On the Activation of Quantum Nonlocality

Andrés Felipe Ducuara García

Trabajo de Investigación presentado al programa de Maestría en Ciencias-Física como requisito para optar al título de Magister en Ciencias-Física

Supervisor
Prof. Dr. John Henry Reina

Co-supervisor
Prof. Dr. Javier Madroñero

Universidad del Valle
Facultad de Ciencias Naturales y Exactas
Departamento de Física
Cali-Colombia
July 15, 2016

Acknowledgements

I would like to thank those who have been of an influence to me in both personal and scientific levels. First of all, I would like to thank my family;

To my parents, Luz Mary García and Jairo Ducuara, for their unconditional love and support, for their belief that science is a noble subject worthy to pursue.

To my sister Angélica García, for introducing me to the world of literature, through which I have found beautiful things, being science one of them.

To my aunt Nancy Moreno, for her terribly contagious optimistic approach toward life which keeps us always going on forward.

To my granny Excelenia Osorio, who taught us about life.

To my brother-in-law Alejandro Ortíz, for his contagious passion towards scientific and geeky subjects.

To my niece Juliana Ortíz, because I wish to be an inspiration to her, as her mother has been for me.

To the Ortíz-Gonzalez family; Mr. Guillermo, Mrs. Nancy, and Catalina Ortíz, for their willingness to help me when I've required of it.

I would now like thank my alma mater, the *Universidad del Valle*, particularly, the *Departamento de Física*.

To all of the professors who, in one way or another, have influenced me. To my colleagues during the Bachelor degree in physics. To my colleagues during the master; Claribel Dominguez, Marcela Quintero, and Josué Motoa.

To my highschool friends; Chester, Miguel, Thamix, Banana, and Natalia Valencia whom, I wish, I could hang out with more often. For their ability to make me forget about physics and, somehow, enjoy other aspects of life.

To the *Quantum Technologies, Information, and Complexity (QuanTIC)* group and the *Centre for Bioinformatics and Photonics (CIBioFI)*. To Professor John Henry Reina for given me the opportunity of being part of the group, but mostly, for giving me the freedom to choose and pursue what I felt passionate about. To Professor Javier Madroñero, for his support and guidance. To my academical brothers Cristian Susa and Carlos Melo. To the QuanTIC/CIBioFi-gang; Manuel Muñoz, Erick Ipus, Alejandro González, Isabel Guzmán, Carlos Parra, Claudia Valdés, Arturo Argüelles, Carlos Galíndez, Luis Giraldo, and Cristian Gómez, for making of the laboratory a cozy place where to work in.

Lastly but not least, I would like to thank my girlfriend Fanny Ortíz. In particular, for being my private English teacher, and for having read this Thesis cover to back, looking for grammatical mistakes and typos.

To Fanny Ortíz, for her unconditional love and support during this last and a half year, toward the fulfilment of our dreams.

Abstract

Quantum nonlocality is a cornerstone property for the development of the so-called quantum protocols as well as for the characterisation, and possible generalisation of quantum theory. However, there are still fundamental aspects of this property that need to be addressed; in particular, its relation with the property of *quantum entanglement*. It is well known that every nonlocal state is an entangled one, yet there exist entangled local states. Therefore, within their original definitions, these two properties are *not* equivalent. The study of this relationship has been put forward by means of the generalisation of nonlocality through the so-called *activation of nonlocality* scenarios. The idea is to only work with these entangled local states in order to try to recover (activate) nonlocality. In this Thesis, we have numerically addressed the properties of entanglement, nonlocality (mostly by means of the violation of the CHSH-inequality), as well as the following generalisations of nonlocality: *k-copy nonlocality*, *hidden nonlocality*, and the combination of these two scenarios, namely, *hidden k-copy nonlocality* and *tensor/filter activation of hidden nonlocality*. We have studied these properties for some bipartite entangled local states, and some two-qubit *not* CHSH-inequality violating states of interest. In particular, we have analysed these properties for two-qubit states that come from the study of quantum games' strategies and the dynamics of open quantum systems.

Resumen

Nolocalidad Cuántica es una propiedad crucial para el desarrollo de los llamados protocolos cuánticos, así como para la caracterización y posible generalización de la teoría cuántica. Sin embargo, todavía permanecen aspectos fundamentales de ésta propiedad los cuales necesitan ser abordados; en particular, su relación con la propiedad de *Entrelazamiento Cuántico*. Es bien sabido que cada estado nolocal es uno entrelazado, aún así, existen estados entrelazados-locales. Por lo tanto, dentro de sus definiciones originales, éstas dos propiedades *no* son equivalentes. El estudio de ésta relación ha sido extendido a través de la generalización de la nolocalidad cuántica a través de los llamados escenarios de *activación de la nolocalidad*. La idea es trabajar unicamente con estos estados entrelazados-locales para tratar de recuperar (activar) nolocalidad. En esta Tesis, hemos abordado numericamente las propiedades de: entrelazamiento, nolocalidad (principalmente través de la violación de la desigualdad CHSH), así como las siguientes generalizaciones de nolocalidad: *nolocalidad a k-copias*, *nolocalidad oculta*, y la combinación de éstos dos escenarios, a decir, *nolocalidad a k-copias oculta*, y *activación de la nolocalidad oculta* a través de *producto tensorial y filtrado local*. Hemos estudiado estas propiedades en algunos estados bipartitos entrelazados-locales, y algunos estados de dos qubits que *no* violan la desigualdad CHSH. En particular, hemos analizado estas propiedades para sistemas de dos qubits en: el estudio de estrategias de juegos cuánticos y la dinámica de sistemas cuánticos abiertos.

