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On compactifications and G-compactifications of Tychonoff spaces

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Abstract

This article deals with a characterization of those topological groups G acting on a Tychonoff space X and on a compactification $\widehat{X}$ of X, in such a way that $\widehat{X}$ turns out to be a G-compactification of X.

Keywords: Uniform spaces, G-spaces, compactificacions, G-compactifications.

MSC(2000): 54H11, 54E15

1 Introduction

In [3] V. A. Chatyrko and K. L. Kozlov proposed the following question: *"Can every compactification of a Tychonoff space be obtained as a G-compactification for some acting group G on X?"*. Formally speaking, this question has a trivial answer, namely, the trivial group acting on X and on any given compactification $\widehat{X}$ of X. In this paper we find out non-trivial solutions to the problem and study their relationships.

It is shown that if X is a Tychonoff space and $\widehat{X}$ is a compactification of X, then X is equipped with a unifom structure such that the group G of uniform homeomorphisms from X to itself, with the topology of uniform convergence, acts on X and on $\widehat{X}$ in such a way that $\widehat{X}$ turns out to be a G-compactification of the G-space X. Furthermore, it is shown that if $\widehat{X}$ is an H-compactification of the H-space X, for some topological group H, then H is homomorphic to a subgroup of G and its topology is finer than the topology of uniform convergence. This answers the above-mentioned question.

2 Terminology and preliminary considerations

For the purpose of establishing terminology and notation we state a few definitions and make two remarks to recall well known results that are indispensable in what follows. All topological spaces in this paper are assumed to be Tychonoff.

Let G be an arbitrary topological group. As usual, an action of G on a set X is a mapping $\psi : G \times X \longrightarrow X$ satisfying the following conditions:

1. $\psi(e, x) = x$, where e denotes the unit element of G, and
2. $\psi(g, \psi(h, x)) = \psi(gh, x)$,

for all $x \in X$ and $g,\ h \in G$.

By a G-space it is meant a Tychonoff space with a continuous action of the group G.

Let X_1 be a G_1-space, X_2 be a G_2-space and $\varphi : G_1 \longrightarrow G_2$ be a topological group embedding. A continuous function $\psi : X_1 \longrightarrow X_2$ is *equivariant with respect to* φ (or simply equivariant, if φ is understood from the context) provided that $\psi(gx) = \varphi(g)\psi(x)$, for all $g \in G_1$ and $x \in X_1$.

A morphism of G-spaces is a continuous, equivariant mapping.

If X is a G-space and there exists an equivariant dense embedding of X into a compact G-space $\widehat{X}$, then $\widehat{X}$ is called a G-compactification of X.

We require the following well known property of complete uniform spaces, cf. Theorem 8.3.10 [4].

Remark 2.1. *Every uniformly continuous function f from a subset A of a uniform space X into a complete uniform space Y can be extended to $\overline{A}$. Besides, if Y is a Hausdorff space, then the extension $\overline{f}$ of f is unique.*

Let $(X, \mathcal{U})$ be a uniform space and G the group of all uniform homeomorphisms from X onto X. By a uniform homeomorphism of X it is meant a uniformly continuous homeomorphism of X, such that its inverse is also uniformly continuous.

The topology of the uniform convergence over G is described as the one such that a fundamental system of neighborhoods of a point $f \in G$ is given by the sets of the form

$$O(f) = \{g \in G : (f(x), g(x)) \in O \text{ for every } x \in X\},$$

where $O \in \mathcal{U}$.

Proposition 2.2. *X is a G-space.*

Proof. The map $\mu : G \times X \longrightarrow X$ defined by $\mu(f, x) = f(x)$ is an action of G on X. To verify the continuity of μ, notice that if $O[f(x)]$ is a basic neighborhood of $f(x)$, with O a symmetric entourage of X, if E is a symmetric entourage of X such that $E \circ E \subset O$ and if V is a neighborhood of x such that $f(y) \in E[f(x)]$ for each $y \in V$, then for each $g \in E(f)$ and each $y \in V$, $(g(y), f(y)) \in E$ and $(f(y), f(x)) \in E$, then $(g(y), f(x)) \in O$, thus $\mu(g, y) = g(y) \in O[f(x)]$. $\square$

3 G-Compactifications

In this section it is shown that a compactification of a Tychonoff space X is a G-compactification of X, with respect to an appropriated uniform structure.

Let X be a Tychonoff space and $\varphi : X \longrightarrow \widehat{X}$ be a compactification of X.

