Vol. XII № 2 Dic. (2004)
Matemáticas: 17–35

**Matemáticas:**
**Enseñanza Universitaria**


# On the unsteady flow of an Oldroyd liquid B in a straight circular tube

D. Owen B.S. Bhat

Received Feb. 23, 2004 Revised Jun. 24, 2004

**Abstract**

Time dependent flow of an Oldroyd liquid B in a straight circular tube is examined. An approach to describe the flow when a general pressure-gradient and general initial conditions are given, is considered. The initial conditions imposed in the present work include the initial distributions of velocity and acceleration. Some general results are derived and three applications are considered.

**Keywords:** Fluid flow, Oldroyd liquid B, unsteady flow, circular tube.

**AMSC(2000):** Primary: 76F10, Secondary: 76D07

## 1 Introduction

The problem of flow of an Oldroyd liquid B in a straight circular tube has been examined by several authors, for example, Ghosh [1], Waters and King [2], Bhatnagar [3], Ramkissoon and Shifang [4] and Ramkissoon and Majumdar [5]. (Other references to other works on non-Newtonian flows may be found in the book by Kapur, Bhatt and Sacheti [6].)

Ref.[1] deals with the liquid in steady Poiseuille flow on which a general pressure-gradient is superimposed. The technique used in its solution is a combination of a finite Hankel and Laplace transformations and this gives a rather compact form for the velocity. (The pressure-gradients in the examples of Ghosh do not satisfy the condition he used to determine his eqn.(12) from eqn.(11), however.)

In ref.[2] the authors examine the generation of Poisuille flow due to a constant pressure-gradient as well as the decay of fully developed Poiseuille flow when the pressure-gradient is removed, using only Laplace transforms. Their primary goal is to explain certain anomalies found in experiments by Oliver and Macsporran [7].

Bhatnagar [3] considers his liquid to be under a pulsating pressure-gradient of non-zero mean. He represents his pressure-gradient, shear-stress and velocity by complex Fourier series and then derives and solves the resulting ordinary differential equations for the coefficients occuring in those Fourier series expansions.

Ramkissoon and Shifang [4] used the approximation of Kantorovich [8], to examine their flow. They considered the liquid starting from rest and give expressions for the first and second approximations.

Ramkissoon and Majumdar [5] used a Fourier-Bessel expansion technique to solve the constant pressure-gradient case and in the sinusoidal and exponential cases they obtained exact solutions, which will be seen in the present work, correspond to specific initial conditions. Here again their liquid starts from rest.

On closer examination we observe that [1] can deal with given initial distributions for the velocity and acceleration and hence the initial shear-stress. Ref.[2] appears not to be able to accommodate a given initial acceleration distribution since the only initial condition is on the velocity. The initial acceleration and shear-stress, if needed, are then obtained after the velocity is found.

Ref. [3] cannot accommodate either an initial velocity or acceleration. These are found afterwards. In [4] and [5] the initial velocity is given explicitly. The other initial condition used can be shown to imply a given initial acceleration.

It is when we eliminate the shear-stress, to get an equation containing the second time derivative for the velocity, that we must find two initial conditions on the velocity. Refs. [1, 4, 5] use this approach.

The object of the present work is to attempt to develop a theory which allows us to determine the behaviour of an Oldroyd liquid B in a straight circular tube given the initial distributions of the velocity, pressure-gradient, shear-stress and its derivative(or initial velocity and acceleration). We use a Fourier-Bessel expansion as in [5]. The present work, we feel, extends [1], [4] and [5] and compliments the others.

Results for the coresponding upper-convected Maxwell liquid may be obtained by setting the constant G, occuring in the present work, equal to zero.

## 2 Formutation of the equations

The flow which takes place in a straight tube of circular cross section $a$ is defined by the velocity field $q = (0, 0, \omega(r, t))$ in a cylindrical coordinate system, the $z$-axis being along the axis of the tube. The equations of motion reduce to

$$\rho \frac{\partial \omega}{\partial t} = -\frac{\partial p}{\partial z} + \frac{1}{r} S_{rz} + \frac{\partial S_{rz}}{\partial r} \tag{1}$$

where $S$ is the extra stress tensor, $p$ is the isotropic pressure and the constitutive equation is given by

$$S + \lambda_1 \tilde{S} = 2\mu D + 2\lambda_2 \tilde{D} \tag{2}$$

where $D$ is the deformation rate tensor, $\lambda_1$ is the relaxation time, $\lambda_2$ the retardation time, $\mu$ a viscosity coefficient and the symbol $\tilde{}$ denotes the upper convected Oldroyd derivative. Using eqn.(2) and defining dimensionless variables,

$$\phi = \frac{\mu\omega}{(\frac{\Delta p}{L})a^2}, \eta = \frac{r}{a}, \tau = \frac{\mu t}{\rho a^2}, G = \frac{\lambda_2}{\rho a^2}, H = \frac{\lambda_1 \mu}{\rho a^2} \equiv \frac{W_e}{R_e},$$

where $a$ is the radius of the tube, $\frac{\Delta p}{L}$ is a characteristic pressure-gradient in the direction of flow, $W_e$ is the Weisenberg number and $R_e$ the Reynolds number, we get the following dimensionless equation for $\phi$

$$H\frac{\partial^2\phi}{\partial\tau^2} + \frac{\partial\phi}{\partial\tau} - (\frac{\partial^2\phi}{\partial\eta^2} + \frac{1}{\eta}\frac{\partial\phi}{\partial\eta}) - G\frac{\partial}{\partial\tau}(\frac{\partial^2\phi}{\partial\eta^2} + \frac{1}{\eta}\frac{\partial\phi}{\partial\eta}) = \Psi(\tau) \tag{3}$$

where

$$\Psi(\tau) = -\frac{L}{\Delta p}\frac{\partial p}{\partial z} + H\frac{\partial}{\partial\tau}(-\frac{L}{\Delta p}\frac{\partial p}{\partial z}). \tag{4}$$

The solution of eqn.(3) will be taken to satisfy the following boundary and initial conditions

$$\begin{cases} \phi(1,\tau) = 0, \frac{\partial\phi(\eta,0)}{\partial\tau} = \Lambda(\eta), \Lambda(1) = 0, \\ \phi(\eta,0) = \Phi(\eta), \frac{\partial\phi(0,\tau)}{\partial\eta} = 0 \end{cases} \tag{5}$$

where $\Phi(\eta)$ may or may not be zero.

The second condition is the extra initial condition mentioned above which, since it is the derivative of a velocity with respect to time at $\tau = 0$, represents an acceleration distribution at time $\tau = 0$. Actually, after making eqn.(1) dimensionless, we get

$$\frac{\partial\phi}{\partial\tau} = -\frac{L}{\Delta p}\frac{\partial p}{\partial z} + \frac{L}{\Delta p}(\frac{1}{a\eta}S_{rz} + \frac{1}{a}\frac{\partial S_{rz}}{\partial\eta})$$

and thus we see that this latter initial condition is equivalent to knowing (a) the dimensionless initial pressure-gradient and (b) the initial shear-stress and its derivative. The fourth condition says that the initial velocity distribution need not be zero. The fifth condition is due to symmetry.