Contents

Introduction

Historical Background

Quantum mechanics alongside with general relativity were two revolutionary theories the origins of which can be dated to the beginning of the 20th century. They both meant independent generalisations of classical mechanics, and changed the way we understand how the universe works at a fundamental level. During the 20th century, these two theories expanded into more fundamental and practical levels, being the development of the standard model of particles, and practically, in one way or another, most of nowadays technologies, respectively, two clear examples of these achievements. However, there are still many fundamental and practical challenges to be addressed. In particular, the unification of these two generalisations, quantum theory and general relativity, stands out as one of these long-standing unsolved fundamental problems (this considered by many, the holy grail of nowadays theoretical physics), whilst the realisation of the so-called quantum computer (a novel way to do computation which is expected to outperform any classical computer), stands out as a current practical challenge. Let us now take a look at these two particular challenges from the point of view of the field of *quantum information theory*

The road toward quantum information can be traced back to an article published in 1935 by A. Einstein, B. Podolsky, and N. Rosen, coined the EPR article [1] (also called the EPR Paradox). There, they were worried about the conceptual implications of the mathematical formalism underlying quantum mechanics. Using current terminology, their result can be stated as follows [2]: Considering together the properties of *completeness* and *locality* within the formalism of quantum mechanics, they generated a contradiction. Since EPR regarded locality as a natural property that quantum mechanics should respect, they concluded that quantum mechanics was not a complete theory. In 1964, J. S. Bell derived an experimentally-testable result showing that the property of locality (alone) within quantum mechanics generated a contradiction. Therefore, either quantum mechanics is a nonlocal theory or there were flaws in the formalism of quantum theory [3]. The very first experimental confrontation of these ideas was carried out by S. J. Freedman and J. F. Clauser [4], which came out in favour of a quantum nonlocal theory. These experiments have been greatly improved since then: in an initial stage by A. Aspect, J. Dalibard and R. Gérard [5], and very recently, by an experimentally free-of-loopholes test [6]. All of these experiments are in accordance with quantum theory and therefore, settling nonlocality as a fact of nature. As a parenthesis, the property of completeness was generalised to the property of ψ-onticity, and it is still focus of intense debate [7], whether quantum mechanics should be refereed as ψ-ontic or not. Although terribly interesting by itself, ψ-onticity is not the focus of this Thesis. We now address the role of quantum information theory with respect to the particular current challenges addressed in the previous paragraph, this, by means of the property of quantum nonlocality.

On the one hand, from a fundamental point of view, quantum nonlocality led to the study of correlations, which has led us to better understand, and could possibly help us to generalise quantum theory. The remarkable result by S. Popescu and D. Rohrlich [8] (which was actually a rediscoverement [9, 10]), establishes the existence of the so-called superquantum, non-signalling correlations (correlations that cannot be achieved by quantum mechanics yet obeying the non-signalling principle from relativity). A natural concern is then, to try to explain why these correlations cannot be produced with quantum mechanics. In this regard, there have been attempts to tighten the constraints imposed by the non-signalling principle with additional ones, namely, information causality [11], local orthogonality [12], almost quantum correlations [13], and triviality of communication [14], as well as to try to generalise quantum theory itself [15, 16]. This is an active area of research, and hopefully, it is going to shed light on the long-standing unification problem.

On the other hand, the discussion around properties like nonlocality and *entanglement* (which as well as nonlocality, is a property inspired by the EPR article) was of a purely fundamental character, until R. Feynman [17] and D. Deutsch [18] (amongst others [19, 20]) proposed the idea of the quantum computer. Even though there are many stages before a physical realisation of a quantum computer (what precisely makes of it a challenging endeavour), they are mainly pushed forward by the scope of what a quantum computer can actually do, this, by means of the so-called quantum protocols. There are quantum protocols that could outperform any of their classical counterparts, namely, quantum cryptography [21, 22], factorisation algorithm (or Shor's algorithm) [23], search algorithm (or Grover's algorithm) [24], as well as protocols which do not even have a classical counterpart to begin with, like quantum teleportation [25]. Most of these protocols explicitly make use of quantum nonlocality as a resource, and therefore, a full understanding of this property is completely necessary.

We have then seen some fundamental and practical goals which have quantum nonlocality as a common factor. In this Thesis, we are going to focus at the very heart of the discussion, by addressing some fundamental aspects of this cornerstone property. In this regard, it was first noted that in order to generate nonlocality with quantum theory, first entanglement was needed [26]. However, these two properties are not equivalent, since there exist entangled but local states [26, 27]. Even though these states might not seem useful to nonlocality-based protocols at a first glance, they are still of interest, because they could potentially help us to clarify the difference between these two properties. Following this line of thought, one might also ask whether there is a way to activate (recover/create) nonlocality by only working with entangled local states. It turned out that this is actually possible, and the procedures in order to achieve this goal have been termed *activation of nonlocality* scenarios [28].