The space $\widehat{X}$ is not only a Hausdorff space but also, due to its compactness, complete and totally bounded. Furthermore, the only uniform structure $\widehat{\mathcal{U}}$ compatible with its topology consists of the set of neighborhoods of the diagonal $\Delta = \{(w, w) : w \in \widehat{X}\}$. Let $\mathcal{U}$ be the initial uniform structure induced by φ over X.

Lemma 3.1. *The uniform structure $\mathcal{U}$ is compatible with the topology of X.*

Proof. If E is a symmetric entourage of $\widehat{\mathcal{U}}$, the continuity of φ implies that $\varphi^{-1}(E[\varphi(x)])$ is a neighborhood of x. Now,

$$\begin{aligned} w \in \varphi^{-1}(E[\varphi(x)]) &\iff \varphi(w) \in E[\varphi(x)], \\ &\iff (\varphi(x), \varphi(w)) \in E, \\ &\iff (x, w) \in (\varphi \times \varphi)^{-1}(E), \\ &\iff w \in (\varphi \times \varphi)^{-1}(E)[x], \end{aligned}$$

then the topology of X is finer than the topology induced by $\mathcal{U}$.

Conversely, let V be a neighborhood of x in X, since

$$\varphi : X \longrightarrow \varphi(X)$$

is a homeomorphism, $\varphi(V)$ is a neighborhood of $\varphi(x)$ in $\varphi(X)$, thus there exists a symmetric entourage E of $\widehat{X}$ such that $E[\varphi(x)] \cap \varphi(X) \subset \varphi(V)$. Since

$$\begin{aligned} w \in (\varphi \times \varphi)^{-1}(E)[x] &\iff \varphi(w) \in E[\varphi(x)] \cap \varphi(X), \\ &\Longrightarrow \varphi(w) \in \varphi(V), \\ &\iff w \in V, \end{aligned}$$

it follows that the topology of X is coarser than the topology induced by $\mathcal{U}$. $\square$

From now on, we will consider the uniform space $(X, \mathcal{U})$, where $\mathcal{U}$ is the initial uniform structure induced by φ over X.

Lemma 3.2. *The map $\varphi : X \longrightarrow \varphi(X)$ is a uniform homeomorphism.*

Proof. Keeping in mind that φ is a uniformly continuous one to one map, it suffices to see that $\varphi^{-1} : \varphi(X) \longrightarrow X$ is uniformly continuous too. A basic entourage of the uniformity defined over X has the form $(\varphi \times \varphi)^{-1}(E)$, where E is an entourage of $\widehat{X}$. It holds that

$$(\varphi \times \varphi)((\varphi \times \varphi)^{-1}(E)) = E \cap (\varphi(X) \times \varphi(X)),$$

which is an entourage of $\varphi(X)$. This shows the uniform continuity of φ^{-1}.

One concludes that $\varphi : X \longrightarrow \varphi(X)$ is a uniform homeomorphism. $\square$

Let G be the group of uniform homeomorphisms of X with the topology of uniform convergence. It was shown that G acts on X by means of the action μ defined by $\mu(f, x) = f(x)$.

Proposition 3.3. *$\widehat{X}$ is a G-space.*

Proof. The group G acts on $\widehat{X}$ by means of the action $\widehat{\mu} : G \times \widehat{X} \longrightarrow \widehat{X}$ defined by $\widehat{\mu}(f, w) = \widehat{f}(w)$, where $\widehat{f} : \widehat{X} \longrightarrow \widehat{X}$ is the only one uniformly continuous extension of the function $\varphi(x) \longmapsto \varphi(f(x)) : \varphi(X) \longrightarrow \widehat{X}$ to $\widehat{X}$. Indeed,

1. If f is the identity map from X onto itself, then $\widehat{f}$ is the identity map from $\widehat{X}$ onto $\widehat{X}$ and if $w \in X$, then $\widehat{\mu}(f, w) = w$.

2. Let $g,\ h \in G$ and $w \in \widehat{X}$. One has that

$$\begin{aligned}\widehat{\mu}(g, \widehat{\mu}(h, w)) &= \widehat{\mu}(g, \widehat{h}(w))\\ &= \widehat{g}(\widehat{h}(w))\\ &= (\widehat{g} \circ \widehat{h})(w)\\ &= \widehat{g \circ h}(w)\\ &= \widehat{\mu}(g \circ h, w).\end{aligned}$$

   Note that the function $\widehat{g} \circ \widehat{h}$ is a uniformly continuous extension of the function $\varphi(x) \longmapsto \varphi(g(h(x))) : \varphi(X) \longrightarrow \widehat{X}$, hence $\widehat{g} \circ \widehat{h} = \widehat{g \circ h}$.

