verify that $P(r)(s) = P(r+s)$ for any integers r and s .
- (c) We say that P is a **free module** in the category **E-grMod** if $P = \bigoplus_i E(m_i)$ for some integers m_i . A trivial example is the exterior algebra E as a graded E -module with a single generator of degree 0. Moreover, $E(r)$ corresponds to the free module over E with a single generator of homogeneous degree $-r$.
- (d) We define the **dual module** of P as $P^* = \bigoplus_i P_i^*$ where $P_i^* := \text{Hom}_k(P_{-i}, k)$. Note that this dual is taken over k . The dual module P^* is a right graded E -module with the action given by

$$\begin{aligned} \text{Hom}_k(P_{-i}, k) \times E_{-j} &\longrightarrow \text{Hom}_k(P_{j-i}, k) \\ (f, e) &\longmapsto fe: P_{j-i} \rightarrow k \\ p &\mapsto f(ep). \end{aligned}$$

- (e) Given a left E -module P , we define the **associated graded right E -module** denoted by P^r with the action given by $p \cdot e := (-1)^{\alpha\beta} ep$ with $p \in P_\alpha$ and $e \in E_{-\beta}$.

We denote by **grMod-E** the category of right graded E -modules. It is straightforward to verify that the assignment $P \mapsto P^r$ defines a functor

$$(-)^r: \mathbf{E}\text{-grMod} \longrightarrow \mathbf{grMod-E},$$

from the category of left graded E -modules to the category of right graded E -modules. Moreover, this functor is an equivalence of categories.

- (f) We can combine the last two items to define a left E -module action on the dual P^* of any graded module P . Specifically, the action is given by

$$(ef)(p) := (-1)^{\alpha\beta}(fe)(p) = (-1)^{\alpha\beta}f(ep) \quad \text{with } p \in P_\alpha \text{ and } e \in E_{-\beta}.$$

In the particular case of the left module E over itself, we can consider its dual E^* equipped with a left E -module action, instead of the natural right action given above. This works because elements of E can multiply from both sides. Explicitly, the left-action is given by $\tilde{e}f(e) := f(e \wedge \tilde{e})$ with $e, \tilde{e} \in E$ and $f \in E^*$.

- (g) Let $P \in \mathbf{E}\text{-grmod}$ be a finitely generated graded E -module, and $v \in V \subseteq E$. Since v is in homogeneous degree -1 , the multiplication maps $P_j \rightarrow P_{j-1}$ with $p \mapsto vp$ are well-defined. Additionally note that $v^2 = 0$ and thus we obtain a **complex associated to P** denoted as $L_v(P)$

$$L_v(P): \cdots \longrightarrow P_{j+1} \xrightarrow{v} P_j \xrightarrow{v} P_{j-1} \longrightarrow \cdots,$$

where P_j is considered being homological degree $-j$.

- (h) Lastly, we define the grading for the following basic operations between E -modules:

- The **direct sum of graded modules** is defined as the graded module with components $(P \oplus Q)_j := P_j \oplus Q_j$ where the action by E is defined componentwise.
- The **tensor product of graded modules** is defined as the graded module with components $(P \otimes_k Q)_j := \bigoplus_i P_i \otimes_k Q_{j-i}$ where the E -action is defined as: $e(p \otimes q) := ep \otimes q + (-1)^{\alpha\beta} p \otimes eq$ with $p \in P_\alpha$, $q \in Q$ and $e \in E_{-\beta}$.
- The **Hom-module of graded modules** is defined as the graded module with components $\text{Hom}_k(P, Q)_j := \bigoplus_i \text{Hom}_k(P_{i-j}, Q_i)$ where the action by E is defined as: $ef(p) := (-1)^{\alpha\beta} f(ep)$ with $p \in P_\alpha$ and $e \in E_{-\beta}$.

Note that the tensor and Hom are taken over k .

Remark 2.1.3 (Basic definitions and operations of **S-grMod**). The graded structure of the symmetric algebra S , along with the objects in the categories **S-grMod** and **S-grmod** allows us to define the items (a)-(f) and (h) analogous to the case of E -modules.

Now, similar to the item (g), in the case of **S-grMod** we can define a sequence of objects and maps of S -modules as follows. Let $M \in \mathbf{S-grMod}$. Since any $w \in W$ is in homogeneous degree 1, the multiplication maps $M_j \rightarrow M_{j+1}$ defined as $m \mapsto wm$ are well-defined. However, this sequence of morphisms does not define a chain complex as in the case of E -modules.

In the following proposition we recompile some useful properties of E -modules.

Proposition 2.1.4. Let $P, Q \in \mathbf{E-grMod}$ and r any integer. Then:

- (1) For any nonzero $e \in E$, there exists an element $\tilde{e} \in E$ such that $\tilde{e} \wedge e \in E_{-n-1}$.
- (2) $P \otimes_k Q(r) = P(r) \otimes_k Q = (P \otimes_k Q)(r)$.
- (3) $\text{Hom}_k(P(r), Q) = (\text{Hom}_k(P, Q))(-r)$ and $\text{Hom}_k(P, Q(r)) = (\text{Hom}_k(P, Q))(r)$.
- (4) $E^* \cong E(-n-1)$. Moreover, $E(r)^* \cong E(-r-n-1)$.
- (5) E as a free E -module is indecomposable.

Proof. (1) Let $e \in E$ be any nonzero element. As we do in the Lemma 2.1.1, e can be uniquely expressed as

$$e = \sum_{S \subseteq \{0,1,\dots,n\}} \alpha_S \left(\bigwedge_{j \in S} e_j \right), \quad \alpha_S \in k.$$

Let $d = \min \{|S| : \alpha_S \neq 0\}$. Take any $I \subseteq \{0, 1, \dots, n\}$ such that $|I| = d$ and $\alpha_I \neq 0$. Note that

$$\left(\bigwedge_{j \in \{0,1,\dots,n\} \setminus I} e_j \right) \wedge e = \pm \alpha_S (e_0 \wedge \dots \wedge e_n),$$

because

$$\left(\bigwedge_{j \in \{0,1,\dots,n\} \setminus I} e_j \right) \wedge \alpha_S \left(\bigwedge_{j \in S} e_j \right) = 0,$$

for all $S \neq I$. Since $e_0 \wedge \dots \wedge e_n \in E_{-n-1}$ we finish the proof by taking $\tilde{e} = \bigwedge_{\{0,1,\dots,n\} \setminus I} e_j$.

(2) We prove $P \otimes_k Q(r) = (P \otimes_k Q)(r)$, the other equality follows similarly. For any j :

$$\begin{aligned} (P \otimes_k Q)(r)_j &= (P \otimes_k Q)_{r+j} \\ &= \bigoplus_i P_i \otimes Q_{r+j-i} = \bigoplus_i P_i \otimes Q(r)_{j-i} = (P \otimes Q(r))_j. \end{aligned}$$

(3) We prove $\text{Hom}_k(P(r), Q) = (\text{Hom}_k(P, Q))(-r)$, the other equality follows similar. For any j :

$$\begin{aligned} \text{Hom}_k(P(r), Q)_j &= \bigoplus_i \text{Hom}_k(P(r)_{i-j}, Q_i) \\ &= \bigoplus_i \text{Hom}_k(P_{i-j+r}, Q_i) = \text{Hom}_k(P, Q)_{j-r} = \text{Hom}_k(P, Q)(-r)_j. \end{aligned}$$

(4) Define a pairing $\langle -, - \rangle : E \times E(-n-1) \rightarrow k$ defined by

$$\langle e, x \rangle = \text{coefficient of } e_0 \wedge \cdots \wedge e_n \text{ in the decomposition of } e \wedge x.$$

We show that the pairing is k -bilinear and perfect, meaning it induces an isomorphism

$$E^* = \text{Hom}_k(E, k) \cong E(-n-1).$$

- Linearity in the second argument: If x, x' and $\lambda, \mu \in k$, then

$$\langle e, \lambda x + \mu x' \rangle = \text{coefficient of } e_0 \wedge \cdots \wedge e_n \text{ in the decomposition of } e \wedge (\lambda x + \mu x').$$

Since the wedge multiplication distributes over addition we have $\lambda(e \wedge x) + \mu(e \wedge x')$ and then we have $\lambda \langle e, x \rangle + \mu \langle e, x' \rangle$. The linearity in the first argument follows similar.

- Non-degeneracy: It suffices to show that for any nonzero $e \in E$ there exists some $x \in E$ such that $\langle e, x \rangle \neq 0$. This is precisely the item (1).

It remains to show that $E(r)^* = E(-r-n-1)$, this follows easily by shifting E . Indeed,

$$(E(r))^* = \text{Hom}_k(E(r), k) = \text{Hom}_k(E, k)(-r) = E^*(-r) = E(-n-1)(-r) = E(-r-n-1).$$

(5) Suppose that we have $E = P \oplus Q$ for any $P, Q \in \mathbf{E}\text{-grmod}$. Since P and Q are finite dimensional as k -vector spaces and $\dim E(m)_0 = 1$ we have $\dim_k P_0 + \dim_k Q_0 = 1$. If $\dim_k P_0 = 1$ and $\dim_k Q_0 = 0$ then each $e \in E$ can be written as $e \cdot 1$ and $1 \in P$ and thus $E = V$. $\square$

Remark 2.1.5 (The exterior algebra is a self-injective algebra). *It is a well-known fact that k is itself an injective module. By [4, Lemma 3.1.6] $\text{Hom}_k(E, k)$ is an injective E -module. Thus by item (4) of Proposition 2.1.4, E is an injective E -module. Therefore, the exterior algebra is a self-injective algebra.*

Proposition 2.1.6. *For any $P \in \mathbf{E}\text{-grmod}$ the following conditions are equivalent:*

- (1) P is free.
- (2) P is projective.

(3) P is injective.

Proof. We will prove that (1) and (2) are equivalent. First it is clear that (1) implies (2). Now, for the reverse implication, suppose that P is a projective module. Since any projective module can be decomposed into a direct sum of indecomposable projective modules, it suffices to prove the statement for indecomposable projective modules.

By definition, a projective module P is a direct summand of some free module F . Thus, there exists a module $M \in \mathbf{E}\text{-grmod}$ such that $F = P \oplus M$ where $F = \bigoplus_i E(m_i)$ for some integers m_i . We may assume that M is indecomposable. Since the exterior algebra E is a finitely dimensional k -algebra by the Krull-Schmidt Theorem [1, Theorem 4.10]. The decomposition of F into indecomposable summands is unique up to isomorphism and order. Consequently, one of the summands $E(m_i)$ must be isomorphic to P .

Now we prove that (1) is equivalent to (3). First consider the dual-functor

$$(-)^* = \text{Hom}_k(-, k): \mathbf{E}\text{-grmod} \longrightarrow \text{grmod-}\mathbf{E}.$$

It is straightforward to see that given any projective module P , then P^* is an injective module and vice-versa, since duality reverses all arrows. Moreover, since we are taken P as a vector space we have $(P^*)^* = P$. Therefore, duality interchanges projective and injective modules. Now take $N \in \mathbf{E}\text{-grmod}$ be an injective module. We already know that N^* is projective and thus free with the form $N^* = E(m_i)$ for some integer m_i . By the item (4) of Proposition 2.1.4 we have

$$N \cong (N^*)^* = (E(m_i))^* \cong E(-m_i - n - 1),$$

and then N is free. For the converse, suppose that N is free and thus projective so $N = E(m_i)$ for some integer m_i . As we know $N^* \cong E(-m_i - n - 1)$ and by Remark 2.1.5 E is an injective E -module. $\square$

Lemma 2.1.7. *The category $\mathbf{E}\text{-grmod}$ has enough projectives and injectives objects.*

Proof. Let $P \in \mathbf{E}\text{-grmod}$ and P^* its dual.

- Enough projective objects: Let $\{m_i\}_{i=0}^t$ be a finite set of generators for P . Consider the homomorphism $\bigoplus_{i=0}^t E(-\deg m_i) \rightarrow P$ where each summand $E(-\deg m_i) \rightarrow P$ is defined $1(\deg m_i) \mapsto m_i$ and $\deg m_i$ indicates which graded piece P_{m_i} the basis element m_i belongs to. By construction, this map is surjective. Moreover, by Proposition 2.1.6, the module $\bigoplus_{i=0}^t E(-\deg m_i)$ is free, and hence projective.
- Enough injective objects: By the previous item, there exists a surjective homomorphism $\bigoplus_{i=0}^t E(-\deg w_i) \rightarrow P^*$ from a free module to the dual module P^* . Taking the dual of this map yields an injective homomorphism $P \rightarrow \bigoplus_{i=0}^t E(-\deg w_i - n - 1)$. Again, by Proposition 2.1.6, the module $\bigoplus_{i=0}^t E(-\deg w_i - n - 1)$ is free, and hence injective. $\square$

2.1.2 Category of complexes associated to E-grMod and S-grMod. In this subsection we define the category of complexes associated to the abelian categories **E-grMod** and **S-grMod**.