Generalities of this Thesis

We have already narrowed the aim of this thesis to the activation of nonlocality scenarios. Let us now clarify some generalities regarding our particular goals here. First, we do not attempt to cover all of these scenarios, but rather those which we can numerically approach. Second, we have focused on (though not limited to) the simplest-non-trivial quantum systems, namely, two-qubit systems. We have worked under these two constrains, mainly because measures are usually easier for two-qubit systems, and because our group has mostly worked with two-qubit states that come from the dynamics of open quantum systems, so part of our motivation was to study this activation phenomenon over these sort of systems. Therefore, this Thesis should be seen as a theoretical-computational approach to the study of nonlocality and its activation, with special focus on (though not limited to) arbitrary two-qubit states. All the numerical calculations were performed with MATLAB, and we have left several two-qubit properties ready to be calculated for arbitrary two-qubit states. The latter allowed us to make a couple of collaborations, by studying the activation of nonlocality for two-qubit states that come from both the study of game strategies and the dynamics of an open quantum system. Without further ado, let us present the aimroad of this Thesis.

In **Chapter 1**, we introduce the standard formalism of quantum theory, and its generalisation to the density matrix formalism, this, with a focus on entanglement. We then move on to the concept of correlations (which have also been called behaviours or boxes), featuring; non-signalling, quantum, local, and deterministic correlations. Finally, we address the main theorem that relates the two worlds of; states-measurements and correlations. In **Chapter 2**, we start by introducing some two-qubit measures, namely, the *entanglement of formation*, the *CHSH-inequality violation*, and the *usefulness for teleportation*, and we then list some entangled local states. In **Chapter 3**, we discuss some of the known activation of nonlocality scenarios, and focus on the four which we explicitly work with, namely; *k-copy nonlocality*, *hidden nonlocality*, *hidden k-copy nonlocality*, and *tensor/filter activation of hidden nonlocality*, taking down to earth these scenarios to the particular case of two-qubit systems. In **Chapter 4**, we analyse these scenarios for some bipartite entangled local states (those from the third chapter) as well as for some not CHSH-inequality violating states. We finally present some **Conclusions**, some **Appendices**, and a couple of collaborative **Publications & Preprints** with some of the results presented here.

Chapter 1

Quantum Mechanics & Correlations

In this chapter, we introduce the physical objects we are interested in, with their mathematical representation. We first introduce the standard formalism of quantum mechanics, followed by its generalisation to the so-called density matrix formalism, focusing on the concepts of *states*, *measurements*, and *entanglement*. We then move on to the objects called *correlations*, which have also been called behaviours or boxes. The concept of quantum nonlocality is first defined for these correlations, and then extended to the quantum state-measurement domain. Finally, we address the main theorem that relates these two worlds of; states-measurements and correlations. Even though quantum nonlocality has been the subject of study for philosophers of science, physicists, and mathematicians, we will focus on a pragmatical mathematical approach to the subject, as it is also depicted in references [28, 29], rather than in a historical-philosophical approach (see a nice unification of the notation and a general discussion around nonlocality from these branches in [30]). Let us start by introducing quantum mechanics.

1.1 Quantum Mechanics

In this section, we first review the standard formalism of quantum mechanics, followed by its generalisation to the density matrix formalism, and the introduction of the concept of entanglement. For a more detailed discussion see references [31, 32, 33, 34].

1.1.1 Standard Formalism

Quantum mechanics deals with *Hilbert Spaces*. A Hilbert space is a *Complete Inner Product Space*. An inner product space is a structure $(\mathbb{H}, +, *, \mathbb{K}, \langle | \rangle)$, being $(\mathbb{H}, +, *, \mathbb{K})$ a *Vector Space* over a *Field* $\mathbb{K}$, and $\langle | \rangle$ an *Inner Product*. Taking into account that the inner product $\langle | \rangle$ induces a norm $||.||$, that consequently induces a metric $d(,)$, the inner product space in question is *Complete*, if this metric $d(,)$ is *Complete* [31, 32]. For simplicity, we will denote this complete inner product space (Hilbert space) by $\mathbb{H}$.