In this work we shall consider cases where the fluid starts from rest, $\Phi = 0$, and also when the fluid is already in motion and is observed to have some velocity distribution, $\Phi(\eta)$, and acceleration distribution, $\Lambda(\eta)$, at a particular instant of time which we take as $\tau = 0$.

## 3 General solution

We shall now seek the general solution, $\phi(\eta,\tau)$, of eqn.(3) and to this end we let

$$\phi(\eta,\tau) = \phi_1(\eta,\tau) + \phi_0(\eta,\tau), \ where \ \phi_1(1,\tau) + \phi_0(1,\tau) = 0.$$

When substituted into eqn.(3) we get

$$\begin{aligned}\Psi(\tau) = H\frac{\partial^2\phi_1}{\partial\tau^2} + \frac{\partial\phi_1}{\partial\tau} - \left(\frac{\partial^2\phi_1}{\partial\eta^2} + \frac{1}{\eta}\frac{\partial\phi_1}{\partial\eta}\right) \\ - G\frac{\partial}{\partial\tau}\left(\frac{\partial^2\phi_1}{\partial\eta^2} + \frac{1}{\eta}\frac{\partial\phi_1}{\partial\eta}\right) + H\frac{\partial^2\phi_0}{\partial\tau^2} + \frac{\partial\phi_0}{\partial\tau} \\ - \left(\frac{\partial^2\phi_0}{\partial\eta^2} + \frac{1}{\eta}\frac{\partial\phi_0}{\partial\eta}\right) - G\frac{\partial}{\partial\tau}\left(\frac{\partial^2\phi_0}{\partial\eta^2} + \frac{1}{\eta}\frac{\partial\phi_0}{\partial\eta}\right).\end{aligned}$$

Now choose $\phi_0$ to be a particular of eqn.(3) and $\phi_1$ to be the general solution of the homogenous equation

$$\begin{aligned}H\frac{\partial^2\phi_1}{\partial\tau^2} + \frac{\partial\phi_1}{\partial\tau} - \left(\frac{\partial^2\phi_1}{\partial\eta^2} + \frac{1}{\eta}\frac{\partial\phi_1}{\partial\eta}\right) \\ -G\frac{\partial}{\partial\tau}\left(\frac{\partial^2\phi_1}{\partial\eta^2} + \frac{1}{\eta}\frac{\partial\phi_1}{\partial\eta}\right) = 0.\end{aligned} \tag{6}$$

The general solution of eqn.(6) will be obtained by expressing $\phi_1(\eta,\tau)$ as a Fourier-Bessel series and we shall obtain this solution first.

## 4 General solution of the homogenous equation

We assume that in general we can express $\phi_1(\eta,\tau)$ by a Fourier-Bessel expansion

$$\phi_1(\eta,\tau) = \sum_{m=1}^{\infty} C_m(\tau)J_0(k_m\eta) \tag{7}$$

where

$$\begin{aligned}C_m(\tau) = \frac{2}{J_1(k_m)^2}\int_0^1 \eta\phi_1(\eta,\tau)J_0(k_m\eta)d\eta, \\ C'_m(\tau) = \frac{2}{J_1(k_m)^2}\int_0^1 \eta\frac{\partial\phi_1(\eta,\tau)}{\partial\tau}J_0(k_m\eta)d\eta,\end{aligned} \tag{8}$$

$k_m$ being the positive zeros of $J_0(k_m)$. Substituting eqn.(7) into the homogenous eqn.(6), using eqn.(8) and using

$$\int_0^1 \eta J_0(k_m\eta)(\frac{\partial^2\phi_1}{\partial\eta^2} + \frac{1}{\eta}\frac{\partial\phi_1}{\partial\eta})d\eta = -\frac{1}{2}k_m^2 J_1(k_m\eta)^2 C_m(\tau) \tag{9}$$

we get

$$HC''_m(\tau) + (1 + Gk_m^2)C'_m(\tau) + k_m^2 C_m(\tau) = 0 \tag{10}$$

The solution of eqn.(10) is

$$C_m(\tau) = A_m e^{\gamma_+(m)\tau} + B_m e^{\gamma_-(m)\tau}, \tag{11}$$

where $\gamma_\pm(m) = \frac{-(1+Gk_m^2)\pm\sqrt{(1+Gk_m^2)^2-4Hk_m^2}}{2H}$, $H \neq 0$.

Eqns.(8) and (11) give

$$\begin{aligned} C_m(0) &= A_m + B_m \\ &= \frac{2}{J_1(k_m)^2}\int_0^1 \eta\phi_1(\eta,0)J_0(k_m\eta)d\eta \equiv -D_m(0), \end{aligned} \tag{12}$$

and

$$\begin{aligned} C'_m(0) &= A_m\gamma_+(m) + B_m\gamma_-(m) \\ &= \frac{2}{J_1(k_m)^2}\int_0^1 \eta\Lambda(\eta)J_0(k_m\eta)d\eta \equiv \Gamma(m) \end{aligned} \tag{13}$$

where

$$\begin{aligned} \Lambda(\eta) &= \frac{\partial\phi_1(\eta,0)}{\partial\tau}, \\ D_m(\tau) &= -\frac{2}{J_1(k_m)^2}\int_0^1 \eta\phi_1(\eta,\tau)J_0(k_m\eta)d\eta. \end{aligned} \tag{14}$$

Solving eqns.(12) and (13) for $A_m$ and $B_m$ we get

$$A_m = \frac{\Gamma(m) + \gamma_-(m)D_m(0)}{\gamma_+(m) - \gamma_-(m)} \tag{15}$$

and

$$B_m = -\frac{\Gamma(m) + \gamma_+(m)D_m(0)}{\gamma_+(m) - \gamma_-(m)}. \tag{16}$$

The formal general solution, $\phi_1(\eta,\tau)$, of the homogenous equation (6) is thus

$$\phi_1(\eta,\tau) = \sum_{m=1}^{\infty}(A_m e^{\gamma_+(m)\tau} + B_m e^{\gamma_-(m)\tau})J_0(k_m\eta) \tag{17}$$

where $A_m$ and $B_m$ are given by eqns.(15) and (16). (We observe that $\gamma_\pm$ will become complex when $k_m$ gets into the interval

$$\left(\frac{1}{G}\sqrt{(2H-G)-2\sqrt{H(H-G)}},\ \frac{1}{G}\sqrt{(2H-G)+2\sqrt{H(H-G)}}\right)$$

and hence $A_m$ and $B_m$ will also be complex. However, it is easy to show that in this interval the quantity $A_m e^{\gamma_+(m)\tau} + B_m e^{\gamma_-(m)\tau}$ is real and is equal to

$$2e^{\tau R_m}(D_m(0)Q_m \cos(Q_m\tau) - (\Gamma(m) + R_m D_m(0)) \sin(Q_m\tau))$$

where $R_m = -\frac{(1+Gk_m^2)}{2H}$ and $Q_m = \frac{\sqrt{4Gk_m^2-(1+Gk_m^2)^2}}{2H}$. Outside the above interval $\gamma_\pm(m)$ are real and negative so $\phi(\eta,\tau)$ is always real.)