3. To be seen that $\widehat{\mu}$ is a continuous function. Let $(g, w) \in G \times \widehat{X}$ and let $O[\widehat{g}(w)]$ be a basic neighborhood of $\widehat{\mu}(g, w) = \widehat{g}(w)$, where O is a symmetric entourage of $\widehat{X}$. Let E be a symmetric entourage of $\widehat{X}$ such that $E \circ E \circ E \circ E \subset O$ and consider $f \in (\varphi \times \varphi)^{-1}(E)(g)$ and $z \in \widehat{g}^{-1} \times \widehat{g}^{-1}(E)[w]$.

   In the first place, one has that $(f(x), g(x)) \in (\varphi \times \varphi)^{-1}(E)$, for every $x \in X$, thus $(\varphi(f(x)), \varphi(g(x))) \in E$, for every $x \in X$.

   In the second place, one has that $(z, w) \in \widehat{g}^{-1} \times \widehat{g}^{-1}(E)$, hence $(\widehat{g}(z), \widehat{g}(w)) \in E$.

   Consider a Cauchy net $(\varphi(x_\lambda))_{\lambda \in \Lambda}$ in $\varphi(X)$ converging to z. The Cauchy net $(\varphi(g(x_\lambda)))_\lambda$ converges in $\widehat{X}$ to $\widehat{g}(z)$ while $(\varphi(f(x_\lambda)))_\lambda$ converges to $\widehat{f}(z)$.

   Since Λ is a directed set, one can choose $\lambda_0 \in \Lambda$ such that if $\lambda > \lambda_0$, then $(\varphi(g(x_\lambda)), \widehat{g}(z)) \in E$ and $(\varphi(f(x_\lambda)), \widehat{f}(z)) \in E$. Let $\lambda > \lambda_0$, one has that $(\varphi(f(x_\lambda)), \widehat{f}(z)) \in E$ and $(\varphi(f(x_\lambda)), \varphi(g(x_\lambda))) \in E$, therefore $(\widehat{f}(z), \varphi(g(x_\lambda))) \in E \circ E$. Furthermore, $(\varphi(g(x_\lambda)), \widehat{g}(z)) \in E$, hence $(\widehat{f}(z), \widehat{g}(z)) \in E \circ E \circ E$ and since $(\widehat{g}(z), \widehat{g}(w)) \in E$, then

$$(\widehat{f}(z), \widehat{g}(w)) \in E \circ E \circ E \circ E \subset O,$$

   thus $\widehat{\mu}(f, z) = \widehat{f}(z) \in O[\widehat{\mu}(g, w)]$. This proves the continuity of $\widehat{\mu}$.

□

The next proposition answers the question proposed by V. A. Chatyrko and K. L. Kozlov cf. [3]: "Can every compactification of a Tychonoff space be obtained as a G-compactification for some acting group G on X?"

Proposition 3.4. *$\widehat{X}$ is a G-compactification of X.*

Proof. One has that for each $f \in G$ and each $x \in X$

$$\begin{aligned}\varphi(\mu(f,x)) &= \varphi(f(x))\\ &= \widehat{f}(\varphi(x))\\ &= \widehat{\mu}(f,\varphi(x)),\end{aligned}$$

it follows that φ is a dense equivariant inmersion of X in $\widehat{X}$, thus $\widehat{X}$ is a G-compactification of the G-space X. $\square$

Now suppose that X is an H-space and that $\widehat{X}$ is an H-compactification of X. Our aim is to find the relation between H and G.

Lemma 3.5. *For each $h \in H$, the function $g_h : X \longrightarrow X$ defined by $g_h(x) = hx$ is a uniform homeomorphism.*

Proof. The function $g_h : X \longrightarrow X$ is continuous, because it is the composition of the action of H on X with the function defined from X in $H \times X$ sending x into (h,x), and it is also bijective since if $x \in X$ then $x = g_h(h^{-1}x)$ and furthermore, if $g_h(x) = g_h(y)$, then $hx = hy$ and therefore $x = y$.

To prove that g_h is uniformly continuous, first we prove that if O is a symmetric entourage of $\widehat{X}$ and if $h \in H$, then $hO = \{(hw,hz) : (w,z) \in O\}$ is also an entourage of $\widehat{X}$. Note that the function $\psi : \widehat{X} \longrightarrow \widehat{X}$ defined by $\psi(w) = h^{-1}w$ is continuous, because it is the composition of the action of H on $\widehat{X}$ with the function defined from $\widehat{X}$ to $H \times \widehat{X}$ sending w into (h^{-1},w).