We now set a couple of restrictions on the Hilbert spaces we will work with, and establish the notation we will use throughout the text. By $\mathbb{H}$, we will mean one of the following Hilbert spaces. First, we use complex Hilbert spaces, i. e., $\mathbb{K} = \mathbb{C}$. Second, we put the restriction of finite-dimensional spaces, therefore, our Hilbert spaces are going to be isomorphic to the Hilbert space $\mathbb{C}^d$ with $d \geq 2$. The vectors of these Hilbert spaces are denoted by greek letters inside the so-called "ket"-"Dirac" notation, namely, $|\psi\rangle \in \mathbb{H}$. Since we are dealing with vector spaces, there always exists an orthonormal basis (orthogonal normalised basis) for these Hilbert spaces, which we write as $\{|i\rangle\}_{i=0}^{d-1}$. By *basis*, we will always mean one of these orthonormalised basis. Third, we use the so-called *Hilbert-Schmidt* inner product, which for two arbitrary vectors; $|\phi\rangle = \sum_{i=0}^{d-1} \phi_i |i\rangle$, and $|\psi\rangle = \sum_{j=0}^{d-1} \phi_j |j\rangle$, being ϕ_i, ψ_j complex numbers, it reads; $\langle\phi|\psi\rangle := \sum_{i=0}^{d-1} \phi_i^* \psi_i$. We also make use of the set of linear operators on $\mathbb{H}$, denoted $LO(\mathbb{H})$. In particular, we define the linear operator $|\psi\rangle\langle\psi|$ as: $|\psi\rangle\langle\psi|(|\phi\rangle) := |\psi\rangle(\langle\psi|\phi\rangle)$. For further details and more generalities about these topics, see references [31, 32]. Let us now assign some of these mathematical objects to the physical concepts we are interested in.

States: In order to define a *quantum state*, we consider a vector $|\psi\rangle \in \mathbb{C}^d$, written in a particular basis; $|\psi\rangle := \sum_{i=0}^{d-1} \alpha_i |i\rangle$, with $\alpha_i \in \mathbb{C}$. We want to interpret $|\psi\rangle$ as a quantum state, being in a superposition of states $\{|i\rangle\}$, where $|\alpha_i|^2$ represents the probability for the physical system to be in the state $|i\rangle$, therefore, we need to impose the normalisation condition $\langle\psi|\psi\rangle = \sum_{i=0}^{d-1} |\alpha_i|^2 = 1$. Thus, the vectors of interest form an equivalence-class of vectors $[\psi] := \{|\phi\rangle \in \mathbb{C}^d |\, |\phi\rangle = \beta |\psi\rangle, \langle\phi|\phi\rangle = 1 \text{ or } |\beta| = 1\}$, commonly called a *ray*. A quantum state is going to be an arbitrary vector of that ray, say $|\psi\rangle$, that is going to represent that whole equivalence-class of vectors $[\psi]$.

Observables, Measurements, and Projections: We now consider the set of linear operators $LO(\mathbb{H})$, and define some special kinds of operators. Given a linear operator $A \in LO(\mathbb{H})$, its *adjoint operator*, denoted as $A^\dagger$, is defined as the operator such that $(\langle\phi|A^\dagger)|\psi\rangle = \langle\phi|(A|\psi\rangle)$, $\forall\, |\phi\rangle, |\psi\rangle \in \mathbb{H}$ (this operator always exists, see [31]). A linear operator A is a *normal operator* if $[A, A^\dagger] := AA^\dagger - A^\dagger A = 0$. A linear operator A is a *self-adjoint* operator if $A^\dagger = A$ (also called an *Hermitian* operator). We can check that any self-adjoint is a normal operator but not viceversa. Given O^x, a self-adjoint operator (the x acts only as a counter), we have the so-called *spectral decomposition theorem* which establishes that O^x can be written as:

$$O^x = \sum_{a=0}^{d-1} o_a^x P_a^x, \tag{1.1}$$

being $\{o_a^x\}$ its eigenvalues that turn out to be real, and $P_a^x = |a\rangle\langle a|$ with $\{|a\rangle\}_{a=0}^{d-1}$ a basis. We remark that the set $M^x := \{P_a^x\}$ also satisfy the properties:

$$\sum_{a=0}^{d-1} P_a^x = \mathbb{1}, \qquad P_a^x \geq 0,\ \ \forall a, \qquad P_a^x P_{a'}^x = P_a^x \delta_{a,a'}, \quad \forall a, a'. \tag{1.2}$$

Self-adjoint operators are important because they give us real eigenvalues, and therefore, they are suitable to represent *properties* of physical systems, that is why, they receive the title of *observables*. The set $M^x = \{P_a^x\}$ is called a *measurement*, and the operators P_a^x are called *projections*. We now present the standard interpretation of quantum mechanics or Copenhagen interpretation.

Standard Interpretation: Given an observable O^x, and an arbitrary state $|\psi\rangle$ written in the eigenbasis of O^x, that is $|\psi\rangle = \sum_a \psi_a |a\rangle$, this state has probability $|\psi_a|^2$ of having a value of o_a^x for the property represented by O^x. If we define $\psi := |\psi\rangle\langle\psi|$, we can check that this probability can also be written as:

$$|\psi_a|^2 = \langle\psi|P_a^x|\psi\rangle = \mathrm{Tr}(P_a^x \psi) =: p(a|x), \tag{1.3}$$

which is also called the *Born Rule*. We can also interpret this probability as the conditional probability function $p(a|x)$ (the probability of obtaining output a, given input measurement x), notation which will come in handy in the next section. This is then, a purely statistical interpretation of quantum mechanics [33, 34]. There are however, interesting alternatives, and several consequences have been derived by means of the study of these concerns, being precisely nonlocality one of these consequences. However, we are going to further address this when we define nonlocality.