The formal general solution of the inhomogenous eqn.(3) for a liquid starting with initial velocity $\phi(\eta,0) = \phi_1(\eta,0) + \phi_0(\eta,0)$ and initial acceleration $\frac{\partial\phi(\eta,0)}{\partial\tau} = \Lambda(\eta)$ is thus

$$\begin{aligned}\phi(\eta,\tau) &= \phi_1(\eta,\tau) + \phi_0(\eta,\tau)\\ &= \sum_{m=1}^{\infty}(A_m e^{\gamma_+(m)\tau} + B_m e^{\gamma_-(m)\tau})J_0(k_m\eta) + \phi_0(\eta,\tau)\end{aligned} \tag{18}$$

We note that the general solution depends on the initial values $\phi_1(\eta,0) = \phi(\eta,0) - \phi_0(\eta,0)$ and $\frac{\partial\phi_1(\eta,0)}{\partial\tau} = \frac{\partial\phi(\eta,0)}{\partial\tau} - \frac{\partial\phi_0(\eta,0)}{\partial\tau}$ from eqns.(12) and (13) and (15) and (16).

It is also important to remember that knowledge of the initial acceleration, $\frac{\partial\phi(\eta,0)}{\partial\tau}$, means knowledge of the initial pressure-gradient and the initial shear-stress and its derivative as is seen from the dimensionless form of eqn.(1):

$$\frac{\partial\phi}{\partial\tau} = -\frac{L}{\Delta p}\frac{\partial p}{\partial z} + \frac{L}{\Delta p}(\frac{1}{a\eta}S_{rz} + \frac{1}{a}\frac{\partial S_{rz}}{\partial\eta}).$$

So knowing the initial velocity, the initial pressure-gradient, the initial shear-stress and its derivative (or equivalently the initial velocity and acceleration) and a particular solution of the inhomogenous equation we can solve the general problem.

### 4.1 Particular solution of the inhomogenous equation

To complete the general solution we need to find a particular solution $\phi_0(\eta,\tau)$ of the inhomogenous equation (3).

Let us now return to eqn.(3) and seek a particular solution. Eqn.(3) is

$$H\frac{\partial^2\phi_0}{\partial\tau^2} + \frac{\partial\phi_0}{\partial\tau} - (\frac{\partial^2\phi_0}{\partial\eta^2} + \frac{1}{\eta}\frac{\partial\phi_0}{\partial\eta}) - G\frac{\partial}{\partial\tau}(\frac{\partial^2\phi_0}{\partial\eta^2} + \frac{1}{\eta}\frac{\partial\phi_0}{\partial\eta}) = \Psi(\tau).$$

We shall first seek solutions of separable form i.e. $\phi_0(\eta,\tau) = \Theta(\eta)T(\tau)$ where we want $\Theta(\eta)$ to vanish at $\eta = 1$. In some cases we shall have to use other methods to find a particular solution.

On substituting $\phi_0 = \Theta T$ into eqn.(3) we get

$$\Theta(H\ddot{T} + \dot{T}) - (T + G\dot{T})(\Theta'' + \frac{\Theta'}{\eta}) = \Psi(\tau)$$

or

$$\Theta'' + \frac{\Theta'}{\eta} - (\frac{H\ddot{T}+\dot{T}}{G\dot{T}+T})\Theta = -\frac{\Psi(\tau)}{G\dot{T}+T},\ G\dot{T}+T \neq 0. \tag{19}$$

We now make the following assumptions:

1.

$$\frac{H\ddot{T}+\dot{T}}{G\dot{T}+T} = -\nu_1 \equiv const. \tag{20}$$

where $G\dot{T}+T \neq 0$,

2.

$$\frac{\Psi(\tau)}{G\dot{T}+T} = \frac{y + H\frac{\partial y}{\partial \tau}}{G\dot{T}+T} = \nu_2 \equiv const. \tag{21}$$

where $y = -\frac{L}{\Delta p}\frac{\partial p}{\partial z},\ \Psi(\tau) = y + H\frac{\partial y}{\partial \tau},\ G\dot{T}+T \neq 0.$
$\Theta$ will then satisfy

$$\Theta'' + \frac{1}{\eta}\Theta' + \nu_1\Theta = -\nu_2. \tag{22}$$

If $\Psi(\tau) \neq 0$ we can write eqns.(20), (21) as

$$H\ddot{T}+\dot{T} = -\tilde{\nu}_1\Psi(\tau),\ \tilde{\nu}_1 = \frac{\nu_1}{\nu_2},\ \nu_2 \neq 0 \tag{23}$$

and

$$G\dot{T}+T = \tilde{\nu}_2(y + H\frac{\partial y}{\partial \tau}),\ \tilde{\nu}_2 = \frac{1}{\nu_2},\ \nu_2 \neq 0. \tag{24}$$

We can now show that $\Psi(\tau)$ satisfies

$$H\tilde{\nu}_2\frac{\partial^2\Psi(\tau)}{\partial\tau^2} + (G\tilde{\nu}_1 + \tilde{\nu}_2)\frac{\partial\Psi}{\partial\tau} + \tilde{\nu}_1\Psi(\tau) = 0. \tag{25}$$

If $\Psi(\tau) \neq 0$ the solution, $T$, of eqn.(24) is

$$T = ce^{-\frac{\tau}{G}} + \frac{e^{-\frac{\tau}{G}}}{G\nu_2}\int e^{\frac{\tau}{G}}\Psi(\tau)d\tau. \tag{26}$$

For a given $y$, i.e. a given pressure-gradient, eqn.(25) may give a relation between the constant parameters or we may get an inconsistent equation which would imply that we cannot use separation of variables.