Definition 2.1.8 (Complexes of E -modules). A **complex over E-grMod** is a sequence

$$P : \quad \dots \longrightarrow P^{i-1} \xrightarrow{d^{i-1}} P^i \xrightarrow{d^i} P^{i+1} \longrightarrow \dots$$

of objects and morphisms in the category **E-grMod** such that $d^i \circ d^{i-1} = 0$ for all i . Additionally, these differentials commute with the multiplication maps defined in Definition 2.1.2, specifically in item (g). That is, for any nonzero $v \in V$ and any $p \in P^i$, we have $vd^i(p) = d^i(vp)$.

Given two chain complexes P and Q of graded E -modules, a **chain map** $f: P \rightarrow Q$ is a sequence of morphisms $f^i: P^i \rightarrow Q^i$ in **E-grMod** satisfying $f^{i+1} \circ d_P^i = d_Q^i \circ f^i$ for all i .

In the above definition, each P^i is a graded E -module, so we denote by P_j^i the j -th homogeneous component of the i -th homological degree of P . We write **C**(E) for the **category of complexes over E-grMod** and **C^b**(E) for the **subcategory of bounded complexes of finitely generated modules over E** . The corresponding definitions for S -modules are as follows.

Definition 2.1.9 (Complexes of S -modules). A **complex over S-grMod** is a sequence

$$M : \quad \dots \longrightarrow M^{i-1} \xrightarrow{d^{i-1}} M^i \xrightarrow{d^i} M^{i+1} \longrightarrow \dots$$

of objects and morphisms in the category **S-grMod** such that $d^i \circ d^{i-1} = 0$ for all i . Additionally, these differentials commute with the multiplication maps defined in Remark 2.1.3. That is, for any nonzero $w \in V$ and any $m \in M^i$, we have $wd^i(m) = d^i(wm)$.

Given two chain complexes M and N of graded S -modules, a **chain map** $f: M \rightarrow N$ is a sequence of morphisms $f^i: M^i \rightarrow N^i$ in **S-grMod** satisfying $f^{i+1} \circ d_M^i = d_N^i \circ f^i$ for all i .

As in the previous definition, each M^i is a graded S -module, so we denote by M_j^i the j -th homogeneous component of the i -th homological degree of M . We write **C**(S) for the **category of complexes over S-grMod** and **C^b**(S) for the **subcategory of bounded complexes of finitely generated modules over S** .

Definition 2.1.10 (Bigraded E -modules). We will denote by **E-BigrMod**, the **category of bigraded E -modules**, where each bigraded E -module $P = \bigoplus_{i,j} P_{i,j}$ is equipped with a linear map $d_P: P \rightarrow P$ which satisfies the following conditions:

- (a) The multiplication map defined in item (g) of the Definition 2.1.2 has bidegree $(0, -1)$.
- (b) The map d_P has bidegree $(1, 0)$.
- (c) The relation $vd_P = d_P v$ holds, for each $v \in V$.
- (d) The map d_P satisfies $d_P^2 = 0$.

Given two bigraded E -modules P and Q , a morphism $f: P \rightarrow Q$ is a bidegree preserving map, that is, $f(P_{ij}) \subseteq Q_{ij}$ such that $d_P \circ f = f \circ d_Q$.

Definition 2.1.11 (Bigraded S -modules). We will denote by **S-BigrMod**, the category of bigraded S -modules, where each bigraded S -module $M = \bigoplus_{i,j} M_{ij}$ is equipped with a linear map $d_M: M \rightarrow M$ which satisfies the following conditions:

- (a') The multiplication map defined in Remark 2.1.3 has bidegree $(0, 1)$.
- (b') The map d_M has bidegree $(1, 0)$.
- (c') The relation $wd_M = d_M w$ holds, for each $w \in W$.
- (d') The map d_M satisfies $d_M^2 = 0$.

Given two bigraded S -modules M and N , a morphism $f: M \rightarrow N$ are bidegree preserving maps, that is, $f(M_{ij}) \subseteq N_{ij}$ such that $d_M \circ f = f \circ d_N$.

Proposition 2.1.12. The categories $\mathbf{C}(E)$ and **E-BigrMod** are equivalent. Similarly, the categories $\mathbf{C}(S)$ and **S-BigrMod** are equivalent.

Proof. We begin by defining the functor $\alpha: \mathbf{C}(E) \rightarrow \mathbf{E-BigrMod}$, which maps a complex $P^\bullet = (P^i, d_P^i)$ to the object $(P = \bigoplus_{i,j} P_j^i, d_P = \bigoplus_i d_P^i)$, and a chain map $f: P^\bullet \rightarrow Q^\bullet$ to $\alpha(f) = \bigoplus_i f^i: (P, d_P) \rightarrow (Q, d_Q)$.

Let $P^\bullet$ be a complex $\mathbf{C}(E)$, we assign the bidegree (i, j) in $P = \bigoplus_{i,j} P_j^i$ where i corresponds to the homological degree and j to the homogeneous degree of the complex $P^\bullet$. In order to prove that α is well-defined we must check the conditions (a) - (d) provided in Definition 2.1.10.

- (a) Let $v \in V$ and $p \in P_j^i$. By definition of multiplication maps we have that $vp \in P_{j-1}^i$ which means that the multiplication maps over P has bidegree $(0, -1)$.
- (b) By definition, the differential maps of $P^\bullet$ are homogeneous degree preserving, that is, $d_P^i(P_j^i) \subseteq P_j^{i+1}$. This is exactly to say that d_P has bidegree $(1, 0)$.
- (c) This is obvious since the differential maps of $P^\bullet$ commutes with multiplication maps given by V .
- (d) By definition, the differentials of $P^\bullet$ satisfy $d_P^i \circ d_P^{i-1} = 0$ and then $d_P^2 = 0$.

We claim that that $\alpha: \mathbf{C}(E) \rightarrow \mathbf{E-BigrMod}$ is a fully faithful functor and essentially surjective. For the following let $P^\bullet$ and $Q^\bullet$ be complexes in $\mathbf{C}(E)$.

- Essential surjectivity: Let $\bigoplus_{i,j} R_{ij}$ be bigraded E -module equipped with a linear map $d_R: R \rightarrow R$. Define the complex $R^\bullet$ by setting $R^i = \bigoplus_j R_{ij}$ with differentials $d_R^i = d_R|_{R^i}$. We can check easily that the conditions (a) - (d) provided in Definition 2.1.10 makes $R^\bullet$ a well-defined complex in $\mathbf{C}(E)$ and thus $\alpha(R^\bullet) = (R, d_R)$.
- Faithfulness: Let $f: P^\bullet \rightarrow Q^\bullet$ be a chain map such that $\alpha(f) = 0$. By definition $\alpha(f) = \bigoplus_i f^i = 0$ this implies $f^i = 0$ and thus $f = 0$.
- Fullness: Let $f: P \rightarrow Q$ be any map between bigraded E -modules. We define a chain map $f^\bullet: P^\bullet \rightarrow Q^\bullet$ by setting $f^i = f|_{P^i}: P^i \rightarrow Q^i$. It is easy to show that f is indeed a chain map in $\mathbf{C}(E)$ and clearly $\alpha(f^\bullet) = f$. $\square$

In the next section, we will strongly rely on the following remark to establish the independence of certain maps with respect to the choice of the dual basis.

Remark 2.1.13 (Relations between different dual basis). *Let $\{e'_m\}_{m=0}^n$ and $\{x'_m\}_{m=0}^n$ be basis of V and W respectively. We write each e'_m and x'_m in terms of the original basis $\{e_m\}_{m=0}^n$ and $\{x_m\}_{m=0}^n$ as the sums*

$$e'_m = \sum_{k=0}^n a_{km} e_k, \quad x'_m = \sum_{k=0}^n b_{km} x_k \quad \text{with} \quad 0 \leq m \leq n. \quad (2.1)$$

The basis $\{e_m\}_{m=0}^n$ and $\{x_m\}_{m=0}^n$ have the relation $x_i(e_j) = \delta_{ij}$ where δ_{ij} stands for the Kronecker delta function. Similarly, we have the relation $x'_i(e'_j) = \delta_{ij}$ for the basis $\{x'_m\}_{m=0}^n$ and $\{e'_m\}_{m=0}^n$. Replacing the equations (2.1) we obtain

$$\delta_{ij} = x'_i(e'_j) = \sum_{k=0}^n b_{ki} x_k \left(\sum_{l=0}^n a_{lj} e_l \right) = \sum_{k=0}^n b_{ki} a_{kj}. \quad (2.2)$$

On the other hand, from the equations

$$x'_m(e_j) = \left(\sum_{k=0}^n b_{km} x_k \right) (e_j) = \sum_{k=0}^n b_{km} x_k(e_j) = b_{jm}, \quad x_m(e'_j) = x_m \left(\sum_{k=0}^n a_{kj} e_k \right) = a_{mj},$$

we get that each e_m and x_m can be expressed in terms of $\{e'_m\}_{m=0}^n$ and $\{x'_m\}_{m=0}^n$ as the sums

$$e_m = \sum_{k=0}^n b_{mk} e'_k, \quad x_m = \sum_{k=0}^n a_{mk} x'_k.$$

Then we can describe the relation $x_i(e_j) = \delta_{ij}$ as the sum

$$\delta_{ij} = x_i(e_j) = \sum_{k=0}^n a_{ik} x'_k \left(\sum_{l=0}^n b_{lj} e'_l \right) = \sum_{k=0}^n a_{ik} b_{jk}. \quad (2.3)$$

2.2 The BGG correspondence

In this section, we introduce and prove the **Bernstein–Gelfand–Gelfand (BGG) correspondence**. To do so, we first define the relevant functors.

2.2.1 L functor. In this subsection, we construct an additive functor $\mathbf{L}: \mathbf{C}(E) \rightarrow \mathbf{C}(S)$. To achieve this, we associate each $P \in \mathbf{C}(E)$ with two complexes: a horizontal complex and a vertical complex. These together form a double complex, whose total complex defines the assignation for $\mathbf{L}(P)$. We begin by defining the horizontal and vertical complexes.

Recall the dual basis $\{e_m\}_{m=0}^n$ and $\{x_m\}_{m=0}^n$ of V and W respectively.

Definition 2.2.1 (Horizontal and vertical complex for $\mathbf{L}$). *Let $P \in \mathbf{C}(E)$ be a complex of graded E -modules. We define two complexes:*

(1) **horizontal complex:**

$$\cdots \longrightarrow S \otimes_k P_j^{i-1} \xrightarrow{F_j^{i-1}} S \otimes_k P_j^i \xrightarrow{F_j^i} S \otimes_k P_j^{i+1} \longrightarrow \cdots$$

with differentials given by $F_j^i: s \otimes p \mapsto s \otimes d_P^i(p)$.

Here $S \otimes P_j^i$ corresponds to the i -th homological component of the horizontal complex.

(2) **Vertical complex:**

$$\cdots \longrightarrow S \otimes_k P_{j+1}^i \xrightarrow{\psi_{j+1}^i} S \otimes_k P_j^i \xrightarrow{\psi_j^i} S \otimes_k P_{j-1}^i \longrightarrow \cdots$$

with differential given by $\psi_j^i: s \otimes p \mapsto \sum_m x_m s \otimes e_m p$.

Here $S \otimes P_j^i$ corresponds to the $-j$ -th homological component of the vertical complex.

Note that the horizontal complex depends in the homological degree of P , and the vertical complex depends in the homogeneous components of each P^i .

Lemma 2.2.2. Let $P \in \mathbf{C}(E)$ be a complex of graded E -modules. Consider the horizontal and vertical complexes given in Definition 2.2.1, from these we have the following:

- (1) In both complexes, every homological component can be viewed, with the trivial grading, as an element of **S-grMod**.
- (2) For every i, j , we have $F_j^{i+1} \circ F_j^i = 0$ and $\psi_j^i \circ \psi_{j+1}^i = 0$.
- (3) The differentials in the vertical complex does not depend in the choice of the given for V and W .
- (4) The differentials of the horizontal and vertical complex commutes each other, that is for every i, j we have $F_j^i \circ \psi_{j+1}^i = \psi_{j+1}^{i+1} \circ F_{j+1}^i$.

Proof. (1) Let $\tilde{s} \in S$. Over each $S \otimes_k P_j^i$ we impose the S -action by taking $\tilde{s}(s \otimes p) := (\tilde{s}s) \otimes p$. The module axioms can be easily checked. Clearly each $S \otimes_k P_j^i$ can be viewed as a graded S -module with only one homogeneous degree. Since $S \otimes_k P_j^i$ corresponds to i -th homological degree of the horizontal complex and to the $-j$ -th homological degree of the vertical complex thus both sequences of objects lies in $\mathbf{C}(S)$.