Remark: The standard formalism of quantum mechanics usually also invokes a postulate which indicates how to introduce space-time, and how to derive a temporal evolution, for either states or observables in the Schrödinger or Heisenberg pictures, respectively, usually arriving to the Schrödinger equation (in the non-relativistic regime) [34]. However, we opted to skip this part, since we are not going to explicitly make use of either space or time (with the little exception of the temporal evolution of a state in the last chapter).

So far, we have only considered *one* physical system. However, we might also consider the fact that measurements are somehow representing another system (be it a measurement apparatus, an experimentalist, or simply the rest of the universe), which should also be addressed as a *system*, and therefore, modelled with a Hilbert space too. Furthermore, the physical system of interest could also be composed of subsystems. These considerations lead us to introduce multipartite systems.

Multipartite Systems: In order to assign a Hilbert space to a multipartite system, a natural way to do so, it is by considering the tensor product of individual systems. Considering an n-partite system represented by $\mathbb{H}_{\otimes} = \bigotimes_{i=1}^{n} \mathbb{C}^{d_i}$ with $d = d_1 d_2 ... d_n$, where $\otimes$ stands for the tensor product, it can be shown that the structure $(\mathbb{H}_{\otimes}, +_{\otimes}, *_{\otimes}, \mathbb{C}, \langle | \rangle_{\otimes})$, with $+_{\otimes}, *_{\otimes}$ element-wise extensions, and $\langle \psi_1 \otimes \psi_2 | \phi_1 \otimes \phi_2 \rangle_{\otimes} := \langle \psi_1 | \phi_1 \rangle \langle \psi_2 | \phi_2 \rangle$, is also a Hilbert space, and therefore, suitable and desirable to represent the n-partite system in question. In particular, we are going to deal with *bipartite systems*, i. e., $\mathbb{H} = \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$, (or $\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$). Considering bases for each system as $\{|i\rangle\}_{i=0}^{d_A-1}$, and $\{|j\rangle\}_{i=0}^{d_B-1}$, we write the general basis as $\{|ij\rangle\}_{i,j=0}^{d_A-1,d_B-1}$, with $|ij\rangle := |i\rangle \otimes |j\rangle$. We are going to work with *qubit* states, or states in $\mathbb{C}^2$, and particularly, with *two-qudit* states $\mathbb{C}^d \otimes \mathbb{C}^d$, and *two-qubit* states $\mathbb{C}^2 \otimes \mathbb{C}^2$. Let us now address the motivation for multipartite systems, in the so-called density matrix formalism.

1.1.2 Density Matrix Formalism

Considering multipartite systems, we now focus on the case when we are not interested in the whole system, but rather in subsystem of it. We will rename the two parts as *system* and *environment*. In doing so, we are going to see that we have to generalise the concepts of states and measurements to *density matrices* and *Positive-Operator-Valued-Measurements* (POVM's), respectively [33, 34] (the temporal evolution of the system needs also to be modified [34]). Finally, we also consider the case when the system is composed by more than one part, and introduce the property of entanglement for these general bipartite density matrices. Let us address these generalisations.

Density Matrix and POVM's: We start by introducing Hilbert spaces for the system and environment as $\mathbb{H}_S$ and $\mathbb{H}_E$, respectively. The Hilbert space for the whole system is then $\mathbb{H} := \mathbb{H}_S \otimes \mathbb{H}_E$, and a general state for the whole system-environment as $|\psi_{SE}\rangle$. If we consider bases for the system and environment as $\{|i\rangle\}$,$\{|j\rangle\}$, respectively, we have that any state in $\mathbb{H}$ can be written as: $|\psi_{SE}\rangle = \sum_{ij} \alpha_{ij} |ij\rangle, \alpha_{ij} \in \mathbb{C}, \forall i, j$. However, we are now only interested in the system S. Inspired by Eq. 1.3, we would like to find a matrix, ρ_s, such that after local measurements $\{P_a^x \otimes \mathbb{1}\}$, being P_a^x a projection on $\mathbb{H}_S$, we get:

$$\langle \psi_{SE} | P_a^x \otimes \mathbb{1} | \psi_{SE} \rangle = \mathrm{Tr}\left[(P_a^x \otimes \mathbb{1})\, \psi_{SE} \right] \stackrel{!}{=} \mathrm{Tr}(P_a^x \rho_s).$$

After some algebra, one finds that if we propose $\rho_S := \sum_{ik} \beta_{ik} |i\rangle \langle k|$ with $\beta_{ik} := \sum_j \alpha_{ij} \alpha_{kj}^*$, and calculate $\mathrm{Tr}(P_a^x \psi_s)$, we obtain this desired probability [34] (this can also be obtained by means of the so-called *Partial Trace* $\mathrm{Tr}_{\mathrm{E}}(\psi_{SE}) = \rho_S$). This matrix ρ_S is called a *density matrix*. Analysing ρ_S, we can check that it satisfies the following two properties: $\mathrm{Tr}(\rho) = 1$, and that is positive semidefinite