A formal particular solution, of separable form, of eqn.(3) is

$$\begin{aligned}\phi_0(\eta,\tau) = &-\frac{1}{\nu_1}\left(1-\frac{J_0(\eta\sqrt{\nu_1})}{J_0(\sqrt{\nu_1})}\right)\left(ce^{-\frac{\tau}{G}}+\frac{H}{G}y\right. \\ &\left.+\left(\frac{G-H}{G^2}\right)e^{-\frac{\tau}{G}}\int e^{\frac{\tau}{G}}y(\tau)d\tau\right),\ \nu_1\neq 0,\ \nu_2\neq 0\end{aligned}$$

and hence we may write the general solution of eqn.(3) as

$$\begin{aligned}\phi(\eta,\tau) = &\sum_{m=1}^{\infty}(A_m e^{\gamma_+(m)\tau}+B_m e^{\gamma_-(m)\tau})J_0(k_m\eta) \\ &-\frac{1}{\nu_1}\left(1-\frac{J_0(\eta\sqrt{\nu_1})}{J_0(\sqrt{\nu_1})}\right)\left(ce^{-\frac{\tau}{G}}+\frac{H}{G}y\right. \\ &\left.+\left(\frac{G-H}{G^2}\right)e^{-\frac{\tau}{G}}\int e^{\frac{\tau}{G}}y(\tau)d\tau\right),\ \nu_1\neq 0,\ \nu_2\neq 0.\end{aligned}$$

## 5 Some general results

We now derive some general results.

### 5.1 Liquid starting from rest

We shall as a first example, consider a fluid starting from rest. This means that we have $\phi(\eta,0)=\phi_1(\eta,0)+\phi_0(\eta,0)=0$ i.e. $\phi_1(\eta,0)=-\phi_0(\eta,0)$.

**Case with $\phi_0\neq 0,\ \frac{\partial\phi_0(\eta,0)}{\partial\tau}=0$**

In this case, since $\phi_1(\eta,0)=-\phi_0(\eta,0)$, we have from eqn.(14)

$$D_m(\tau)=\frac{2}{J_1(k_m)^2}\int_0^1 \eta\phi_0(\eta,\tau)J_0(k_m\eta)d\eta \tag{27}$$

and

$$\Lambda(\eta)=\frac{\partial\phi_1(\eta,0)}{\partial\tau}=-\frac{\partial\phi_0(\eta,0)}{\partial\tau}. \tag{28}$$

The general solution, eqn.(18), of the homogenous equation which reduces to $-\phi_0(\eta,0)$ at $\tau=0$ is thus

$$\phi_1(\eta,\tau)=\sum_{m=1}^{\infty}(A_m e^{\gamma_+(m)\tau}+B_m e^{\gamma_-(m)\tau})J_0(k_m\eta)$$

where $D_m$ and $\Lambda$ are given by eqns.(27) and (28).

Therefore the velocity distribution for a liquid starting from rest with initial acceleration

$$\Lambda(\eta)=-\frac{L}{\Delta p}\frac{\partial p(0)}{\partial z}+\frac{L}{\Delta p}(\frac{S_{rz}(\eta,0)}{a\eta}+\frac{1}{a}\frac{\partial S_{rz}(\eta,0)}{\partial\eta})$$

is

$$\phi(\eta,\tau)=\sum_{m=1}^{\infty}(A_m e^{\gamma_+(m)\tau}+B_m e^{\gamma_-(m)\tau})J_0(k_m\eta)+\phi_0(\eta,\tau). \tag{29}$$

**Case with $\phi_0(\eta,0)=0$**

We note that if the particular solution of the inhomogenous equation, $\phi_0(\eta,\tau)$, is such that $\phi_0(\eta,0)=0$, then for a liquid starting from rest the condition $\phi(\eta,\tau)=0$ implies that $\phi_1(\eta,0)=0$ also and from eqn.(7) $C_m(0)=A_m+B_m=0$.

Now from eqn.(18)

$$\frac{\partial\phi(\eta,0)}{\partial\tau}=\sum_{m=1}^{\infty}C'_m(0)J_0(k_m\eta)+\frac{\partial\phi_0(\eta,0)}{\partial\tau}, \tag{30}$$

and if we choose the initial acceleration $\frac{\partial\phi(\eta,0)}{\partial\tau}=\frac{\partial\phi_0(\eta,0)}{\partial\tau}$ then from eqns.(11) and (30)

$$C'(0)=A_m\gamma_+(m)+B_m\gamma_-(m)=0$$

and hence $A_m=0=B_m$ i.e. $C_m=0$ for all $\tau\geq 0$. This gives $\phi(\eta,\tau)=\phi_0(\eta,\tau)$ for all $\tau\geq 0$.

We can interpret this as the following Special Result I:

If the liquid starts from rest at $\tau=0$ i.e. $\phi(\eta,0)=\phi_1(\eta,0)+\phi_0(\eta,0)=0$ with $\phi_0(\eta,0)=0$, ($\phi_0(\eta,\tau)$ a particular solution of the inhomogenous equation) and an initial acceleration $\frac{\partial\phi(\eta,0)}{\partial\tau}=\frac{\phi_0(\eta,0)}{\partial\tau}$ i.e. initial shear-stress given by

$$PS_{rz}=\frac{1}{\eta}\int_0^{\eta}u(\frac{L}{\Delta p}\frac{\partial p(0)}{\partial z}+\frac{\partial\phi_0(u,0)}{\partial\tau})du, \tag{31}$$

where $P=\frac{L}{a\Delta p}$, then the velocity distribution $\phi(\eta,\tau)=\phi_0(\eta,\tau)$ for all $\tau\geq 02$.

## 5.2 Liquid in motion

**Case whit $\phi(\eta,0)\neq 0$**

For a liquid already in motion at $\tau=0$ we have $\phi(\eta,0)\neq 0$. If $\phi(\eta,0)=f(\eta)\neq 0$, that is, the liquid is not starting from rest but is observed to have a velocity distribution $f(\eta) \neq 0$ at $\tau = 0$, then we must have from eqn.(18)

$$\phi(\eta, 0) = f(\eta) = \sum_{m=1}^{\infty} C_m(0) J_0(k_m \eta) + \phi_0(\eta, 0)$$

so

$$\sum_{m=1}^{\infty} C_m(0) J_0(k_m \eta) = \phi(\eta, 0) - \phi_0(\eta, 0).$$

If $\phi(\eta, 0) = \phi_0(\eta, 0)$ then

$$C_m(0) = A_m + B_m = 0 \tag{32}$$

and since $\frac{\partial\phi(\eta,0)}{\partial\tau} = \sum_{m=1}^{\infty} C'_m J_0(k_m\eta) + \frac{\partial\phi_0(\eta,0)}{\partial\tau}$, i.e. since $\sum_{m=1}^{\infty} C'_m J_0(k_m\eta) = \frac{\partial\phi(\eta,0)}{\partial\tau} - \frac{\partial\phi_0(\eta,0)}{\partial\tau}$, then if the initial acceleration satisfies $\frac{\partial\phi(\eta,0)}{\partial\tau} = \frac{\phi_0(\eta,0)}{\partial\tau}$ we get from eqn.(11) $C'_m(0) = A_m\gamma_+(m) + B_m\gamma_-(m) = 0$ which with eqn.(32) gives $C_m(\tau) = 0$ for all $\tau \geq 0$ and hence $\phi(\eta, \tau) = \phi_0(\eta, \tau)$ for all $\tau \geq 0$.