(2) For the horizontal complex let $s \otimes p \in S \otimes_k P_j^i$ and note that

$$(F_j^{i+1} \circ F_j^i)(s \otimes p) = F_j^{i+1}(s \otimes d_P^i(p)) = s \otimes d_P^{i+1}(d_P^i(p)) = 0.$$

For the vertical complex let $s \otimes p \in S \otimes_k P_{j+1}^i$. By definition we have

$$(\psi_j^i \circ \psi_{j+1}^i)(s \otimes p) = \sum_{k,m} (x_k x_m) s \otimes (e_k e_m) p$$

we can split the sums to get

$$(\psi_j^i \circ \psi_{j+1}^i)(s \otimes p) = \sum_{k=m} (x_m x_m) s \otimes (e_m e_m) p + \sum_{k < m} (x_k x_m) s \otimes (e_k e_m) p + \sum_{m > k} (x_k x_m) s \otimes (e_k e_m) p$$

where the first summand is clearly zero since $e_m^2 = 0$. Now, since $e_k e_m = -e_m e_k$ and because both remaining sums run over $0 \leq k \neq m \leq n$ we have that the product $e_k e_m$ over $k < m$ is equivalent to have the product $-e_m e_k$ over $k < m$. Following this and reorganizing the sums, at the end we get

$$(\psi_j^i \circ \psi_{j+1}^i)(s \otimes p) = \sum_{k < m} (x_k x_m) \otimes (e_k e_m - e_m e_k) p = 0.$$

(3) Let $s \otimes p \in S \otimes P_j^i$ and let $\{e'_m\}_{m=0}^n$ and $\{x'_m\}_{m=0}^n$ be basis of V and W respectively. From this basis, the differential maps for the vertical complex is given by

$$\tilde{\psi}_j^i: S \otimes_k P_j^i \longrightarrow S \otimes_k P_{j-1}^i, \quad s \otimes p \longmapsto \sum_m x'_m s \otimes e'_m p.$$

We show that $\tilde{\psi}_j^i$ defined with respect $\{e_m\}_{m=0}^n$ and $\{x_m\}_{m=0}^n$ coincides over $S \otimes P_j^i$. Start by writing e'_m and x'_m in terms of the basis $\{e_m\}_{m=0}^n$ and $\{x_m\}_{m=0}^n$ as the sums $e'_m = \sum_k a_{km} e_k$ and $x'_m = \sum_k b_{km} x_k$. Replacing this into the definition of $\tilde{\psi}_j^i$ we get

$$\begin{aligned} \tilde{\psi}_j^i(s \otimes p) &= \sum_{m=0}^n \left(\sum_{k=0}^n b_{km} x_k s \otimes \sum_{\ell=0}^n a_{\ell m} e_\ell p \right) \\ &= \sum_{m=0}^n \left(\sum_{k,\ell} b_{km} a_{\ell m} (x_k s \otimes e_\ell p) \right) && \text{(Taking out the scalars)} \\ &= \sum_{m=0}^n \left(\sum_{\substack{k=1 \\ k \neq \ell}}^n b_{km} a_{km} (x_k s \otimes e_k p) + \sum_{k \neq \ell} b_{km} a_{\ell m} (x_k s \otimes e_\ell p) \right) && \text{(Splitting the sums)} \\ &= \sum_{k=0}^n \underbrace{\left(\sum_{m=0}^n b_{km} a_{km} (x_k s \otimes e_k p) \right)}_1 + \sum_{k \neq \ell} \underbrace{\left(\sum_{m=0}^n b_{km} a_{\ell m} (x_k s \otimes e_\ell p) \right)}_0 && \text{(By (2.2) and (2.3))} \\ &= \sum_{k=0}^n x_k s \otimes e_k p = \psi_j^i(s \otimes p) \end{aligned}$$

(4) Let $s \otimes p \in P \otimes_k P_{j+1}^i$. Clearly

$$\begin{aligned} (\psi_{j+1}^{i+1} \circ F_{j+1}^i)(s \otimes p) &= \psi_{j+1}^{i+1}(s \otimes d^i(p)) = \sum_{m=0}^n x_m s \otimes e_m d^i(p) \\ &= \sum_{m=0}^n x_m s \otimes d^i(e_m p) \\ &= F_j^i \left(\sum_{m=0}^n x_m s \otimes e_m p \right) = (F_j^i \circ \psi_{j+1}^i)(s \otimes p) \quad \square \end{aligned}$$

By the previous lemma, the horizontal and vertical complexes fit into the following well-defined double complex

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \\
 \cdots & \longrightarrow & S \otimes P_j^i & \xrightarrow{F_j^i} & S \otimes P_j^{i+1} & \longrightarrow & \cdots \\
 & & \uparrow \psi_{j+1}^i & & \uparrow \psi_{j+1}^{i+1} & & \\
 \cdots & \longrightarrow & S \otimes P_{j+1}^i & \xrightarrow{F_{j+1}^i} & S \otimes P_{j+1}^{i+1} & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & &
 \end{array} \tag{2.4}$$

Now we can get the total complex to get the functor assignation for $\mathbf{L}$.

Definition 2.2.3 (L functor). Let $P \in \mathbf{C}(E)$ be a complex of graded modules over E . We define $\mathbf{L}(P)$ as the corresponding total complex of the double complex defined in Diagram (2.4). Explicitly,

$$\mathbf{L}(P)^k := \bigoplus_{i-j=k} S \otimes_k P_j^i \quad \text{with differentials} \quad d_{\mathbf{L}(P)}^k := \bigoplus_{i-j=k} F_j^i + (-1)^i \psi_j^i.$$

For the morphism assignation, let $f: P \rightarrow Q$ be a chain map in $\mathbf{C}(E)$. We define $\mathbf{L}(f): \mathbf{L}(P) \rightarrow \mathbf{L}(Q)$ as $s \otimes p \mapsto s \otimes f(p)$.

Remark 2.2.4. For the following let $P \in \mathbf{C}(E)$.

- The minus sign in the assignment of $\mathbf{L}(P)$ appears because $S \otimes_k P_j^i$ corresponds to the $-j$ -th homological component of the vertical complex.
- Since S is graded, in the definition of $\mathbf{L}(P)$ we can obtain a formula that relates the homogeneous components of both S and P . We can use the homogeneous grading of $S \otimes P^i$ to get the formula:

$$\mathbf{L}(P)_j^i := \bigoplus_m S_{j-m} \otimes P_m^{i+m}.$$

- A concise way to summarize the previous remark is:

If $s \in S_k$ and $p \in P_j^i$, then $s \otimes p \in \mathbf{L}(P)_{j+k}^{i-j}$.

This equation is useful for simplifying calculations. We will refer to this formula as the **slogan for the L functor**.

- Using the preceding slogan, we can easily deduce that the map $\mathbf{L}(f)$ is well-defined in the sense that it preserves both the homological and homogeneous degrees. Indeed, given any $s \in S_k$ and $p \in P_j^i$, we have $s \otimes p \in \mathbf{L}(P)_{j+k}^{i-j}$. Since $f(p) \in Q_j^i$, we deduce that $s \otimes f(p) \in \mathbf{L}(Q)_{j+k}^{i-j}$.
- Let $P \in \mathbf{E}\text{-grMod}$. Considering this module as concentrated in degree 0 complex, it is straightforward to see that the vertical complex given in Definition 2.2.1 is precisely $\mathbf{L}(P[0])$. This follows because for each $i \neq 0$, the complex satisfies $P[0]_j^i = 0$.

Proposition 2.2.5. *The assignment $\mathbf{L}: \mathbf{C}(E) \rightarrow \mathbf{C}(S)$ is a well-defined additive functor. Moreover, the restriction $\mathbf{L}: \mathbf{C}^b(E) \rightarrow \mathbf{C}^b(S)$ to the bounded categories of complexes is well-defined too.*

Proof. Let P, Q and R be complexes in $\mathbf{C}(E)$ and let $f, h: P \rightarrow Q$ and $g: Q \rightarrow R$ be chain maps. We check the functoriality first so for any $s \otimes p \in S \otimes P$ we check:

$$\begin{aligned}\mathbf{L}(g \circ f)(s \otimes p) &= s \otimes (g \circ f)(p) = \mathbf{L}(g)(s \otimes f(p)) = (\mathbf{L}(g) \circ \mathbf{L}(f))(s \otimes p) \\ \mathbf{L}(1_P)(s \otimes p) &= s \otimes p = 1_{\mathbf{L}(P)}(s \otimes p)\end{aligned}$$

By Lemma 1.1.8, to prove the additivity it suffices to show that $\mathbf{L}(P \oplus Q) = \mathbf{L}(P) \oplus \mathbf{L}(Q)$.

$$\begin{aligned}\mathbf{L}(P \oplus Q)^k &= \bigoplus_{i-j=k} S \otimes (P \oplus Q)_j^i = \bigoplus_{i-j=k} S \otimes (P_j^i \oplus Q_j^i) \\ &= \bigoplus_{i-j=k} (S \otimes P_j^i) \oplus (S \otimes Q_j^i) = \bigoplus_{i-j=k} (S \otimes P_j^i) \oplus \bigoplus_{i-j=k} (S \otimes Q_j^i) \\ &= \mathbf{L}(P)^k \oplus \mathbf{L}(Q)^k\end{aligned}$$

To facilitate the writing we omit the indexes. Let $s \otimes (p \oplus q) \in \mathbf{L}(P \oplus Q)^k$. Then

$$\begin{aligned}d_{\mathbf{L}(P \oplus Q)}^k(s \otimes (p \oplus q)) &= F_j^i(s \otimes (p \oplus q)) + (-1)^i \psi_j^i(s \otimes (p \oplus q)) \\ &= s \otimes d_{P \oplus Q}(p \oplus q) + (-1)^i \sum_m x_m s \otimes e_m(p \oplus q) \\ &= (s \otimes d_P(q) \oplus (-1)^i \sum_m x_m s \otimes e_m p) \oplus (s \otimes d_Q(p) + (-1)^i \sum_m x_m s \otimes e_m q) \\ &= (d_{\mathbf{L}(P)}^k(s \oplus p)) \oplus (d_{\mathbf{L}(Q)}^k(s \oplus q)).\end{aligned}$$

From this, we conclude that the functor $\mathbf{L}$ is well-defined.

For the second statement, let $P \in \mathbf{C}^b(E)$ be a bounded complex of finitely graded E -modules. By definition, $\mathbf{L}(P)$ is the total complex of double complex (2.4). Note that the horizontal complex is bounded, as it depends on the homological components of P , which are finite. Additionally, since each homological component is a finite-dimensional k -vector space, each P^i has only finitely many homogeneous components, and thus the vertical complex is bounded. Consequently, the entire double complex is bounded.

Now, given that the bicomplex is bounded, for any integer k , the sum $\mathbf{L}(P)_k = \bigoplus_{i-j=k} S \otimes_k P_j^i$ is finite because only finitely many terms contribute. Furthermore, since each P^i is finitely generated as an E -module, and S is trivially finitely generated as an S -module, each $S \otimes_k P_j^i$ is finitely generated S -module. Therefore, $\mathbf{L}(P)$ is a bounded complex of finitely generated S -modules. $\square$

2.2.2 R functor. In this subsection, we construct an additive functor $\mathbf{R}: \mathbf{C}(S) \rightarrow \mathbf{C}(E)$. To achieve this, we associate each $M \in \mathbf{C}(S)$ with two complexes: a horizontal complex and a vertical complex, analogous to the construction of the functor $\mathbf{L}$. Together, these form a double complex. Once again,

by taking the corresponding total complex of this double complex, we obtain the assignment of $\mathbf{R}(M)$. We begin by defining these vertical and horizontal complexes.

Definition 2.2.6 (Row and vertical complex for $\mathbf{R}$). *Let $M \in \mathbf{C}(S)$ be a complex of graded S -modules. We define two complexes:*

(1) **Horizontal complex:**

$$\cdots \longrightarrow \mathrm{Hom}_k(E, M_j^{i-1}) \xrightarrow{G_j^{i-1}} \mathrm{Hom}_k(E, M_j^i) \xrightarrow{G_j^i} \mathrm{Hom}_k(E, M_j^{i+1}) \longrightarrow \cdots,$$

with differentials given by $G_j^i: f \mapsto d_M^i \circ f$.

Here $\mathrm{Hom}_k(E, M_j^i)$ corresponds to the i -th homological component of the horizontal complex.