$\rho \geq 0$, which means that $\forall \left|\phi\right\rangle \in \mathbb{H} : \left\langle\phi\right| \rho \left|\phi\right\rangle \geq 0$ (which also implies that ρ is self-adjoint). We then define the set of density matrices as:

$$D(\mathbb{H}) := \{\rho \in LO(\mathbb{H}) | \rho \geq 0 \wedge \mathrm{Tr}(\rho) = 1\}. \tag{1.4}$$

Since we have a self-adjoint matrix, it can also be written in its spectral decomposition (Eq. 1.1) as $\rho_s = \sum_l p_l \left|l\right\rangle \left\langle l\right|$ with $\sum_l p_l = 1$, and $p_l \geq 0$. We now consider another representation of these matrices by considering:

$$\rho = \sum_i p_i \psi_i, \qquad \psi_i := \left|\psi_i\right\rangle \left\langle\psi_i\right|. \tag{1.5}$$

being $\{\left|\psi_i\right\rangle, p_i\}$ an *ensemble* of the states that we initially introduced, and that we now redefined as *pure* states, with $p_i > 0\ \ \forall i$, and $\sum_i p_i = 1$. Then, Eq. 1.5 can be seen as a statistical mixture of those pure states, and that is why they are called *mixed states*. We can check that this matrix also belongs to the set $D(\mathbb{H})$, so we can address general density matrices as mixed states by considering Eq. 1.5. A similar construction can be made in order to generalise projective measurements to the so-called Positive-Operator-Valued-Measurements (POVM's) as the set $M^x = \{E_a^x\}$, satisfying the properties [34]:

$$\sum_a E_a = \mathbb{1}, \qquad E_a^x \geq 0. \tag{1.6}$$

These operators can be seen as generalisations of the projections, without considering the third condition in Eq. 1.2. Now that we have generalised states and measurements, let us address a property that comes out from considering multipartite systems.

Entanglement for General Mixed Systems: We consider that the system of interest is now composed by two subsystems, and define the property of entanglement. Given a general bipartite mixed state as in Eq. 1.5: $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ with $p_i \geq 0$ and $\sum p_i = 1$, we say that this bipartite state is *separable*, if each one of the pure states from the mixture is separable, or that $\left|\psi_i\right\rangle = \left|\psi_i^1\right\rangle \otimes \left|\psi_i^2\right\rangle$, or $|\psi_i\rangle\langle\psi_i| = \psi_i^1 \otimes \psi_i^2\ \forall i$, and therefore having:

$$\rho = \sum_i p_i \psi_i^1 \otimes \psi_i^2, \tag{1.7}$$

with $\psi_i^1 := |\psi_i^1\rangle\langle\psi_i^1|$ and $\psi_i^2 := |\psi_i^2\rangle\langle\psi_i^2|$. If we cannot write ρ as in Eq. 1.7, we then say that ρ is *entangled*, or that it has *entanglement*. Given a general quantum state, it is not a trivial task to know whether it is possible to decompose it as in Eq. 1.7. There are *criteria* to detect whether there is entanglement or not, and *measures* to additionally determine the *amount* of entanglement there is [29].

In this subsection, we have then introduced density matrices, POVM's, and the property of entanglement for multipartite quantum states. We now aim to introduce the property of nonlocality for quantum states. However, nonlocality is originally a property of some objects called *correlations*, and from then, it can be extended to the quantum states domain. That is why we now first address the concept of correlations.

1.2 Correlations

In this section, we are going to introduce the concept of *correlations*, which have also been called behaviours or boxes. Since we have already introduced quantum mechanics, we first introduce *quantum* correlations. We then extend this structure to *general* and *non-signalling* correlations, *local* and *deterministic* correlations, and we finally describe some geometrical properties of all of these correlations, and extend the concept of nonlocality from correlations to states. This section is mainly based on the references [28, 35, 36, 37]. Let us then first introduce quantum correlations.

1.2.1 Quantum Correlations $Q(s)$

We have already seen that the quantum-mechanical probabilities given by Eq. 1.3, can be seen as conditional probability functions. Let us further formalise this. Given a general bipartite system $\rho \in D(\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B})$, the first part of the whole system is sent to experimentalist A (Alice), whilst the second to experimentalist B (Bob). Each of them has a set of local measurements given by local POVM's $M^x := \{E_a^x\}$, and $M^y := \{F_b^y\}$, where x and y (called *Inputs*) act as counters which could arbitrarily run as $x = 1, ..., I_A$, and $y = 1, ..., I_B$, with $I_A, I_B \in \mathbb{N}$, and $a = 1, ..., O_A$, $b = 1, ..., O_B$ (called *Outputs*), $O_a, O_B \in \mathbb{N}$. The amount of inputs and outputs define a *scenario* as:

$$s := (I_A, I_B, O_A, O_B), \tag{1.8}$$

and the magnitude $|s| := I_A I_B O_A O_B$. After receiving their respective part of the quantum state ρ, Alice and Bob make measurements x and y, obtaining outcomes a and b, respectively. After repeating this experiment enough times, eventually, they are able to establish the probability of obtaining outcomes a, b after measuring x and y, which could be written as the joint conditional probability function $p_\rho(ab|xy)$. This procedure is usually called a *Bell test*, and we can see a sketch of it in Fig. 1.1. According to Quantum Mechanics, these statistics follows from the Born Rule Eq. 1.3 as:

$$p_\rho(ab|xy) := \mathrm{Tr}\left[(E_a^x \otimes F_b^y)\rho\right]. \tag{1.9}$$