We can interpret this as the following Special Result II:

If initially the liquid has a velocity distribution $\phi(\eta, 0) = \phi_0(\eta, 0) \neq 0$ ($\phi_0(\eta, \tau)$ a particular solution of the inhomogenous equation) and an acceleration distribution $\frac{\partial\phi(\eta,0)}{\partial\tau} = \frac{\partial\phi_0(\eta,0)}{\partial\tau}$ i.e. an initial shear-stress given by

$$PS_{rz} = \frac{1}{\eta}\int_0^{\eta} u\left(\frac{L}{\Delta p}\frac{\partial p(0)}{\partial z} + \frac{\partial\phi_0(u,0)}{\partial\tau}\right)du,$$

then the velocity distribution is given by $\phi(\eta, \tau) = \phi_0(\eta, \tau)$ for all $\tau \geq 0$.

**Case with $\phi(\eta, 0) = \sigma(\eta) \neq \phi_0(\eta, 0)$**

In this case

$$\begin{aligned} C_m(0) &= A_m + B_m \\ &= \frac{2}{J_1(k_m\eta)^2}\int_0^1 \eta(\sigma(\eta) - \phi_0(\eta, 0)) J_0(k_m\eta)d\eta \end{aligned}$$

and

$$\begin{aligned} C'_m(0) &= A_m\gamma_+(m) + B_m\gamma_-(m) \\ &= \frac{2}{J_1(k_m)^2}\int_0^1 \eta\left(\Lambda(\eta) - \frac{\partial\phi_0(\eta,0)}{\partial\tau}\right) J_0(k_m\eta)d\eta. \end{aligned}$$

We can now find the velocity distribution $\phi(\eta, \tau)$ when $\Lambda(\eta)(= \frac{\partial\phi(\eta,0)}{\partial\tau})$ is equal, or not equal, to $\frac{\partial\phi_0(\eta,0)}{\partial\tau}$.

## 6 Applications for $H > G$

We shall now consider some flows defined by specific pressure-gradients and obtain their general solutions .

### 6.1 A liquid starting from rest under a constant pressure-gradient

As a first example we consider the liquid starting from rest with an initial acceleration $\Lambda(\eta)$, under a constant pressure-gradient. The constant pressure-gradient is expressed by $y = -\frac{L}{\Delta p}\frac{\partial p}{\partial z} = k$, and from eqn.(25) we find that $\tilde{\nu}_1 = 0$. Taking $c = 0$ in eqn.(26) we get $T = k\tilde{\nu}_2 = \frac{k}{\nu_2}$. From eqn.(22) $\Theta = \frac{\nu_2}{4}(1-\eta^2)$, hence

$$\phi_0(\eta,\tau) = \frac{\nu_2}{4}(1-\eta^2)\frac{k}{\nu_2} = \frac{k}{4}(1-\eta^2), \quad \frac{\partial\phi_0(\eta,0)}{\partial\tau} = 0.$$

Thus from Sec.5.1.(a) the general velocity distribution for a liquid starting from rest under a constant pressure-gradient and initial acceleration $\Lambda(\eta) = k + \frac{L}{\Delta p}(\frac{S_{rz}(\eta,0)}{a\eta} + \frac{1}{a}\frac{\partial S_{rz}(\eta,0)}{\partial\eta})$ is

$$\begin{aligned}\phi(\eta,\tau) = &\sum_{m=1}^{infty}(A_m e^{\gamma_+(m)\tau} + B_m e^{\gamma_-(m)\tau})J_0(k_m\eta) \\ &+ \frac{k(1-\eta^2)}{4}\end{aligned} \tag{33}$$

where $A_m$ and $B_m$ are given by eqns.(15) and (16). This result generalises the corresponding expression for $\Phi(\eta,\tau)$ in [5]. (Actually, in [5], the authors use the condition $(1+Gk_m^2)$ $C'_m(0)$ $+k_m^2 C_m(0) = 0$ to give $C'_m(0)$. This actually implies that they have imposed a particular initial distribution for the acceleration.)

From Sec. 5.2. we note the following Special Result:

If the liquid is in motion at time $\tau = 0$, under a constant pressure-gradient with initial velocity distribution $\phi(\eta,0) = \frac{k}{4}(1-\eta^2)$ and its initial acceleration is zero(i.e. its initial shear-stress $S_{rz}(\eta,0) = -\frac{k}{2}a\frac{\Delta p}{L}\eta)$, then its velocity distribution $\phi(\eta,\tau) = \frac{k}{4}(1-\eta^2)$ for all $\tau \geq 0$.

The corresponding result for the upper-convected Maxwell flow is obtained from eqn.(18) by setting $G = 0$. The corresponding classical solution is obtained from eqn.(18) by ignoring $\gamma_-$, setting $G = 0$ and replacing $\gamma_+(m)$ by $\lim_{H\to 0}\gamma_+(m) = -k_m^2$.

### 6.2 Liquid in motion under an exponential pressure-gradient

We consider next the case of a liquid already in motion subject to an exponential pressure-gradient given by $-\frac{L}{\Delta p}\frac{\partial p}{\partial z} = ke^{\alpha\tau}$ so $\Psi(\tau) = k(1+H\alpha)e^{\alpha\tau}$

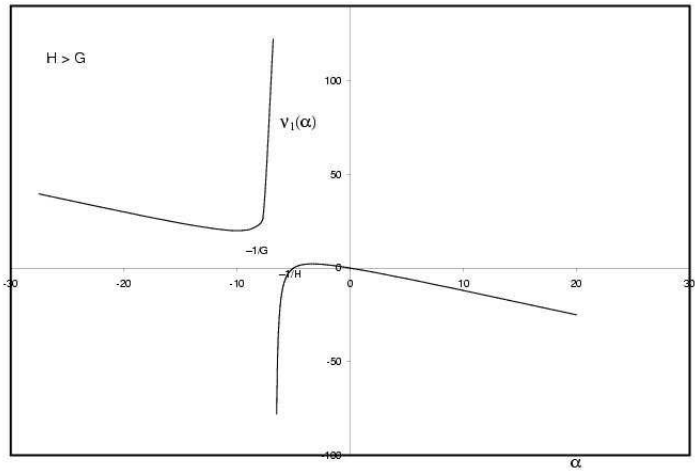


**Figure 1:** $\nu_1$ against $\alpha$

where $\alpha$ may be positive or negative. Eqn.(25) gives $\nu_1 = -\alpha(\frac{1+\alpha H}{1+\alpha G})$, and eqn.(26) gives for $T$, with $c = 0$, $T = \frac{k}{\nu_2}(\frac{1+\alpha H}{1+\alpha G})e^{\alpha\tau}$, $1 + \alpha G \neq 0$, $\nu_2 \neq 0$, hence from Sec.4.1

$$\phi_0(\eta, \tau) = \Theta T = \frac{k}{\alpha}(1 - \frac{J_0(\eta\sqrt{\nu_1})}{J_0(\sqrt{\nu_1})})e^{\alpha\tau}.$$

Now on examining the above expression for $\nu_1$ we consider the following cases

$H > G > 0$

Plotting $\nu_1$ against $\alpha$ we get the curve in Fig.1 We consider the following two cases:

1. (i) $\alpha > 0$, (ii) $-\frac{1}{G} < \alpha < -\frac{1}{H}$.