(2) **Vertical complex:**

$$\cdots \longrightarrow \mathrm{Hom}_k(E, M_{j-1}^i) \xrightarrow{\varphi_{j-1}^i} \mathrm{Hom}_k(E, M_j^i) \xrightarrow{\varphi_j^i} \mathrm{Hom}_k(E, M_{j+1}^i) \longrightarrow \cdots,$$

with differential given by $\varphi_j^i: f \mapsto [e \mapsto \sum_m x_m f(e_m e)]$

Here $\mathrm{Hom}_k(E, M_j^i)$ corresponds to the i -th homological component of the vertical complex. Note that the horizontal complex depends in the homological degree of M , and the vertical complex depends in the homogeneous components of each M^i .

Lemma 2.2.7. *Let $M \in \mathbf{C}(S)$ be a complex of graded S -modules. Consider the row and vertical complexes given in Definition 2.2.6, from these we have the following:*

- (1) *In both complexes, every homological component can be viewed, with the trivial grading, as an element of $\mathbf{E}\text{-grMod}$.*
- (2) *For every i, j we have $G_j^{i+1} \circ G_j^i = 0$ and $\varphi_{j+1}^i \circ \varphi_j^i = 0$.*
- (3) *The differentials in the vertical complex does not depend in the choice of the basis given by V and W .*
- (4) *The differentials of the horizontal and vertical complex commute each other. That is for every i, j $G_j^i \circ \varphi_{j-1}^i = \varphi_{j-1}^{i+1} \circ G_{j-1}^i$.*

Proof. (1) Let $\tilde{e} \in E$. We define an action of E on each $f \in \mathrm{Hom}_k(E, M_j^i)$ by the formula $\tilde{e}f(e) := f(e\tilde{e})$. The module axioms can be easily checked. Clearly each $\mathrm{Hom}_k(E, M_j^i)$ can be viewed as a graded E -module with only one homogeneous degree. Since $\mathrm{Hom}_k(E, M_j^i)$ corresponds to i -th homological degree of the horizontal and vertical complex thus both sequence of objects lies in $\mathbf{C}(E)$.

(2) Let $f \in \mathrm{Hom}_k(E, M_j^i)$. For the horizontal complex note that

$$(G_j^{i+1} \circ G_j^i)(f) = (d^{i+1} \circ d^i) \circ f = 0.$$

Now, for the vertical complex let $f \in \text{Hom}_k(E, M_j^i)$. By definition we have

$$[(\varphi_{j+1}^i \circ \varphi_j^i)(f)](e) = \sum_{m=0}^n \left(\sum_{k,j} b_{km} a_{jm} x_k f(e_j e) \right),$$

we can split the sums to get

$$[(\varphi_{j+1}^i \circ \varphi_j^i)(f)](e) = \sum_{m=k} x_m x_m f(e_m e_m e) + \sum_{m < k} x_m x_k f(e_k e_m e) + \sum_{m > k} x_k x_m f(e_m e_k e),$$

now, proceeding as in the proof of item 2 of Lemma 2.2.2 we get that $[(\varphi_{j+1}^i \circ \varphi_j^i)(f)](e) = 0$.

(3) Let $f \in \text{Hom}_k(E, M_j^i)$ and let $\{e'_m\}_{m=0}^n$ and $\{x'_m\}_{m=0}^n$ be dual basis of V and W respectively. From this basis, the differential maps for the vertical complex are given by

$$\tilde{\varphi}: \text{Hom}_k(E, M_j^i) \longrightarrow \text{Hom}_k(E, M_{j+1}^i), \quad f \longmapsto [e \mapsto \sum_m x'_m f(e'_m e)].$$

We show that φ_j^i defined with respect to $\{e_m\}_{m=0}^n$ and $\{x_m\}_{m=0}^n$ coincides over $\text{Hom}_k(E, M_j^i)$. As we do in the proof of item 3 of Lemma 2.2.2, we start by writing e'_m and x'_m in terms of the basis $\{e_m\}_{m=0}^n$ and $\{x_m\}_{m=0}^n$ as the sums $e'_m = \sum_k a_{km} e_k$ and $x'_m = \sum_k b_{km} x_k$. Replacing this into the definition of $\tilde{\varphi}_j^i$ we get

$$\begin{aligned} [\tilde{\varphi}_j^i(f)](e) &= \sum_{m=0}^n \left(\sum_{k,j} b_{km} a_{jm} x_k f(e_j e) \right) \\ &= \sum_{m=0}^n \left(\sum_{\substack{k=1 \\ k=j}}^n b_{km} a_{km} x_k f(e_k e) + \sum_{k \neq j} b_{km} a_{jm} x_k f(e_j e) \right) && \text{(Splitting the sums)} \\ &= \sum_{k=0}^n \underbrace{\left(\sum_{m=0}^n b_{km} a_{km} x_k f(e_k e) \right)}_1 + \sum_{k \neq j} \underbrace{\left(\sum_{m=0}^n b_{km} a_{jm} x_k f(e_j e) \right)}_0 && \text{(By (2.2) and (2.3))} \\ &= \sum_{k=0}^n x_k f(e_k e) = [\varphi_d(f)](e). \end{aligned}$$

(4) Let $f \in \text{Hom}_k(E, M_j^i)$ and note that:

$$\begin{aligned} [(\varphi_{j-1}^{i+1} \circ G_{j+1}^i)(f)](e) &= \varphi_{j-1}^{i+1}(d_M^i(f(e))) = \sum_{k=0}^n x_k d^i(f(e_k e)) \\ &= d^i \left(\sum_{k=0}^n x_k f(e_k e) \right) = G_j^i \left(\sum_{k=0}^n x_k f(e_k e) \right) = [(G_j^i \circ \varphi_{j-1}^i)(f)](e). \quad \square \end{aligned}$$

By the previous lemma, the horizontal and vertical complexes fit into the following well-defined

double complex

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \\
 \cdots & \longrightarrow & \text{Hom}(E, M_j^i) & \xrightarrow{G_j^i} & \text{Hom}(E, M_j^{i+1}) & \longrightarrow & \cdots \\
 & & \uparrow \varphi_{j-1}^i & & \uparrow \varphi_{j-1}^{i+1} & & \\
 \cdots & \longrightarrow & \text{Hom}(E, M_{j-1}^i) & \xrightarrow{G_{j-1}^i} & \text{Hom}(E, M_{j-1}^{i+1}) & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & &
 \end{array} \tag{2.5}$$

Now we can get the total complex to get the functor assignation for $\mathbf{R}$.

Definition 2.2.8 (R functor). *Let $M \in \mathbf{C}(S)$ be a complex of graded modules over S . We define $\mathbf{R}(M)$ as the corresponding total complex of the double complex defined in (2.5). Explicitly,*

$$\mathbf{R}(M)^k := \bigoplus_{i+j=k} \text{Hom}_k(E, M_j^i) \quad \text{with differentials} \quad d_{\mathbf{R}(M)}^k := \bigoplus_{i+j=k} G_j^i + (-1)^i \varphi_j^i.$$

For the morphism assignation, let $g: M \rightarrow N$ be a chain map in $\mathbf{C}(S)$. We define $\mathbf{R}(g): \mathbf{R}(M) \rightarrow \mathbf{R}(N)$ as $\varphi \mapsto g \circ \varphi$.

Remark 2.2.9. For the following let $M \in \mathbf{C}(S)$.

- Since E is graded, in the definition of $\mathbf{R}(M)$ we can obtain a formula that relates the homogeneous components of both E and M . We can use the homogeneous grading of $\text{Hom}_k(E, M^i)$ to get the formula

$$\mathbf{R}(M)_j^i := \bigoplus_m \text{Hom}_k(E_{m-j}, M_m^{i-m}).$$

- A concise way to summarize the previous remark is:

If $e \in E_{-k}$ and $f(e) \in M_j^i$ then $f \in \mathbf{R}(M)_{j+k}^{i+j}$.

This equation is useful to simplifying calculations. We will refer to this formula as the **slogan for R functor**.

- Using the preceded slogan we can easily deduce that the map $\mathbf{R}(g)$ is well-defined in the sense that $\mathbf{R}(f)$ preserves both homological and homogeneous degree. Indeed, let $h \in \mathbf{R}(M)_{j+k}^{i+j}$ and $e \in E_{-k}$. Clearly $h(e) \in M_j^i$ and thus $(g \circ h)(e) \in N_j^i$ by the above slogan we get that $g \circ h \in \mathbf{R}(N)_{j+k}^{i+j}$.
- Let $M \in \mathbf{S}\text{-grMod}$. Considering this module as concentrated in degree 0 complex, it is straightforward to see that the vertical complex given in the Definition 2.2.6 is precisely $\mathbf{R}(M[0])$. This follows because for each $i \neq 0$, the complex satisfies $M[0]_j^i = 0$.

Proposition 2.2.10. *The assignation $\mathbf{R}: \mathbf{C}(S) \rightarrow \mathbf{C}(E)$ is a well-defined additive functor. Moreover, the restriction $\mathbf{R}: \mathbf{C}^b(S) \rightarrow \mathbf{C}^b(E)$ to the bounded categories of complexes is well-defined too.*

Proof. The proof is analogous to the Proposition 2.2.5. $\square$

2.2.3 Main theorem. In this subsection we enunciate and prove that the functors $\mathbf{L}$ and $\mathbf{R}$ defined above are adjoint. Moreover we extend this to an equivalence between the bounded derived categories $\mathbf{D}^b(E)$ and $\mathbf{D}^b(S)$.

Definition 2.2.11 (Bigraded maps over k). *Let P and M be complexes of graded modules of E and S respectively and regard them as bigraded vector spaces over k . We denote by $\text{Hom}_{\text{bigrd}}(P, M)$ the **space of linear bigraded maps over k** such that satisfies the following conditions:*

(1) *The elements $\phi \in \text{Hom}_{\text{bigrd}}(P, M)$ are of the form $\phi_j^i: P_j^i \rightarrow M_j^{i-j}$.*

(2) *For every $\phi \in \text{Hom}_{\text{bigrd}}(P, M)$ we have*

$$\phi_j^{i+1} \circ d_P^i - d_M^{i-j} \circ \phi_j^i = \left(\sum_m x_m \otimes e_m \right) \phi_{j-1}^i, \quad (2.6)$$

where $(\sum_m x_m \otimes e_m) \phi_{j-1}^i$ stands for the map that takes an element $p \in P_j^i$ and send it to the sum $(-1)^{i+1} \sum_{m=0}^n x_m \phi_{j-1}^i(e_m p)$ where $\{e_m\}_{m=0}^n$ and $\{x_m\}_{m=0}^n$ are dual basis for V and W respectively.

Theorem 2.2.12 (Bernstein-Gelfand-Gelfand correspondance). *The functor $\mathbf{L}: \mathbf{C}(E) \rightarrow \mathbf{C}(S)$ is left adjoint to $\mathbf{R}: \mathbf{C}(S) \rightarrow \mathbf{C}(E)$. That is, we have an isomorphism*

$$\text{Hom}_{\mathbf{C}(E)}(P, \mathbf{R}(M)) \cong \text{Hom}_{\mathbf{C}(S)}(\mathbf{L}(P), M),$$

for every $P \in \mathbf{C}(E)$ and $M \in \mathbf{C}(S)$.