We now define the *quantum correlation* generated by the state ρ and local measurements $M^x = \{E_a^x\}$ and $N^x = \{F_b^y\}$, as the vector $\vec{p}_\rho = [p_\rho(ab|xy)]$ with

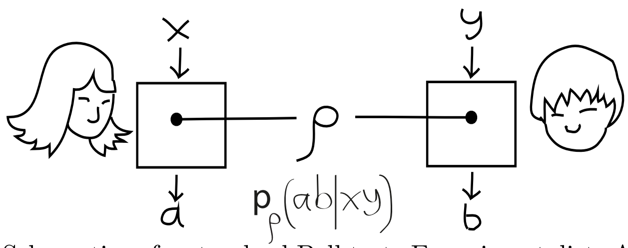

Figure 1.1: Schematics of a standard Bell test. Experimentalists Alice and Bob, share a bipartite quantum state ρ. They then make measurements x and y and obtain outcomes a and b, respectively. After repeating the experiment enough times, they determine the joint conditional probability function $p_\rho(ab|xy)$.

elements given by Eq. 1.9, which should be understood as:

$$\vec{p}_\rho = \begin{pmatrix} p_\rho\left(00|00\right) \\ \vdots \\ p_\rho\left(ab|xy\right) \\ \vdots \\ p_\rho\left(O_A O_B|I_A I_B\right) \end{pmatrix} \in \mathbb{R}^{|s|}. \tag{1.10}$$

We will later see that we can further constrain the dimension of this vector, by considering only the independent values. For the moment, let us define the set of quantum correlations generated by the quantum state ρ as:

$$Q_\rho(s) := \{\vec{p}_\rho \in \mathbb{R}^{|s|} \,|\, \exists M^x \wedge N^x \text{ s. t. Eq. 1.9 holds}\}, \tag{1.11}$$

with $M^x = \{E_a^x\}$, and $N^x = \{F_b^y\}$. We will also distinguish the set $Q_\rho^P(s)$ when these POVM's are only projections. We now define the set of all quantum correlations as:

$$Q(s) := \{\vec{p}_\rho \in \mathbb{R}^{|s|} \,|\, \exists \mathbb{H} \wedge \rho \wedge M^x \wedge N^x \text{ s. t. Eq. 1.9 holds}\}. \tag{1.12}$$

Let us now extract the fundamental properties from these quantum correlations, and extend them to *general* and *non-signalling* correlations [35, 36, 38, 39].

1.2.2 General $C(s)$ & Non-Signalling $NS(s)$ Correlations

Considering the set of quantum correlations (Eq. 1.12), we now first extract its general properties in order to define general correlations, and then, by imposing a natural condition, we introduce the non-signalling correlations.

General Correlations $C(s)$

Considering the scenario $s := (I_A, I_B, O_A, O_A)$, we consider vectors as in Eq. 1.10, with values given by the joint conditional probability functions $p_\rho(ab|xy)$ (Eq. 1.9).

These joint conditional probability functions satisfy the properties:

$$0 \leq p(ab|xy) \leq 1, \quad \forall x, y, a, b. \tag{1.13}$$

$$\sum_{a,b} p(ab|xy) = 1, \quad \forall x, y. \tag{1.14}$$

Whilst the first condition (Eq. 1.13) defines a hyperplane. The second condition (Eq. 1.14) constrains the vector $\vec{p}$ to live in an hypercube in $\mathbb{R}^{|s|}$, which is a *polytope* (a *convex* set with a finite amount of *vertices*). Therefore, $\vec{p}$ lives in the intersection of these sets, which is still a polytope. Even though these vectors initially live in $\mathbb{R}^{|s|}$, the second condition (Eq. 1.14) reduces the effective dimensionality of the vectors to $|c| := I_A I_B (O_A O_B - 1)$. We now define the the set of general correlations as:

$$C(s) := \{\vec{p} \in [0,1]^{|c|} | \text{ Eq. 1.14 holds}\}. \tag{1.15}$$

Coming back to the quantum set, we first can check that $Q(s) \subset C(s)$. Even though it is possible to show that $Q(s)$ is a convex set, it is not a polytope [35, 36, 39]. Let us now introduce a property for these general correlations inspired from the theory of relativity.