   In both of these situations $\nu_1 < 0$.

2. (i) $-\frac{1}{G} < -\frac{1}{H} < \alpha < 0$, (ii) $\alpha < -\frac{1}{G} < -\frac{1}{H}$.

   In both of these situations $\nu_1 > 0$.

We now examine these cases.

$\alpha > 0$, **(ii)** $-\frac{1}{G} < \alpha < -\frac{1}{H}$

Here $\nu_1 < 0$ hence $\sqrt{\nu_1} = \imath\beta, \beta = \sqrt{\alpha(\frac{1+\alpha H}{1+\alpha G})}$ and

$$\phi_0(\eta,\tau) = \Theta(\eta,\tau)T(\tau) = \frac{k}{\alpha}(1 - \frac{J_0(\imath\beta\eta)}{J_0(\imath\beta)})e^{\alpha\tau}.$$

The general solution here is

$$\phi(\eta,\tau) = \sum_{m=1}^{\infty}(A_m e^{\gamma_+(m)\tau} + B_m e^{\gamma_-(m)\tau})J_0(k_m\eta) + \phi_0(\eta\tau).$$

From Sec.5.2 above we can write the following Special Result:

For a liquid already in motion at time $\tau = 0$ under the exponential pressure-gradient and with initial acceleration $\frac{\partial\phi(\eta,0)}{\partial\tau} = \frac{\partial\phi_0(\eta,0)}{\partial\tau}$ i.e. initial shear-stress given by $PS_{rz} = \frac{1}{\eta}\int_0^\eta u(-k + k(1 - \frac{J_0(\imath\beta u)}{J(\imath\beta)}))du$ and initial velocity $\phi(\eta,0) = \phi_0(\eta,0) = \frac{k}{\alpha}(1 - \frac{J_0(\imath\beta\eta)}{J_0(\imath\beta)})$, the velocity distribution is $\phi(\eta,\tau) = \phi_0(\eta,\tau) = \frac{k}{\alpha}(1 - \frac{J_0(\imath\beta\eta)}{J_0(\imath\beta)})e^{\alpha\tau}$ for all $\tau \geq 0$.

This explains the eqn.(18) of [5].

It can be shown that by taking $|\beta|$, $|\beta\eta|$ large enough and considering the behaviour of $J_0$, [9], that

$$\phi(\eta,\tau) \approx \frac{k}{\alpha}e^{\alpha\tau}(1 - \frac{1}{\sqrt{\eta}}e^{-\beta(1-\eta)}),\ k = -\frac{L}{\Delta p}\frac{\partial p}{\partial z}.$$

For $\eta$ near 1 and $|\beta|$ large enough we see that the velocity distribution will depend on the wall distance because of the presence of the term $exp(-\beta(1-\eta))$ thus implying a boundary layer effect. We also note that this will happen when we have either an exponential increasing or decreasing pressure-gradient since $\nu_1$ can be negative when $\alpha$ is negative and satisfies $-\frac{1}{G} < \alpha < -\frac{1}{H}$. This is in contrast to what was mentioned in Sec.3 (3.2) in [5].

**Example** As an example of the Special Result B when $\alpha > 0$ or $-\frac{1}{G} < \alpha < -\frac{1}{H}$, we take $H = 0.2, G = 0.15$ and display the velocity and shear-stress distributions $\phi(\eta,\tau)$, $PS_{rz}(\eta,\tau)$ when $\alpha = 1,\ \tau = 0.5$. The results are shown in Fig.2. We also considered the case where $\alpha = -6$ but since the results were similar we do not show them. We did not go beyond $\tau = 0.5$ because the exponential terms become either small or too large. We plot both the velocity distribution, $\phi$, and the corresponding shear-stress, $S_{rz}$.

$-\frac{1}{G} < -\frac{1}{H} < \alpha < 0$ **and** $\alpha < -\frac{1}{G} < -\frac{1}{H}$

Here $\nu_1 = |\alpha|(\frac{1-|\alpha|H}{1-|\alpha|G}) = \beta_1 > 0$ and $\phi_0(\eta,\tau) = \frac{k}{\alpha}(1 - \frac{J_0(\sqrt{\beta_1}\eta)}{J_0(\sqrt{\beta_1})})e^{\alpha\tau}$.

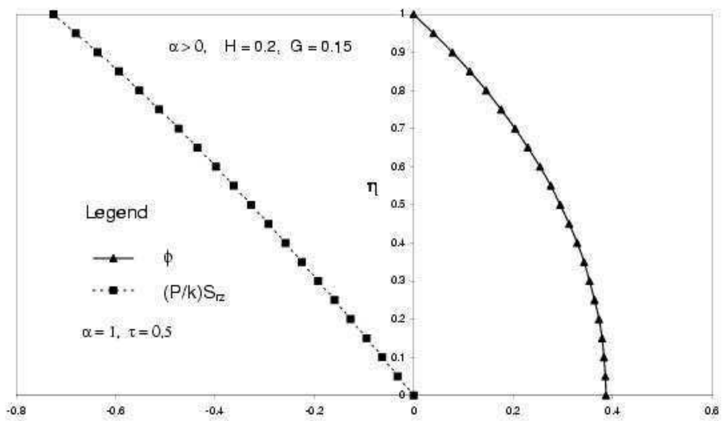


**Figure 2:** Velocity, Shear-Stress; $\alpha = 1$

The general solution is

$$\phi(\eta, \tau) = \sum_{m=1}^{\infty} (A_m e^{\gamma_+(m)\tau} + B_m e^{\gamma_-(m)\tau}) J_0(k_m \eta) + \phi_0(\eta, \tau).$$

We get the following Special Result C from Sec. 5.2(a):

For a liquid already in motion at $\tau = 0$ under the given exponential pressure-gradient and initial acceleration

$$\frac{\partial \phi(\eta, 0)}{\partial \tau} = \frac{\partial \phi_0(\eta, 0}{\partial \tau}$$

i.e. initial shear-stress given by $PS_{rz} = \frac{1}{\eta} \int_0^\eta u(-k + k(1 - \frac{J_0(\beta_1 u)}{J_0(\beta_1)}))du$ and initial velocity distribution $\phi_0(\eta, 0) = \frac{k}{\alpha}(1 - \frac{J_0(\beta_1 \eta)}{J_0(\beta_1)})$, the velocity distribution is $\phi(\eta, \tau) = \phi_0(\eta, \tau) = \frac{k}{\alpha}(1 - \frac{J_o(\beta_1 \eta)}{J_0(\beta_1)})e^{\alpha\tau}$ for all $\tau \geq 0$.