Proof. The idea of the proof is to show that given any $P \in \mathbf{C}(E)$ and $M \in \mathbf{C}(S)$, the Hom-sets of complexes $\text{Hom}_{\mathbf{C}(E)}(P, \mathbf{R}(M))$ and $\text{Hom}_{\mathbf{C}(S)}(\mathbf{L}(P), M)$ are isomorphic to $\text{Hom}_{\text{bigrd}}(P, M)$. To achieve this consider the following diagram.

$$\begin{array}{ccc} \text{Hom}_{\mathbf{C}(S)}(\mathbf{L}(P), M) & \xrightarrow{\alpha_{P,M}} & \text{Hom}_{\mathbf{C}(E)}(P, \mathbf{R}(M)) \\ & \searrow \Gamma \quad \nearrow \Phi & \\ & \text{Hom}_{\text{bigrd}}(P, M) & \end{array} \quad (2.7)$$

We start defining the maps Γ and Γ^{-1} as:

$$\begin{aligned} \Gamma: \text{Hom}_{\mathbf{C}(S)}(\mathbf{L}(P), M) &\longrightarrow \text{Hom}_{\text{bigrd}}(P, M), \\ \varphi &\longmapsto \Gamma(\varphi): P \rightarrow M \\ p &\longmapsto \varphi(1 \otimes p), \end{aligned}$$

$$\begin{aligned}
\Gamma^{-1}: \operatorname{Hom}_{\mathbf{bigrd}}(P, M) &\longrightarrow \operatorname{Hom}_{\mathbf{C}(S)}(\mathbf{L}(P), M) \\
\psi &\longmapsto \Gamma^{-1}(\psi): \mathbf{L}(P) \rightarrow M \\
s \otimes p &\longmapsto s\psi(p).
\end{aligned}$$

The maps Γ and Γ^{-1} are inverses of each other. Indeed, let $\varphi \in \operatorname{Hom}_{\mathbf{C}(S)}(\mathbf{L}(P), M)$ and $\psi \in \operatorname{Hom}_{\mathbf{bigrd}}(P, M)$. For any $s \in S$ and $p \in P$,

$$[(\Gamma^{-1} \circ \Gamma)(\varphi)](s \otimes p) = \Gamma^{-1}(\Gamma(\varphi))(s \otimes p) = s\Gamma(\varphi)(p) = s\varphi(1 \otimes p) = \varphi(s \otimes p)$$

$$[(\Gamma \circ \Gamma^{-1})(\psi)](p) = \Gamma(\Gamma^{-1}(\psi))(p) = \Gamma^{-1}(\psi)(1 \otimes p) = \psi(p).$$

We now check that the maps Γ and Γ^{-1} are well-defined. Let $\varphi \in \operatorname{Hom}_{\mathbf{C}(S)}(\mathbf{L}(P), M)$ and let us see that $\Gamma(\varphi)$ satisfy the conditions (1) and (2) of the Definition 2.2.11. So, consider the following:

- We check that $\Gamma(\varphi)$ maps elements of P_j^i to M_j^{i-j} . Let $p \in P_j^i$. By the slogan for the $\mathbf{L}$ functor, we see that $1 \otimes p \in \mathbf{L}(P)_j^{i-j}$. Since φ preserves both the homological and homogeneous degrees, it follows that $\varphi(1 \otimes p) \in M_j^{i-j}$.
- We check that, for every i, j , the map $y(\varphi)$ satisfies the equation (2.6). For that note that by definition, the chain map φ commutes with $d_{\mathbf{L}(P)}$ and d_M , then given any $1 \otimes p \in S \otimes P$ we have

$$\begin{aligned}
0 &= \varphi_j^{i-j+1}(d_{\mathbf{L}(P)}^{i-j}(1 \otimes p)) - d_M^{i-j}(\varphi_j^{i-j}(1 \otimes p)) \\
&= \varphi_j^{i-j+1}(1 \otimes d_P^i(p) + (-1)^i \sum_{m=0}^n x_m \otimes e_m p) - d_M^{i-j}(\Gamma(\varphi)_j^i(p)) \\
&= \varphi_j^{i-j+1}(1 \otimes d_P^i(p)) + (-1)^i \sum_{m=0}^n x_m \varphi_{j-1}^{i-j+1}(1 \otimes e_m p) - d_M^{i-j}(\Gamma(\varphi)_j^i(p)) \\
&= \Gamma(\varphi)_j^{i+1}(d_P^i(p)) - (-1)^{i+1} \sum_{m=0}^n x_m \Gamma(\varphi)_{j-1}^i(e_m p) - d_M^{i-j}(\Gamma(\varphi)_j^i(p)) \\
&= \Gamma(\varphi)_j^{i+1}(d_P^i(p)) - \left(\sum_m x_m \otimes e_m \right) \Gamma(\varphi)_{j-1}^i(p) - d_M^{i-j}(\Gamma(\varphi)_j^i(p)).
\end{aligned}$$

We show Γ^{-1} is well-defined. Let $\psi \in \operatorname{Hom}_{\mathbf{bigrd}}(P, M)$. We check that $\Gamma^{-1}(\psi)$ respects both homological and homogeneous degrees and that commutes with differentials.

- Let $s \in S_k$ and $p \in P_j^i$. By the slogan for $\mathbf{L}$ functor we have that the product $s \otimes p \in \mathbf{L}(P)_{j+k}^{i-j}$ implying that $\psi(p) \in M_j^{i-j}$, since $s\psi(p) \in M_{j+k}^{i-j}$. We conclude that $(\Gamma^{-1}(\psi))(s \otimes p) \in M_{j+k}^{i-j}$ as desired.
- In this item we show that the map $\Gamma^{-1}(\psi)$ commutes with differentials. Note that by the previously item $\Gamma^{-1}(\varphi)$ preserves homological degree, so the maps $\Gamma^{-1}(\varphi): \mathbf{L}(P)^i \rightarrow M^i$ are

well-defined. Now, multiplying by s the equation (2.6) we obtain:

$$\begin{aligned}
0 &= s\psi_j^{i+1}(d_P^i(p)) + (-1)^i \sum_{m=0}^n x_m s\psi_{j-1}^i(e_m p) - d_M^{i-j}(s\psi_j^i(p)) \\
&= \Gamma^{-1}(\psi)_{j+k}^{i-j+1}(s \otimes d_P^i(p)) + (-1)^i \sum_{m=0}^n x_m s\psi_{j-1}^i(e_m p) - d_M^{i-j}(s\psi_j^i(p)) \\
&= \Gamma^{-1}(\psi)_{j+k}^{i-j+1}(s \otimes d_P^i(p)) + (-1)^i \sum_{m=0}^n \Gamma^{-1}(\psi)_{j+k}^{i-j+1}(x_m s \otimes e_m p) - d_M^{i-j}(\Gamma^{-1}(\psi))_{j+k}^{i-j}(s \otimes p) \\
&= \Gamma^{-1}(\psi)_{j+k}^{i-j+1}(s \otimes d_P^i(p)) + (-1)^i \sum_{m=0}^n x_m s \otimes e_m p - d_M^{i-j}(\Gamma^{-1}(\psi))_{j+k}^{i-j}(s \otimes p) \\
&= (\Gamma^{-1}(\psi)_{j+k}^{i-j+1} \circ d_{\mathbf{L}(P)}^{i-j})(s \otimes p) - (d_{\mathbf{L}(P)}^{i-j} \circ \Gamma^{-1}(\psi)_{j+k}^{i-j})(s \otimes p).
\end{aligned}$$

We define the maps Φ and Φ^{-1} as:

$$\begin{aligned}
\Phi: \operatorname{Hom}_{\mathbf{bigrd}}(P, M) &\longrightarrow \operatorname{Hom}_{\mathbf{C}(E)}(P, \mathbf{R}(M)) \\
\psi &\longmapsto \Phi(\psi): P \rightarrow \mathbf{R}(M) \\
p &\longmapsto \Phi(\psi)(p): E \rightarrow M. \\
e &\mapsto \psi(ep),
\end{aligned}$$

$$\begin{aligned}
\Phi^{-1}: \operatorname{Hom}_{\mathbf{C}(E)}(P, \mathbf{R}(M)) &\longrightarrow \operatorname{Hom}_{\mathbf{bigrd}}(P, M) \\
f &\longmapsto \Phi^{-1}(f): P \rightarrow M \\
p &\longmapsto f(p)(1).
\end{aligned}$$

The maps Φ and Φ^{-1} are mutually inverse. Indeed, let $\psi \in \operatorname{Hom}_{\mathbf{bigrd}}(P, M)$ and $f \in \operatorname{Hom}_{\mathbf{C}(E)}(P, \mathbf{R}(M))$. For any $p \in P$ and $e \in E$,

$$\begin{aligned}
[(\Phi^{-1} \circ \Phi)(\psi)](p) &= [\Phi^{-1}(\Phi(\psi))](p) = [\Phi(\psi)(p)](1) = \psi(p) \\
[(\Phi \circ \Phi^{-1})(f)](p) &= [\Phi(\Phi^{-1}(f))](p) = [\Phi^{-1}(f)](ep) = [f(ep)](1) = [f(p)](e).
\end{aligned}$$

As we did before we may check that Φ and Φ^{-1} are well-defined, this follows analogously to the case of Γ and Γ^{-1} .

We conclude that for any $P \in \mathbf{C}(E)$ and $M \in \mathbf{C}(S)$

$$\operatorname{Hom}_{\mathbf{C}(E)}(P, \mathbf{R}(M)) \cong \operatorname{Hom}_{\mathbf{C}(S)}(\mathbf{L}(P, M)),$$

via the composition $\Phi \circ \Gamma$. □

2.3 Equivalence between $\mathbf{D}^b(E)$ and $\mathbf{D}^b(S)$

In this section we will focus on extending the adjointness of the functors $\mathbf{R}: \mathbf{C}(S) \rightarrow \mathbf{C}(E)$ and $\mathbf{L}: \mathbf{C}(E) \rightarrow \mathbf{C}(S)$ to an equivalence between their respective bounded derived categories.

First, we need to establish the corresponding functor at the level of derived categories. A similar discussion regarding the extension of functors between abelian categories was already discussed in Remark 1.3.24.

In the current setting, we aim to extend the functors $\mathbf{L}: \mathbf{C}(E) \rightarrow \mathbf{C}(S)$ and $\mathbf{R}: \mathbf{C}(S) \rightarrow \mathbf{C}(E)$ to exact functors between their respective bounded homotopy categories. To achieve this, it suffices to show that the functors $\mathbf{L}$ and $\mathbf{R}$ commutes with translation, preserves homotopic maps and sends distinguished triangles into distinguished triangles. The next lemmas achieve this.

Lemma 2.3.1. *The functor $\mathbf{L}: \mathbf{C}(E) \rightarrow \mathbf{C}(S)$:*

- (1) *Commutates with translation functor.*
- (2) *Transforms homotopic maps.*
- (3) *Transforms cone map complexes.*
- (4) *The restriction to the category $\mathbf{C}^b(E)$ preserves quasi-isomorphisms.*
- (5) *Is exact.*

Proof. To facilitate the writing we omit the indexes. Let $P, Q \in \mathbf{C}(E)$.

(1) This follows from:

$$\mathbf{L}(P[1])^k = \bigoplus_{i-j=k} S \otimes_k (P[1])_j^i = \bigoplus_{i-j=k} S \otimes_k P_j^{i+1} = \bigoplus_{i-j=k+1} S \otimes_k P_j^i = \mathbf{L}(P)^{k+1} = \mathbf{L}(P)^k[1].$$

(2) Let $f \sim g: P \rightarrow Q$ be homotopic maps. By definition there exists an homotopy h such that $d_Q \circ h + h \circ d_P = f - g$. We claim that $\tilde{h} := 1_S \otimes h$ is an homotopy between $\mathbf{L}(f)$ and $\mathbf{L}(g)$. Consider $s \otimes p \in L(P)^k$ and assume that $s \otimes p \in S \otimes P_j^i$ with $i - j = k$, in order to calculate $d_{\mathbf{L}(Q)} \circ \tilde{h} + \tilde{h} \circ d_{\mathbf{L}(P)}$ we notice that

$$(i) \quad \tilde{h}(s \otimes p) = s \otimes h^i(p).$$

$$(ii) \quad d_{\mathbf{L}(Q)}(s \otimes h^i(p)) = s \otimes d_Q^{i-1}(h^i(p)) + (-1)^{i-1} \sum_m x_m s \otimes e_m h^i(p).$$

$$(iii) \quad d_{\mathbf{L}(P)}(s \otimes p) = s \otimes d_P^i(p) + (-1)^i \sum_m x_m \otimes e_m p$$

$$(iv) \quad \tilde{h}(s \otimes d_P^i(p) + (-1)^i \sum_m x_m \otimes e_m p) = s \otimes h^i(d_P^i(p)) + (-1)^i \sum_m x_m s \otimes e_m h^i(p).$$

Adding (ii) and (iv), and simplifying by grouping and canceling terms, we obtain at the end that the

expression $(d_{\mathbf{L}(Q)} \circ \tilde{h} + \tilde{h} \circ d_{\mathbf{L}(P)})(s \otimes p)$ is equal to

$$\begin{aligned} s \otimes \left(d_Q^{i-1}(h^i(p)) + s \otimes h^{i+1}(d_P^i(p)) \right) &= s \otimes (f^i(p) - g^i(p)) \\ &= (\mathbf{L}(f)^n - \mathbf{L}(g)^n)(s \otimes p). \end{aligned}$$

(3) Let $f: P \rightarrow Q$ be a chain map and consider the cone complex $\text{Cone}(f)$. We check that $\text{Cone}(\mathbf{L}(f)) \cong \mathbf{L}(\text{Cone}(f))$. Over the objects we have

$$\begin{aligned} \mathbf{L}(\text{Cone}(f))^k &= \bigoplus_{i-j=k} S \otimes (\text{Cone}(f))_j^i = \bigoplus_{i-j=k} S \otimes (P_j^{i+1} \oplus Q_j^i) \\ &= \left(\bigoplus_{i-j=k} S \otimes P_j^{i+1} \right) \oplus \left(\bigoplus_{i-j=k} S \otimes Q_j^i \right) \\ &= \mathbf{L}(P)^{k+1} \oplus \mathbf{L}(Q)^k = \text{Cone}(\mathbf{L}(f))^k. \end{aligned}$$

Now we compute the differentials of both $\mathbf{L}(\text{Cone}(f))$ and $\text{Cone}(\mathbf{L}(f))$ and see that they coincide. Consider $s \otimes (p \oplus q) \in \mathbf{L}(\text{Cone}(f))^k$ so $p \oplus q \in P_j^i$ for $i - j = k$.

$$\begin{aligned} d_{\mathbf{L}(\text{Cone}(f))}^k(s \otimes (p \oplus q)) &= s \otimes \begin{pmatrix} -d_P^{i+1} & 0 \\ f^{i+1} & d_Q^i \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} + (-1)^i \sum_m x_m s \otimes e_m(p \oplus q) \\ &= s \otimes \left(-d_P^{i+1}(p) \oplus (f^{i+1}(p) + d_Q^i(q)) \right) + (-1)^i \sum_{m=0}^n x_m s \otimes e_m(p \oplus q). \\ &= -d_{\mathbf{L}(P)}^{k+1}(s \otimes p) \oplus \mathbf{L}(f)^{k+1}(s \otimes p) + d_{\mathbf{L}(Q)}^k(s \otimes q) \\ &= \begin{pmatrix} -d_{\mathbf{L}(P)}^{k+1} & 0 \\ \mathbf{L}(f)^{k+1} & d_{\mathbf{L}(Q)}^k \end{pmatrix} \begin{pmatrix} s \otimes p \\ s \otimes q \end{pmatrix} = d_{\text{Cone}(\mathbf{L}(f))}^k((s \otimes p) \oplus (s \otimes q)). \end{aligned}$$

(4) Let $P \in \mathbf{C}^b(E)$ be an acyclic bounded complex. As we already mentioned, $\mathbf{L}(P)$ is the total complex of the double complex (2.4) which is bounded since P is a bounded complex of finite dimensional graded E -modules. By Proposition 1.1.38 it suffices to show that either the horizontal or vertical complexes are acyclic, in this case we prove for the horizontal complex.