Let (w,w) be a point of the diagonal Δ of $\widehat{X}$. Since $(w,w) = (hh^{-1}w, hh^{-1}w)$, one has that $\Delta \subset hO$. Let E be a symmetric entourage of $\widehat{X}$ such that $E \circ E \subset O$. The continuity of ψ guarantees the existence of an entourage F of $\widehat{X}$ such that $h^{-1}F[w] = \{h^{-1}z : z \in F[w]\} \subset E[h^{-1}w]$. Now, if $(y,z) \in F[w] \times F[w]$, then $h^{-1}y \in E[h^{-1}w]$ and $h^{-1}z \in E[h^{-1}w]$, thus $(h^{-1}y, h^{-1}w) \in E$ and $(h^{-1}z, h^{-1}w) \in E$. It follows that $(h^{-1}y, h^{-1}z) \in O$, that is, $(y,z) \in hO$. Then $F[w] \times F[w] \subset hO$, therefore hO is a neighborhood of Δ in $\widehat{X} \times \widehat{X}$, hence hO is an entourage of $\widehat{X}$.

To be shown that g_h is uniformly continuous, let E be a symmetric entourage of $\widehat{X}$. It holds that

$$\begin{aligned}(x,y) \in (g_h^{-1} \times g_h^{-1})\left(((\varphi \times \varphi)^{-1})(E)\right) &\iff (g_h(x), g_h(y)) \in ((\varphi \times \varphi)^{-1})(E)\\ &\iff (hx,hy) \in ((\varphi \times \varphi)^{-1})(E)\\ &\iff (\varphi(hx), \varphi(hy)) \in E\\ &\iff (h\varphi(x), h\varphi(y)) \in E\\ &\iff (\varphi(x), \varphi(y)) \in h^{-1}E\\ &\iff (x,y) \in ((\varphi \times \varphi)^{-1}(h^{-1}E).\end{aligned}$$

On the other hand,

$$\begin{aligned} g_h^{-1}(y) = x &\iff g_h(x) = y, \\ &\iff hx = y, \\ &\iff x = h^{-1}y, \\ &\iff x = g_{h^{-1}}(y), \end{aligned}$$

thus $g_h^{-1} = g_{h^{-1}}$ and since $g_{h^{-1}}$ is uniformly continuous, then g_h is a uniform homeomorphism, that is, $g_h \in G$. □

Now define the equivalence relation $\sim$ over H by $h_1 \sim h_2$, if and only if, $h_1x = h_2x$, for every $x \in X$ and define the function $\Gamma : H/\sim \longrightarrow G$ by $\Gamma([h]) = g_h$. The function Γ is well defined, is one to one and Γ is a group homomorphism since if $[h_1],\ [h_2] \in H/\sim$, then

$$\begin{aligned} \Gamma([h_1][h_2])(x) &= \Gamma[h_1h_2](x) \\ &= g_{h_1h_2}(x) \\ &= h_1h_2x \\ &= g_{h_1}(h_2x) \\ &= g_{h_1}(g_{h_2}(x)) \\ &= (g_{h_1} \circ g_{h_2})(x). \end{aligned}$$

It has been shown that if X is an H-space and if $\widehat{X}$ is an H-compactification of X, then $H/\sim$ can be algebraically identified with a subgroup of G.

The following definition introduced in 1975 by J. de Vries [6] plays a crucial role in the proof of the continuity of the function Γ.

Definition 3.6. *An action of a topological group G on a space X is* bounded *if there exists a uniformity $\mathcal{U}$ on X compatible with its topology such that for each $U \in \mathcal{U}$ there exists a neighborhood O of the identity of G such that $(x, gx) \in U$, for each $x \in X$ and each $g \in O$.*

In [6], J. de Vries proved that a G-space X has a G-compactification if and only if the action of G on X is bounded.

Lemma 3.7. *The function Γ is continuous.*

Proof. We will show the continuity of the function Γ by showing that the function $h \longmapsto g_h : H \longrightarrow G$ is continuous: Let E be an entourage of $\widehat{X}$ and call σ the action of H on $\widehat{X}$. Since $\widehat{X}$ has an H-compactification, the action σ is bounded and since there exists only one uniformity compatible with the topology of $\widehat{X}$, then there exists a neighborhood O of the identity e of H such that $(\widehat{x}, h\widehat{x}) \in E$, for each $\widehat{x} \in \widehat{X}$ and each $h \in O$. It follows that for each $h \in O$ and each $x \in X$ one has that $(\varphi(x), h\varphi(x)) \in E$ and since φ is equivariant, it follows that $(\varphi(x), \varphi(hx)) \in E$.