This result explains eqn.(22) of [5].

For $|\beta_1|$, $|\beta_1 \eta|$ large enough ,[9], we get the following asymptotic expansion $\phi(\eta, \tau) \approx \frac{e^{\alpha\tau}}{\alpha}(1 - \frac{\cos(\beta_1 \eta - \pi/4)}{\sqrt{\eta}\cos(\beta_1 - \pi/4)})$ and although this depends on the distance $\eta$ it does not depend on it in such a way as to produce a boundary layer effect at the side of the tube. We also note that this behaviour occurs only with the decreasing pressure-gradient.

**Example** A second example is taken from the Special Result C when $0 > \alpha > -\frac{1}{H} > -\frac{1}{G}$ or $\alpha < -\frac{1}{G} < -\frac{1}{H} < 0$, say $\alpha = -1, -6.7$ with $H = 0.2, G = 0.15$. We obtained the velocity and shear-stress distributions as shown in Figs. (3,4), for $\tau = 0.5$. We obtained graphs for $\tau = 0, 1.0$ but they looked similar to that for $\tau = 0.5$ so they were not included. Also in the shear-stress graphs we used $\frac{P}{k}S_{rz}$.

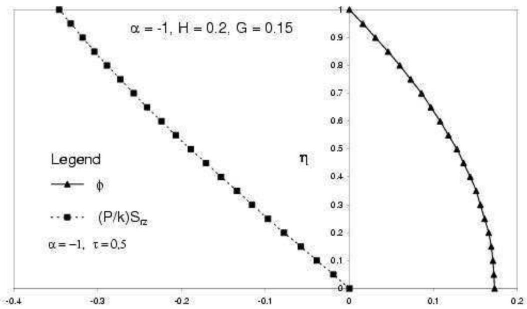


**Figure 3:** Velocity, Shear-Stress; $\alpha = -1$

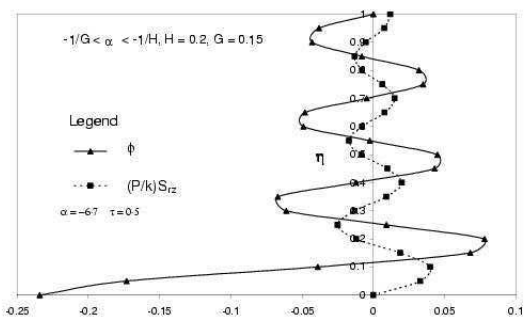


**Figure 4:** Velocity, Shear-stress; $\alpha = -6.7$

We note that oscillations occur in $\phi(\eta, \tau)$ and $S_{rz}$, that is, we have flow reversal in this case.

$0 < H < G$

In this case we have the following two situations when $\nu_1 < 0$.

1. $\alpha > 0$; $-\frac{1}{H} < \alpha < -\frac{1}{G}$
   Both of these situations will produce a boundary layer effect as in the previous case above.

   We also have the following situations:
2. $\alpha < 0, \alpha > -\frac{1}{G} > -\frac{1}{H}$; $\alpha < 0, \alpha < -\frac{1}{H} < -\frac{1}{G}$
   These do not produce a boundary layer effect.

$H = G$

In the case we get $\nu_1 = -\alpha$ by substutiting $H = G$ in $\nu_1 = -\alpha(\frac{1+\alpha H}{1+\alpha G})$ from Sec.4.1 and $\phi_0(\eta, \tau) = \frac{k}{\alpha}(1 - \frac{J_0(\imath\sqrt{\alpha}\eta)}{J_0(\imath\sqrt{\alpha})})e^{\alpha\tau}$. This implies that if $H = G$ the non-Newtonian liquid behaves like a Newtonian liquid.

## 7 Liquid under a sinusoidal pressure -gradient

This type of pressure-gradient is encountered, for example, in the reciprocating piston system.

In this case if we try to use eqns.(25) and (20) we meet some problems(we cannot get a seperable solution) and hence we must look for another method. To this end we note that we can write $y = -\frac{L}{\Delta p}\frac{\partial p}{\partial z} = Re(ke^{\imath\alpha\tau})$ and hence from eqn.(4) $\Psi(\tau) = Re(ke^{\imath\alpha\tau}(1+\alpha H\imath))$. Therefore from eqn.(3) and the form of the left-hand side, since the $\tau$ dependence part contains only derivatives with respect to $\tau$, it becomes natural to seek a solution of the form

$$\phi_0 = Re(ke^{\imath\alpha\tau}\Theta) \equiv Re(T(\tau)\Theta(\eta))$$

where $\Theta$ is now complex. (The derivatives introduce powers of $\imath\alpha$, while $\Theta$ remains unchanged.) Substuting in eqn.(3) we choose $\Theta$ to satisfy

$$\Theta'' + \frac{1}{\eta}\Theta' + \eta\Theta = -\nu_1.$$

We thus have $\phi_0(\eta, \tau) = Re(\frac{k}{\imath\alpha}(1 - \frac{J_0(\eta\sqrt{\nu_1})}{J_0(\sqrt{\nu_1})})e^{\imath\alpha\tau})$ and the general solution

$$\phi(\eta, \tau) = \sum_{m=1}^{\infty}(A_m e^{\gamma_+(m)\tau} + B_m e^{\gamma_-(m)\tau})J_0(k_m\eta) + Re\left(\frac{k}{\imath\alpha}\left(1 - \frac{J_0(\eta\sqrt{\nu_1})}{J_0(\sqrt{\nu_1})}\right)e^{\imath\alpha\tau}\right)$$

where $A_m,\ B_m$ are given in eqns.(15), (16), $\nu_1,\ \nu_2$ are complex and given by $\nu_1 = -\imath\alpha(\frac{1+\imath\alpha H}{1+\imath\alpha G}) = -\imath\alpha\nu_2$.

If we choose $\phi(\eta,0) = \phi_0(\eta,0) = Re(\frac{k}{\imath\alpha}(1 - \frac{J_0(\eta\sqrt{\nu_1})}{J_0(\sqrt{\nu_1})}))$ and $\frac{\partial\phi(\eta,0)}{\partial\tau} = \frac{\partial\phi_0(\eta,0)}{\partial\tau} = Re(k(1 - \frac{J_0(\eta\sqrt{\nu_1})}{J_0(\sqrt{\nu_1})}))$ then we see that $A_m = B_m = 0$ and hence

$$\phi(\eta,\tau) = \phi_0(\eta,\tau) = Re\left(\frac{k}{\imath\alpha}\left(1 - \frac{J_0(\eta\sqrt{\nu_1})}{J_0(\sqrt{\nu_1})}\right)e^{\imath\alpha\tau}\right)$$

for all $\tau \geq 0$.