We check that $\ker F^i \subseteq \text{im } F^{i-1}$. Let $s \otimes p \in S \otimes_k P^i$ such that $F^i(s \otimes p) = 0$ this implies that $d^i(p) = 0$ then $p \in \ker d^i = \text{im } d^{i-1}$. Therefore there exists $p' \in P^{i-1}$ such that $d^{i-1}(p') = p$ getting that $F^{i-1}(s \otimes p') = s \otimes d^{i-1}(p') = s \otimes p$ and then $s \otimes p \in \text{im } F^{i-1}$ as desired.

(5) Let $0 \rightarrow P \xrightarrow{f} Q \xrightarrow{g} R \rightarrow 0$ be a short exact sequence in $\mathbf{C}(E)$. We are going to show that the sequence $0 \rightarrow \mathbf{L}(P)^k \xrightarrow{\mathbf{L}(f)} \mathbf{L}(Q)^k \xrightarrow{\mathbf{L}(g)} \mathbf{L}(R)^k \rightarrow 0$ is exact.

Let $s \otimes p \in S \otimes P_j^i$ with $i - j = k$. By definition $\mathbf{L}(f)(s \otimes p) = s \otimes f(p)$ this is exactly the map $1_S \otimes f$, since both 1_S and f are injective then $\mathbf{L}(f)$ is injective. Similarly, as $\mathbf{L}(g) = 1_S \otimes g$ and both 1_S and g are surjective maps, then $\mathbf{L}(g)$ is surjective.

It remains to prove the exactness in the middle, for that, we prove $\ker \mathbf{L}(g) \subseteq \operatorname{im} \mathbf{L}(f)$. Let $s \otimes q \in S \otimes Q_j^i$ such that $\mathbf{L}(g)(s \otimes q) = s \otimes g(q) = 0$. This implies that $g(q) = 0$, now since $\ker g \subseteq \operatorname{im} f$, there exists $\tilde{p} \in P_j^i$ such that $f(\tilde{p}) = q$ and so $\mathbf{L}(s \otimes \tilde{p}) = s \otimes f(\tilde{p}) = s \otimes q$ and $s \otimes q \in \operatorname{im} \mathbf{L}(f)$ as desired. $\square$

By items (1) and (2) of the previous lemma, the functor $\mathbf{L}$ extends to a functor $\mathbf{L}: \mathbf{K}(E) \rightarrow \mathbf{K}(S)$. Moreover, by items (3) and (5), any distinguished triangle in $\mathbf{K}(E)$ $P \rightarrow Q \rightarrow \operatorname{Cone}(f) \rightarrow P[1]$ is mapped to a distinguished triangle $\mathbf{L}(P) \rightarrow \mathbf{L}(Q) \rightarrow \mathbf{L}(\operatorname{Cone}(f)) \rightarrow \mathbf{L}(P)[1]$ in $\mathbf{K}(S)$. Furthermore, by item (4), the the functor $\mathbf{L}: \mathbf{K}^b(E) \rightarrow \mathbf{K}^b(S)$ preserves quasi-isomorphisms. Consequently, the composition

$$\begin{array}{ccc} \mathbf{K}^b(E) & \xrightarrow{\mathbf{L}} & \mathbf{K}^b(S) \\ & \searrow & \downarrow Q_S \\ & & \mathbf{D}^b(S), \end{array}$$

sends quasi-isomorphisms to isomorphisms via the localization functor Q_S . By the universal property we obtain an induced functor $\mathbf{L}: \mathbf{D}^b(E) \rightarrow \mathbf{D}^b(S)$, which we continue to denote by $\mathbf{L}$.

For the functor $\mathbf{R}$ we have similar results. Because the proof is analogous to Lemma 2.3.1 we omit it.

Lemma 2.3.2. *The functor $\mathbf{R}: \mathbf{C}(S) \rightarrow \mathbf{C}(E)$:*

- (1) *Commutates with translation functor*
- (2) *Transforms homotopic maps.*
- (3) *Transforms cone map complexes.*
- (4) *The restriction to the category $\mathbf{C}^b(S)$ preserves quasi-isomorphisms.*
- (5) *Is exact.*

As we mentioned above from the previous lemma we can obtain an exact functor $\mathbf{R}: \mathbf{K}(S) \rightarrow \mathbf{K}(E)$ between triangulated categories. Furthermore, from this functor we obtain a well-defined functor $\mathbf{R}: \mathbf{D}^b(S) \rightarrow \mathbf{D}^b(E)$.

Proposition 2.3.3. *The isomorphism $\alpha_{P,M}: \operatorname{Hom}_{\mathbf{C}(S)}(\mathbf{L}(P), M) \rightarrow \operatorname{Hom}_{\mathbf{C}(E)}(P, \mathbf{R}(M))$ given by the Theorem 2.2.12 preserves nullhomotopic morphisms. Therefore, we obtain*

$$\operatorname{Ht}(\mathbf{L}(P), M) \cong \operatorname{Ht}(P, \mathbf{R}(M)).$$

Proof. Let $P \in \mathbf{C}(E)$ and $M, N \in \mathbf{C}(S)$. Consider the composition map

$$\pi(1_M)^*: \operatorname{Hom}_{\mathbf{C}(S)}(\operatorname{Cone}(1_M), N) \longrightarrow \operatorname{Hom}_{\mathbf{C}(S)}(M, N),$$

given by the projection $M \xrightarrow{1_M} M \xrightarrow{\pi(1_M)} \text{Cone}(1_M)$. We start by showing $\text{im}(\pi(1_M)^*) = \text{Ht}(M, N)$. Note that, by definition of $\pi(1_M)^*$ every morphism $\varphi: \text{Cone}(1_M) \rightarrow N$ is mapped to the composition

$$M \xrightarrow{\pi(1_M)} \text{Cone}(1_M) \xrightarrow{\varphi} N,$$

which clearly factorizes through $\text{Cone}(1_M)$, then by Lemma 1.1.44 the composition $\varphi \circ \pi(1_M)$ is a nullhomotopic morphism and thus we get the inclusion $\text{im}(\pi(1_M)^*) \subseteq \text{Ht}(M, N)$. For the converse, let $\psi: M \rightarrow N$ be a nullhomotopic morphism, again by Lemma 1.1.44 we have the factorization

$$\begin{array}{ccc} M & \xrightarrow{\psi} & N \\ & \searrow \pi(1_M) & \nearrow \\ & \text{Cone}(1_M) & \end{array}$$

this implies that $\psi \in \text{im}(\pi(1_M)^*)$. For the second statement we have,

$$\begin{aligned} \text{Ht}(\mathbf{L}(P), N) &= \text{im}(\text{Hom}_{\mathbf{C}(S)}(\text{Cone}(1_{\mathbf{L}(P)}), N) \rightarrow \text{Hom}_{\mathbf{C}(S)}(\mathbf{L}(P), N)) \\ &= \text{im}(\text{Hom}_{\mathbf{C}(S)}(\text{Cone}(\mathbf{L}(1_P)), N) \rightarrow \text{Hom}_{\mathbf{C}(S)}(\mathbf{L}(P), N)) \\ &= \text{im}(\text{Hom}_{\mathbf{C}(S)}(\mathbf{L}(\text{Cone}(1_P)), N) \rightarrow \text{Hom}_{\mathbf{C}(S)}(\mathbf{L}(P), N)) \\ &= \text{im}(\text{Hom}_{\mathbf{C}(E)}(\text{Cone}(1_P), \mathbf{R}(N)) \rightarrow \text{Hom}_{\mathbf{C}(E)}(P, \mathbf{R}(N)) = \text{Ht}(P, \mathbf{R}(N)). \end{aligned}$$

The second equality follows because $1_{\mathbf{L}(P)} = \mathbf{L}(1_P)$. The third equality follows by the item (1) of Lemma 2.3.1 and the fourth equality follows by the adjunction between $\mathbf{L}$ and $\mathbf{R}$, and we are done. $\square$

Proposition 2.3.4. *Let $P \in \mathbf{C}^b(E)$ and $M \in \mathbf{C}^b(S)$. Then*

$$\text{Hom}_{\mathbf{D}^b(E)}(\mathbf{L}(P), M) \cong \text{Hom}_{\mathbf{D}^b(S)}(P, \mathbf{R}(M)).$$

Proof. Let $P \in \mathbf{C}^b(E)$ be a bounded finitely generated complex of E -modules. As we noticed before, the complex $\mathbf{L}(P)$ corresponds to a free bounded complex of S -module, and thus $\mathbf{L}(P)$ is a projective object in $\mathbf{C}^b(S)$. Then, we obtain

$$\text{Hom}_{\mathbf{D}^b(S)}(\mathbf{L}(P), M) \cong \text{Hom}_{\mathbf{K}^b(S)}(\mathbf{L}(P), M) = \text{Hom}_{\mathbf{C}^b(S)}(\mathbf{L}(P), M) / \text{Ht}(\mathbf{L}(P), M),$$

where the isomorphism holds by the Remark 1.3.33 and the equality follows by (1.3). Similarly, let $M \in \mathbf{C}^b(S)$ be a bounded finitely generated complex of S -modules, again $\mathbf{R}(M)$ corresponds to a free bounded complex of E -modules and thus is injective. Now, using the Theorem 1.3.28 and the equality (1.3) we get

$$\text{Hom}_{\mathbf{D}^b(E)}(P, \mathbf{R}(M)) \cong \text{Hom}_{\mathbf{C}^b(E)}(P, \mathbf{R}(M)) / \text{Ht}(P, \mathbf{R}(M)).$$

Finally, following Proposition 2.3.3 and the Theorem 2.2.12

$$\begin{aligned} \text{Hom}_{\mathbf{D}^b(E)}(\mathbf{L}(P), M) &\cong \text{Hom}_{\mathbf{C}^b(S)}(\mathbf{L}(P), M) / \text{Ht}(\mathbf{L}(P), M) \\ &\cong \text{Hom}_{\mathbf{C}^b(S)}(P, \mathbf{R}(M)) / \text{Ht}(P, \mathbf{R}(M)) \cong \text{Hom}_{\mathbf{D}^b(S)}(P, \mathbf{R}(M)). \end{aligned} \quad \square$$

We construct a natural transformation between the identity functor $1_{\mathbf{C}(E)}: \mathbf{C}(E) \rightarrow \mathbf{C}(E)$ and the functor composition $\mathbf{R} \circ \mathbf{L}: \mathbf{C}(E) \rightarrow \mathbf{C}(E)$ as follows. Let $P \in \mathbf{C}(E)$ and $M \in \mathbf{C}(S)$, and recall the isomorphism $\alpha_{P,M}: \text{Hom}_{\mathbf{C}(S)}(\mathbf{L}(P), M) \rightarrow \text{Hom}_{\mathbf{C}(E)}(P, \mathbf{R}(M))$ obtained in Theorem 2.2.12. Let $1_{\mathbf{L}(P)}: \mathbf{L}(P) \rightarrow \mathbf{L}(P)$ be the identity morphism over $\mathbf{L}(P)$. Setting $M = \mathbf{L}(P)$ we define

$$i_P := \alpha(1_{\mathbf{L}(P)}): P \longrightarrow \mathbf{R}(\mathbf{L}(P)).$$

Clearly the family of morphisms i_P with $P \in \mathbf{C}(E)$ defines a natural transformation $1_{\mathbf{C}(E)} \Rightarrow \mathbf{R} \circ \mathbf{L}$.