Then $(x,hx)\in(\varphi\times\varphi)^{-1}(E)$, this means that $(x,g_hx)\in(\varphi\times\varphi)^{-1}(E)$, for every $x\in X$, thus $g_h\in(\varphi\times\varphi)^{-1}(E)(id_G)$.

Then the map $h\longmapsto g_h : H\longrightarrow G$ is continuous at e and consequently in all of H. $\square$

Given an entourage E of $\widehat{X}$, there exists a neighborhood O of e such that $(\widehat{x},h\widehat{x})\in E$, for every $\widehat{x}\in\widehat{X}$ and every $h\in O$. It follows that for each $x\in X$, $(\varphi(x),h\varphi(x))\in E$, hence $(x,hx)\in(\varphi\times\varphi)^{-1}(E)$, for every $x\in X$. Then $O\subset\{h\in H:(x,hx)\in(\varphi\times\varphi)^{-1}(E)$ for every $x\in X\}$. This shows that the topology of H is finer than the topology of the uniform convergence.

The next proposition follows from our considerations above.

Proposition 3.8. *If X is an H space and if $\widehat{X}$ is an H-compactification of X, then $H/\sim$ is a subgroup of G and H has a topology finer than the uniform convergence topology.*

Remark 3.9. *The quotient $H/\sim$ has been identify algebraically with a subgroup of G by means of a continuous homomorphism.*

If H is endowed with the uniform convergence topology, that is, with the topology τ such that the basic neighborhoods of the identity e are the sets of the form $\{h\in H:(x,hx)\in(\varphi\times\varphi)^{-1}(E),$ for every $x\in X\}$, where E is an entourage of $\widehat{X}$, then $H/\sim$ and $\Gamma(H/\sim)$ are homeomorphic.

4 Compactifications and completions

The initial unifom structure induced on X by the map $\varphi: X\longrightarrow\widehat{X}$ is such that φ turns out to be a uniformly continuous function. On the other hand, since $\widehat{X}$ is a compact space, it is complete and totally bounded and taking into account that X is a Tychonoff space, one concludes that $\widehat{X}$ is a Hausdorff space.

Suppose that Y is a complete Hausdorff uniform space and that $f: X\longrightarrow Y$ is uniformly continuous. The function $g:\varphi(X)\longrightarrow Y$ defined by $g(\varphi(x))=f(x)$ is well defined, since φ is one to one. Furthermore, g is uniformly continuous because if W is an entourage of Y, then $(f\times f)^{-1}(W)$ is an entourage of X, this means that there exists an entourage E of $\widehat{X}$ such that $(\varphi\times\varphi)^{-1}(E)\subset(f\times f)^{-1}(W)$. One has that $E\cap(\varphi(X)\times\varphi(X))\subset(g\times g)^{-1}(W)$, indeed, if $(\varphi(x),\varphi(y))\in E$, then $(x,y)\in(\varphi\times\varphi)^{-1}(E)$, thus $(f(x),f(y))\in W$, hence $(g(\varphi(x)),g(\varphi(y)))\in W$, therefore $(\varphi(x),\varphi(y))\in(g\times g)^{-1}(W)$. Then $(g\times g)^{-1}(W)$ is an entourage of $\varphi(X)$, it follows that g is uniformly continuous.

There exists a unique uniformly continuous extension $\widehat{g}:\widehat{X}\longrightarrow Y$ of g, this extension is the only one uniformly continuous function from $\widehat{X}$ to Y such that $\widehat{g}\varphi=f$. This shows the following proposition.

Proposition 4.1. *$\widehat{X}$ is the completion of the uniform space $(X,\mathcal{U})$.*

References

[1] Alvarado G.: Sobre compactificaciones de G-espacios, Master thesis, Universidad Nacional de Colombia, Bogotá, 2004.

[2] Arens R.: Topologies for Homeomorphism Groups, American Journal of Mathematics, Vol 68, (1946), no. 4, 593–610.

[3] Chatyrko V. A. and Kozlov K. L.: The Maximal G-compactifications of G-spaces with Special Actions, Proceedings of the Ninth Prague Topological Symposium, (2001), 15–21.

[4] Engelking R.: General topology, Heldermann, 1989.

[5] Ford L. R. Jr.: Homeomorphism groups and coset spaces, Transactions of the American Mathematical Society, Vol 77 (1954), no. 3, 490–497

[6] de Vries J.: On the existence of G-compactifications, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. Vol 26 (1978), no. 3, 275–280. MR 58# 31002.

[7] de Vries J.: On the G-compactifications of products, Pacific Journal of Mathematics Vol 110 (1984), no. 2, 447–470.

[8] Willard S.: General Topology, Addison-Wesley Publishing Company, Inc., Reading, Mass., 1970.

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