We can state this as Special Result D as follows:

For a liquid already in motion at $\tau = 0$ under the given sinusoidal pressure-gradient and initial acceleration $\frac{\partial\phi(\eta,0)}{\partial\tau} = \frac{\partial\phi_0(\eta,0)}{\partial\tau}$, initial shear-stress given by $PS_{rz} = \frac{1}{\eta}\int_0^\eta u(-k + Re(k(1 - \frac{J_0(\sqrt{\nu_1}u)}{J_0(\sqrt{\nu_1})})))du$ and initial velocity distribution $\phi_0(\eta,0) = Re(\frac{k}{\imath\alpha}(1 - \frac{J_0(\sqrt{\nu_1}\eta)}{J_0(\sqrt{\nu_1})})e^{\imath\alpha\tau})$, the velocity distribution is $\phi(\eta,\tau) = \phi_0(\eta,\tau) = Re(\frac{k}{\imath\alpha}(1 - \frac{J_0(\sqrt{\nu_1}\eta)}{J_0(\sqrt{\nu_1})})e^{\imath\alpha\tau})$ for all $\tau \geq 0$.

This result explains eqn.(28) of [5].

In [5] the velocity distribution for a pulsating pressure-gradient at different instants is shown in their Fig.1. The graphs for the cases $H = 0.2,\ G = 0.05$ appear to be incorrect. We recomputed the graphs and show, for example, only those for $H = 0.2,\ G = 0.05,\ a = n = 25,\ \tau = \pi/2n,\ \tau = \pi/n$ in Figs.(5,6) below. We again plotted $\frac{P}{k}S_{rz}$ as the shear-stress.

It can also be seen that if we set $H = G$, an Oldroyd liquid with the corresponding $\phi_0$, initial velocity and acceleration distributions and pressure-gradient will behave as though it were a classical liquid.

## 8 Concluding remarks

We have attempted to examine in detail the flow of an Oldroyd liquid B through a straight circular tube when different initial conditions are specified. Five previous investigations of this problem have not examined this aspect in any detail. Our work used a Fourier-Bessel expansion, as in [5], for the velocity distribution and we were able to obtain a general solution when general initial conditions are specified for the velocity and non-zero acceleration (or equivalently the pressure-gradient and shear-stress.) We applied the theory to problems where the pressure-gradient was (i) a constant, (ii) exponential and (iii) sinusoidal and give the general expressions for the corresponding velocity distributions.

In the analysis it was found that in certain cases if the initial acceleration is chosen to be equal to certain known functions(i.e. derivatives of particular solutions of an inhomogenous equation) then the velocity distributions are given in terms of these known functions for all time. The results agreed with

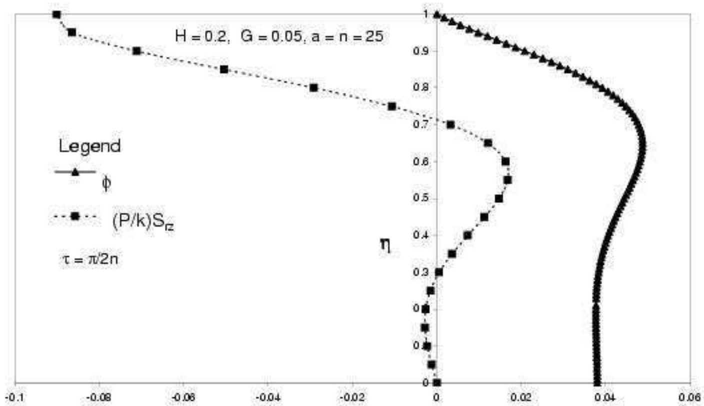


**Figure 5:** Velocity, shear-stress for sinusoidal pressure-gradient at $\tau = \pi/2n$

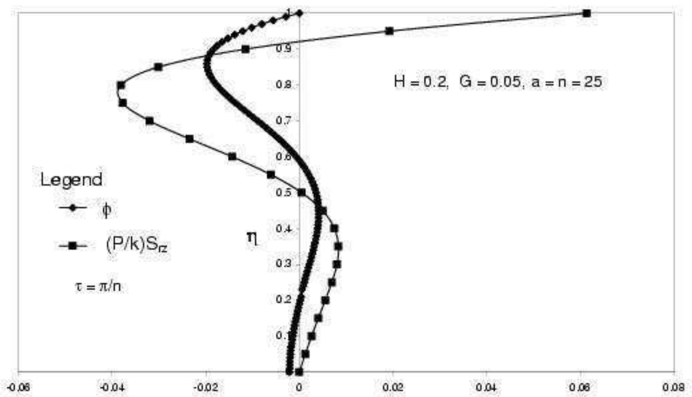


**Figure 6:** Velocity, shear-stress for sinusodial pressure-gradient at $\tau = \pi/n$

the results obtained in [5], however, these authors did not discuss anything about initial conditions.

As an example we considered the exponential pressure-gradient where $-\frac{L}{\Delta p}\frac{\partial p}{\partial z} = ke^{\alpha\tau}$. We considered real values of $\alpha$ in four regions where the parameter $\nu_1(\alpha)$ was either positive or negative. For different values of the parameters $\alpha$, $H$, $G$ we give graphic representations of the velocity and corresponding shear-stress distributions at times close to $\tau = 0$. In some cases, for example, where $\alpha < -\frac{1}{G} < -\frac{1}{H} < 0$ flow reversal was present.

**Acknowledgements** The authors would like to thank Mr. Arvind Mohais for several helpful discussions on Latex Graphics.

## References

[1] Ghosh, A.K. Acta Technica Academiae Scientiarum Hungaricae **66** no.4, p 455, 1969.

[2] Waters, N.D. and King, M.J. J.Phys.D: Appl.Phys., **4**, p 204, 1971.

[3] Bhatnagar, R.K. Appl.Sci.Res., **30**, p 241, 1975.

[4] Ramkissoon, H. and Shifang, H. Proc.China-Japan International conf. in Rheol. Beijing, p 161, 1991.

[5] Ramkissoon, H. and Majumdar, S.R. Proc.3rd. Caribbean conf. on Fluid Mechanics, **1**, 1995.

[6] Kapur, J.N., Bhatt, B.S. and Sacheti N.C. Non-Newtonian Fliud Flows (A survey monograph), Pragati Prakashan, Meerut (India), 1982.

[7] Oliver, D.R. and Macsporran, W.C. Can.J.Chem.Eng. ,**48**, p 243, 1970.

[8] Kantorovich, L.V. and Krilov, V.I. Approximate Methods of Higher Analysis, Interscience Publishers, New York, Chap.IV, 1958.

[9] Jeffreys, H. Asymptotic Approximations, Oxford at the Clarendon Press, Oxford, U.K., p 77, 1952.

*Dirección de los autores:* Owen D. , Department of Mathematics and Computer Science, The University of the West Indies, St. Augustine, Trinidad and Tobago, W.I., — Bhatt B. , Department of Mathematics and Computer Science, The University of the West Indies, St. Augustine, Trinidad and Tobago, W.I., BSBhatt@fans.uwi.tt