Proposition 2.3.5. *If $P \in \mathbf{C}^b(E)$, then $i_P := \alpha(1_{\mathbf{L}(P)}): P \longrightarrow \mathbf{R}(\mathbf{L}(P))$ is a quasi-isomorphism.*

Proof. We proceed by induction on $\dim P$. Let V be the trivial one-dimensional complex, i.e., $V^0 = \mathbb{1}$ and $V^i = 0$ for $i \neq 0$. Define $U = \mathbf{R}(\mathbf{L}(V))$. Using the standard descriptions of the functors $\mathbf{L}$ and $\mathbf{R}$, we can explicitly describe U as follows:

$$U_j^i = \mathbf{R}(\mathbf{L}(V))_j^i = \bigoplus_m \text{Hom}_k(E_{m-j}, \mathbf{L}(V)_m^{i-m}).$$

Expanding $\mathbf{L}(V)_m^{i-m}$, we obtain:

$$U_j^i = \bigoplus_m \text{Hom}_k\left(E_{m-j}, \bigoplus_n S_{m-n} \otimes_k V_n^{i-m+n}\right).$$

Since V_n^{i-m+n} is nonzero only when $n = 0$, setting $n = m - i$ simplifies the expression:

$$U_j^i = \bigoplus_m \text{Hom}_k\left(E_{i-j}, S_i \otimes_k \mathbb{1}_{m-i}\right).$$

Taking $m = i$, we obtain

$$U_j^i = \text{Hom}_k(E_{i-j}, S_i).$$

Thus, we obtain the following complex:

$$U: \cdots \longrightarrow \text{Hom}_k(E_{i-1-j}, S_{i-1}) \longrightarrow \text{Hom}_k(E_{i-j}, S_i) \longrightarrow \text{Hom}_k(E_{i+1-j}, S_{i+1}) \longrightarrow \cdots$$

where the differential is given by

$$d_U^i: f \longmapsto [e \mapsto \sum_m x_m f(e_m e)].$$

This complex U is obtained from the standard Koszul complex (see [5, Chapter 8, Section 4]) for the symmetric algebra S , corresponding to the regular sequence given by the basis $\{x_0, \dots, x_n\}$ of W . By [17, Lemma 15.30.2], we obtain that $H^i(U) = 0$ for all $i \neq 0$. It remains to prove that for $i = 0$, the map $H^0(i_V)$ is an isomorphism. This follows by examining the definition of $i_V = \alpha(1_{\mathbf{L}(V)})$ in (2.7) and noting that the map $i_V: \mathbb{1} \rightarrow U_0^0 = \text{Hom}_k(k, k)$ sends $e \in \mathbb{1}$ to the identity map in $U_0^0 = \text{Hom}_k(k, k)$. Hence $i_V: V \rightarrow U$ is a quasi-isomorphism for the trivial one dimensional complex V . Furthermore, since any one dimensional complex P coincides with V up to shifts, then the base case is proved.

Assume that whenever $P \in \mathbf{C}^b(E)$ with $\dim_k P = n$, the map $i_P: P \rightarrow \mathbf{R}(\mathbf{L}(P))$ is a quasi-isomorphism. So, suppose we have P such that $\dim_k P = n + 1$, since can be embedded into an exact sequence

$$0 \longrightarrow P' \longrightarrow P \longrightarrow P'' \longrightarrow 0,$$

with $\dim_k P' = n$ and $\dim P'' = 1$. Since the functor composition $\mathbf{R} \circ \mathbf{L}$ is exact, from the above exact sequence we obtain the following commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & P' & \longrightarrow & P & \longrightarrow & P'' \longrightarrow 0 \\ & & \downarrow i_{P'} & & \downarrow i_P & & \downarrow i_{P''} \\ 0 & \longrightarrow & \mathbf{R}(\mathbf{L}(P')) & \longrightarrow & \mathbf{R}(\mathbf{L}(P)) & \longrightarrow & \mathbf{R}(\mathbf{L}(P'')) \longrightarrow 0. \end{array}$$

By the inductive hypothesis both $i_{P'}$ and $i_{P''}$ are quasi-isomorphisms, thus applying $H^i(-)$ and using the five lemma we obtain that i_P is a quasi-isomorphism. $\square$

Theorem 2.3.6. *The functor $\mathbf{L}: \mathbf{D}^b(E) \rightarrow \mathbf{D}^b(S)$ determines an equivalence of triangulated categories.*

Proof. As we already mentioned the functor $\mathbf{L}: \mathbf{D}^b(E) \rightarrow \mathbf{D}^b(S)$ is exact. It suffices to prove that $\mathbf{L}$ is fully faithful and essentially surjective:

Fully faithful: Let $P, \tilde{P} \in \mathbf{C}(E)$ and consider the composition

$$\mathrm{Hom}_{\mathbf{C}(E)}(P, \tilde{P}) \xrightarrow{\mathbf{L}} \mathrm{Hom}_{\mathbf{C}(S)}(\mathbf{L}(P), \mathbf{L}(\tilde{P})) \xrightarrow{\alpha} \mathrm{Hom}_{\mathbf{C}(E)}(P, \mathbf{R}(\mathbf{L}(\tilde{P})))$$

where α is the isomorphism in Theorem 2.2.12. We claim that the composition has the form $\psi \mapsto i_{\tilde{P}} \circ \psi$ for any map $\psi: P \rightarrow \tilde{P}$ in $\mathbf{C}(E)$ where $i_{\tilde{P}} = \alpha(1_{\mathbf{L}(\tilde{P})})$. Let $p \in P$, chasing the commutative map (2.7) we may check $\alpha(\mathbf{L}(\psi))(p) = (i_{\tilde{P}} \circ \psi)(p)$. Calculating each side:

$$\begin{aligned} \alpha(\mathbf{L}(\psi))(p) &= \Phi(\Gamma(\mathbf{L}(\psi)))(p) & (i_{\tilde{P}} \circ \psi)(p) &= \Phi(\Gamma(1_{\mathbf{L}(\tilde{P})}))(\psi(p)) \\ &\Rightarrow \left[\Phi(\Gamma(\mathbf{L}(\psi)))(p) \right](e) & &\Rightarrow \left[\Phi(\Gamma(1_{\mathbf{L}(\tilde{P})}))(\psi(p)) \right](e) \\ &= \Gamma(\mathbf{L}(\psi))(ep) & &= \Gamma(1_{\mathbf{L}(\tilde{P})})(e\psi(p)) \\ &= \mathbf{L}(\psi)(1 \otimes ep) & &= 1_{\mathbf{L}(\tilde{P})}(1 \otimes e\psi(p)) \\ &= 1 \otimes \psi(ep). & &= 1 \otimes e\psi(p). \end{aligned}$$

Clearly $1 \otimes \psi(ep) = 1 \otimes e\psi(p)$ and thus $\alpha(\mathbf{L}(\psi)) = i_{\tilde{P}} \circ \psi$. Now, by Proposition 2.3.5 the family of maps $i_{\tilde{P}} \in \mathrm{Hom}_{\mathbf{D}^b(E)}(\tilde{P}, \mathbf{R} \circ \mathbf{L}(\tilde{P}))$ are isomorphisms in the setting of bounded derived categories, this implies that the composition

$$\mathrm{Hom}_{\mathbf{D}^b(E)}(P, \tilde{P}) \xrightarrow{\mathbf{L}} \mathrm{Hom}_{\mathbf{D}^b(S)}(\mathbf{L}(P), \mathbf{L}(\tilde{P})) \xrightarrow{\alpha} \mathrm{Hom}_{\mathbf{D}^b(E)}(P, \mathbf{R}(\mathbf{L}(\tilde{P}))),$$

between bounded derived categories is an isomorphism. On the other hand α is an isomorphism by virtue of Proposition 2.3.4. Hence $L: \mathrm{Hom}_{\mathbf{D}^b(E)}(P, \tilde{P}) \rightarrow \mathrm{Hom}_{\mathbf{D}^b(S)}(\mathbf{L}(P), \mathbf{L}(\tilde{P}))$ is also an isomorphism, we are done.

Essentially surjective: According to the previous discussion, the image category $\mathbf{L}(\mathbf{C}^b(E))$ is a full triangulated subcategory. Moreover, any free complex $W \in \mathbf{D}^b(E)$ with a single generator can be expressed as $\mathbf{L}(P) = W$ where $\dim_k V = 1$. Since any complex of finitely generated graded S -modules can be constructed from modules concentrated in degree 0 by applying shifts, finite direct sums, and cones, the objects $\mathbf{L}(P)$ corresponding to one-dimensional complexes serve as generators in $\mathbf{D}^b(S)$. Therefore, $\mathbf{L}(\mathbf{D}^b(E)) = \mathbf{D}^b(S)$. $\square$

Chapter 3

Applications

In this chapter we will present, without proofs, how we can use the BGG correspondence to characterize the bounded derived category of coherent sheaves over the projective space $\mathbb{P}^n$ associated to an n -dimensional vector space. We will assume that the reader is familiar with the definition of coherent sheaves.

Let $\mathbf{D}^b(\mathbb{P}^n) := \mathbf{D}^b(\mathbf{Coh}(\mathbb{P}^n))$ be the bounded derived category of coherent sheaves for $\mathbb{P}^n$. The steps to obtain the desired equivalence are the following:

- (1) First of all, we will consider two thick subcategories, $\mathcal{I} \subseteq \mathbf{D}^b(E)$ and $\mathcal{F} \subseteq \mathbf{D}^b(S)$, which will induces an equivalence between the quotient categories $\mathbf{D}^b(E)/\mathcal{I}$ and $\mathbf{D}^b(S)/\mathcal{F}$.
- (2) Secondly, we will show that the stable module category $\mathbf{stmod}(E)$ of E and the quotient $\mathbf{D}^b(E)/\mathcal{I}$ are equivalent.
- (3) Finally, the bounded derived categories $\mathbf{D}^b(\mathbb{P}^n)$ and $\mathbf{D}^b(S)/\mathcal{F}$ becomes equivalent.

We start by defining the subcategories $\mathcal{I}$ and $\mathcal{F}$ of $\mathbf{D}^b(E)$ and $\mathbf{D}^b(S)$ respectively.

Definition 3.0.1.

- (1) Let $\mathcal{I}$ to be the full subcategory in $\mathbf{D}^b(E)$ consisting of complexes isomorphic, in $\mathbf{D}^b(E)$ to bounded complexes of the form

$$\cdots \longrightarrow F^{-1} \longrightarrow F^0 \longrightarrow F^1 \longrightarrow \cdots,$$

which all F^i are free E -modules with a finite number of generators.

- (2) Let $\mathcal{F}$ to be the full subcategory in $\mathbf{D}^b(S)$ consisting of complexes isomorphic, in $\mathbf{D}^b(S)$ to bounded complexes of the form

$$\cdots \longrightarrow M^{-1} \longrightarrow M^0 \longrightarrow M^1 \longrightarrow \cdots,$$

which all M^i are finite dimensional graded S -modules.

3.0.1 Equivalence between $\mathbf{D}^b(\mathbb{P}^n)$ and $\mathbf{D}^b(S)/\mathcal{F}$. Recall that $\mathbf{S-grmod}$ denote the category of finitely generated graded S -modules. Let $\tilde{\mathcal{S}}$ be the full subcategory of $\mathbf{S-grmod}$ consisting of finite

dimensional graded S -modules and consider the Serre quotient $\mathbf{S}\text{-grmod}/\tilde{\mathcal{S}}$ (see [17, 12.10 Serre subcategories]) with exact localization functor $q: \mathbf{S}\text{-grmod} \rightarrow \mathbf{S}\text{-grmod}/\tilde{\mathcal{S}}$. By Proposition 1.3.23 we obtain well-defined functor

$$q: \mathbf{D}^b(S) \longrightarrow \mathbf{D}^b(\mathbf{S}\text{-grmod}/\tilde{\mathcal{S}}).$$

Note that the subcategory $\mathcal{F}$ consists of complexes in $\mathbf{D}^b(S)$ whose objects are given by graded S -modules in $\tilde{\mathcal{S}}$, this implies that $\ker q = \mathcal{F}$, and so $\mathcal{F}$ is thick. Furthermore, q induces an equivalence

$$\mathbf{D}^b(S)/\mathcal{F} \cong \mathbf{D}^b(\mathbf{S}\text{-grmod}/\tilde{\mathcal{S}}).$$

Definition 3.0.2 (Quasi-finitely generated modules). *Let $M \in \mathbf{S}\text{-grMod}$. We say that M is **quasi-finitely generated** if for some integer d , $M_{\geq d}$ is finitely generated. Additionally, we say that M has **quasi-finite length** if for some integer d , $M_{\geq d}$ is the zero module.*

Denote by $\mathbf{S}\text{-qmod}$ the category of quasi-finitely generated graded S -modules, and by $\mathbf{qflmod}$ the full subcategory of quasi-finite length S -modules. This last subcategory is a Serre subcategory.

Proposition 3.0.3. *The inclusion functor $\mathbf{S}\text{-mod} \rightarrow \mathbf{S}\text{-qmod}$ induces an equivalence of categories*

$$\mathbf{S}\text{-mod}/\tilde{\mathcal{S}} \cong \mathbf{S}\text{-qmod}/\mathbf{qflmod}.$$

Proof. See [2, Lemma 3.9.11]. □

The next theorem corresponds to a fundamental result established by Serre on the structure of coherent sheaves of $\mathcal{O}_{\mathbb{P}^n}$ -modules on $\mathbb{P}^n$. It describes the relationship between graded modules over A and coherent sheaves. Specifically, coherent sheaves correspond to objects in the localization of the category of finitely generated graded modules by a certain torsion subcategory. For further details the reader may refer to [10, Sheaves of Modules].

Theorem 3.0.4. *There exists an equivalence of categories*

$$\mathbf{S}\text{-qmod}/\mathbf{qflmod} \cong \mathbf{Coh}(\mathbb{P}^n).$$

Proof. See [17, Proposition 30.14.4.] and [10, Sheaves of Modules, Exercise 5.9]. □

Following Proposition 3.0.3 and 3.0.4 we conclude the equivalence

$$\mathbf{D}^b(S)/\mathcal{F} \cong \mathbf{D}^b(\mathbb{P}^n).$$

3.0.2 Equivalence between $\mathbf{D}^b(E)/\mathcal{I}$ and $\mathbf{D}^b(S)/\mathcal{F}$. We start by characterizing the objects $\mathcal{F}$.

Lemma 3.0.5. *Let $W \in \mathbf{D}^b(S)$. $W \in \mathcal{F}$ if and only if $\dim_k H^i(W) < \infty$ for all i .*

Proof. If $W \in \mathcal{F}$, then all of its homologies are subspaces of each W^i , because these spaces are finite dimensional, their homologies are. Suppose now, that all the homologies of W are finite dimensional. Let $W(N)$ the subcomplex of W such that $W(N)^i$ consists of homogeneous elements of degree bigger than N in W^i . Then, there is a short exact sequence in the category of complexes $0 \rightarrow W(N) \rightarrow W \rightarrow W/W(N) \rightarrow 0$ and by hypothesis the middle term has finite dimensional homology, so the last one has and the first one too because the homology functor is homological. Clearly, we can take N big enough such that $\dim H^i(W(N)) = 0$. So, we get that the last two complexes are quasi-isomorphic and we are done. $\square$

Proposition 3.0.6. *The quotient categories $\mathbf{D}^b(E)/\mathcal{I}$ and $\mathbf{D}^b(S)/\mathcal{F}$ are equivalent.*

Proof. It is enough to see that $\mathbf{L}(\mathcal{I}) \subset \mathcal{F}$ and $\mathbf{R}(\mathcal{F}) \subset \mathcal{I}$. Let $W \in \mathcal{F}$, then $\mathbf{R}(W)$ is defined as a tensor product of complexes with finite dimensional graded components. Then all its components are free, and we are done. Let $V \in \mathcal{I}$, then each V^i is a free E -module. Because $\mathbf{L}(V)$ is defined as the Hom functor with respect to E , we must have that the homologies for $\mathbf{L}(V)^i$ are finite dimensional. So, by the previous Lemma, $\mathbf{L}(V) \in \mathcal{F}$. $\square$

3.0.3 Characterization of $\mathbf{D}^b(E)/\mathcal{I}$. In order to characterize the category $\mathbf{D}^b(E)/\mathcal{I}$ we introduce the concept of stable module category. Recall that the exterior algebra E is self-injective finite dimensional k -algebra.

Definition 3.0.7 (Stable module category). *The **stable module category** of E , denoted as $\mathbf{stmod}(E)$, is the additive category whose objects are the same as the objects of $\mathbf{E}\text{-grmod}$ but morphisms*

$$\mathrm{Hom}_{\mathbf{stmod}(E)}(X, Y) := \mathrm{Hom}_{\mathbf{E}\text{-grmod}}(X, Y) / P \mathrm{Hom}(X, Y),$$

where $P \mathrm{Hom}(X, Y)$ it is the subspace consisting of homomorphism $X \rightarrow Y$ that factors through some projective graded E -module. Thus, two maps f, g are equivalent in $\mathbf{stmod}(E)$ if and only if $f - g$ factors through a projective module.

The category $\mathbf{stmod}(E)$ has a triangulated structure, which is defined as follows. Given any morphism $\alpha: E \rightarrow B$ in $\mathbf{stmod}(E)$, choose an injection $i: X \rightarrow I$ into an injective object I . Consider the pushout (in $\mathbf{E}\text{-grmod}$) of α and i , yielding the following commutative diagram:

$$\begin{array}{ccccc} A & \xrightarrow{i} & I & \longrightarrow & \mathrm{coker} i \\ \alpha \downarrow & & \alpha' \downarrow & & \downarrow 1 \\ B & \xrightarrow{\beta} & C & \longrightarrow & \mathrm{coker} i. \end{array}$$

The image of the sequence

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow \mathrm{coker} i.$$

in $\mathbf{stmod}(E)$ define a triangulated structure over the stable module category. Now, considering the functor composition

$$\mathbf{E}\text{-grmod} \longrightarrow \mathbf{D}^b(E) \longrightarrow \mathbf{D}^b(E)/\mathcal{I}.$$

we note that any projective E -module P is mapped to zero, and so it factors through the functor

$$\mathbf{stmod}(E) \longrightarrow \mathbf{D}^b(E)/\mathcal{I}.$$

Theorem 3.0.8. *The functor $\mathbf{stmod}(E) \longrightarrow \mathbf{D}^b(E)/\mathcal{I}$ is exact and defines an equivalence of triangulated categories.*

Proof. See [15, Theorem 2.1.] □

Finally, let V and $(n + 1)$ -dimensional vector space. Then $\mathbb{P}^n = \mathbb{P}(V)$, and so, the previous results allow us to write the following sequence of equivalences:

$$\mathbf{D}^b(\mathbb{P}^n) \cong \mathbf{D}^b(S)/\mathcal{F} \cong \mathbf{D}^b(E)/\mathcal{I} \cong \mathbf{stmod}(E).$$

which characterizes the derived category of coherent sheaves over $\mathbb{P}^n$ as the stable category of modules for the exterior algebra E associated to V .

Chapter A

Graded algebras

This appendix provides a concise overview of graded rings and modules. For the following let k be a field.

Definition A.0.1 (Graded ring). A ring R is said to be **graded** there exists a collection of subgroups $\{R_i\}_{i \in \mathbb{Z}}$ such that: $R = \bigoplus_{i \in \mathbb{Z}} R_i$ and $R_i R_j \subseteq R_{i+j}$ for all $i, j \in \mathbb{Z}$.

Definition A.0.2 (Graded module). A **graded module** over a commutative ring R is an R -module M equipped with a decomposition $M = \bigoplus_{i \in \mathbb{Z}} M_i$ such that each M_i is an R -submodule of M . An element $m \in M_i$ is said to be **homogeneous of degree i** .

A map $\varphi: M \rightarrow N$ between two graded modules M and N is called a **homogeneous map of degree d** if, for every homogeneous element $m \in M_i$, we have $\varphi(m) \in N_{i+d}$. In particular, if φ is a homogeneous map of degree 0, then it is called a **degree-preserving homomorphism**.

For the following let V be a n -dimensional k -vector space. Roughly speaking it suffices for V to be an R -module.

Definition A.0.3 (Tensor algebra). For each $0 < i \in \mathbb{N}$ set $T^i(V) := V^{\otimes i}$ and $T^0(V) := k$. We define the **tensor algebra** of V as $T(V) := \bigoplus_{i \geq 0} T^i(V)$.

The tensor algebra $T(V)$ is a k -algebra via the product obtained from the distributivity and the isomorphism $T^i(V) \otimes T^j(V) = T^{i+j}(V)$. Explicitly, $(v_1 \otimes \cdots \otimes v_r)(v'_1 \otimes \cdots \otimes v'_t) := v_1 \otimes \cdots \otimes v_r \otimes v'_1 \otimes \cdots \otimes v'_t$ and for the scalar product we set $r(v_1 \otimes \cdots \otimes v_r) := rv_1 \otimes \cdots \otimes rv_r =: (v_1 \otimes \cdots \otimes v_r)r$ which is well-defined.

Definition A.0.4 (Exterior algebra). The **exterior algebra** of V , denoted as $\bigwedge(V)$, is the quotient k -algebra $T(V)/A(V)$ where $A(V)$ is the ideal of $T(V)$ generated by all elements of the form $w \otimes w$ for all $w \in V$. The image of $w_1 \otimes \cdots \otimes w_i$ in $\bigwedge(V)$ is called **wedge product** and denoted by $w_1 \wedge \cdots \wedge w_i$.

Remark A.0.5. The exterior algebra has the following properties:

- The ideal $A(V)$ is generated by homogeneous elements, hence, $A(V)$ is a graded ideal. More-

over, the exterior algebra is graded with the homogeneous components defined by $\bigwedge^i(V) = T^i(V)/A^i(V)$ where $A^i(V) = A(V) \cap T^i(V)$.

- The wedge product of $\bigwedge(V)$ anticommutes.
- For all $i > n$, $\bigwedge^i(V) = 0$ since in each summand of the decomposition $w_1 \wedge \cdots \wedge w_i$ through the generators, there are repeated elements. This implies that $\bigwedge V = \bigoplus_{i=0}^n \bigwedge^i V$.
- If $\{v_1, v_2, \dots, v_n\}$ is a basis of V , then a basis for $\bigwedge^i V$ is given by all wedge products of the form

$$v_{j_1} \wedge v_{j_2} \wedge \cdots \wedge v_{j_i},$$

where $1 \leq j_1 < j_2 < \cdots < j_i \leq n$. Consequently, the exterior algebra $\bigwedge(V)$ has a basis consisting of all wedge products $\{1, e_i, e_i \wedge e_j, e_i \wedge e_j \wedge e_k, \dots, e_0 \wedge e_1 \wedge \cdots \wedge e_n\}$. Furthermore, $\dim_k \bigwedge^i V = \binom{n}{i}$

Definition A.0.6 (Symmetric algebra). The **symmetric algebra** of V , denoted as $S(V)$, is the quotient k -algebra $T(V)/I(V)$ where $I(V)$ is the ideal of $T(V)$ generated by all elements of the form $w \otimes w' - w' \otimes w$ for all $w, w' \in V$. The image of $w_1 \otimes \cdots \otimes w_i$ in $\text{Sym}(V)$ is called the **symmetric product** and is denoted by $w_1 \cdots w_i$.

Remark A.0.7. The symmetric algebra has the following properties:

- The ideal $I(V)$ is generated by homogeneous elements, hence, $I(V)$ is a graded ideal. Moreover, the symmetric algebra is graded with the homogeneous components defined by $S^i(V) = T^i(V)/I^i(V)$, where $I^i(V) = I(V) \cap T^i(V)$.
- The symmetric product in $\text{Sym}(V)$ is commutative.
- Unlike the exterior algebra, there is no restriction on the degree of elements in $\text{Sym}(V)$, meaning that $S(V) = \bigoplus_{i=0}^{\infty} S^i(V)$.
- If $\{v_1, v_2, \dots, v_n\}$ is a basis of V , then a basis for $S^i(V)$ is given by all monomials of the form

$$v_{j_1} v_{j_2} \cdots v_{j_i},$$

where $1 \leq j_1 \leq j_2 \leq \cdots \leq j_i \leq n$. Consequently, the symmetric algebra $\text{Sym}(V)$ has a basis consisting of all monomials in $\{v_1, v_2, \dots, v_n\}$.

Definition A.0.8 (Bigraded module). A **bigraded module** over a commutative ring R is an R -module M equipped with a decomposition $M = \bigoplus_{(i,j) \in \mathbb{Z} \times \mathbb{Z}} M_{i,j}$ such that each $M_{i,j}$ is an R -submodule of M . An element $m \in M_{i,j}$ is said to be **homogeneous of bidegree** (i, j) .

A map $\varphi: M \rightarrow N$ between two bigraded modules M and N is called a **homogeneous map of bidegree** (a, b) if, for every homogeneous element $m \in M_{i,j}$, we have $\varphi(m) \in N_{i+a, j+b}$. In particular, if φ is a homogeneous map of bidegree $(0, 0)$, then it is called a **bidegree preserving homomorphism**.

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