Non-Signalling Correlations $NS(s)$

Considering general correlations (Eq. 1.15), we want to model physically motivated correlations, so as a requirement, we should consider that our two systems respect the non-signalling principle, or that they cannot communicate instantaneously. Given a scenario s and $\vec{p} \in C(s)$, we represent this constraint by imposing the properties [28]:

$$\sum_{a} p(ab|xy) = p(b|xy) \stackrel{!}{=} p(b|y), \quad \forall x, \tag{1.16}$$

$$\sum_{b} p(ab|xy) = p(a|xy) \stackrel{!}{=} p(a|x), \quad \forall y. \tag{1.17}$$

The probabilities $p(b|xy)$ (Eq. 1.16) should be independent of x, because if we consider otherwise, it would mean that the input of Alice (x), would be somehow influencing the output of Bob (b). This is not possible because these correlations, in principle, could represent spatially separated systems as far away of each other as desired, so the influence $x \longrightarrow b$ would take place instantaneously, which would enter in conflict with the non-signalling principle from the theory of relativity. We then define the non-signalling correlations set as:

$$NS(s) := \{\vec{p}_{NS} \in C(s) | \text{ Eq. 1.16 and Eq. 1.17 hold}\}. \tag{1.18}$$

The non-signalling conditions define hyperplanes (linear conditions), and since $C(s)$ is already a polytope, we have that $NS(s)$ is still a polytope. However, the dimension of the set gets reduced to certain value $ns < c$ [35]. Let us now introduce the concept of locality and nonlocality for correlations.

1.2.3 Local $L(s)$ & Deterministic $D(s)$ Correlations

In order to introduce locality and nonlocality as properties of correlations, we first need to introduce the concept of an *ontological model*. We then introduce the locality condition, local correlations, and deterministic correlations. Let us then start by addressing a brief historical background about the ontological models, followed by the formalism itself.

Quantum Ontological Models

Historical Background: Let us come back to the interpretation of quantum mechanics. The Einstein-Podolsky-Rosen (EPR) article [1] is arguably the first, but definitely the most popular article addressing the conceptual implications of the mathematical formalism underneath quantum mechanics. In particular, they addressed concepts as *realism*, *determinism*, and *completeness*, concluding that quantum mechanics is *not* a complete theory. In a modern language, following reference [2], their result can be established as follows: Assuming the properties of *locality* and *completeness*, it is possible to derive a contradiction. Therefore, in order to avoid this contradiction, either locality or completeness should be dropped. EPR considered that locality was a natural condition to keep, so completeness was discarded. Let us now briefly review the road that these two properties have taken, since the EPR article until nowadays.

On the one hand, the discussion regarding the property of completeness led to a generalisation of it, which has been called *ψ-ontic realism* [2], where the Pusey-Barrett-Rudolph (PBR) theorem [40] and related ones [7], have addressed this property. However, the reality of the quantum state is still an open issue, and it is still unknown if it could lead to practical applications as the property of nonlocality has already done.

On the other hand, regarding the property of locality, John S. Bell formally introduced it within the so-called ontological models formalism, where it was then taken to a testable form [3]. Even though Bell himself wanted to prove quantum theory wrong, this was not the case, as experiments have shown [4, 5, 6]. Therefore, we have to rather accept a nonlocal quantum theory. After humankind started to digest this result, practical applications have come out [28].

In order to introduce the property of locality, we now take a glimpse on the quantum ontological models formalism [2] (even though the completeness and related properties are also of interest, this is not the focus of this work, see [7]).

Formalism: Given a scenario s, and considering general correlations $\vec{p} \in C(s)$ as in Eq. 1.15, we now want to introduce a *set of hidden variables* $\lambda \in \Lambda$, extra joint conditional probability functions $\xi(ab|xy\lambda)$ and $\mu(\lambda)$ that relate these elements as it is depicted in Fig. 1.2. The triplet (Λ, μ, ξ) is then called a *Quantum Ontological Model* (QOM) [2, 7]. Since we want to address quantum mechanics, so $\mu(\lambda|\rho) = \mu(\lambda)$. General correlations $\vec{p} \in C(s)$, as in Eq. 1.15, can be rewritten here as [2]:

$$p(ab|xy) = \int_\Lambda d\lambda \mu(\lambda)\xi(ab|xy\lambda), \tag{1.19}$$

We can see a diagram of this result also in Fig. 1.2. Let us now address what we

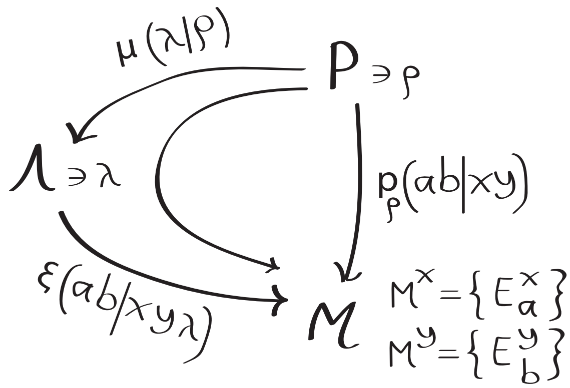

Figure 1.2: Schematics of an ontological model, featuring the ontic states space Λ, the measurements space M, the states space P, and joint conditional probabilities ξ, μ, p that relate these spaces. The joint conditional probability $p_\rho(ab|xy)$ can be written by following the inner arrow as in Eq. 1.19.

can talk about within this formalism. We first note that it can model non-signalling correlations, which conditions now translate to conditions for the functions ξ_1 and ξ_2. We also note that it can reproduce quantum correlations by explicitly defining: $\mu(\lambda|\rho) = \delta(\lambda - \lambda_\rho)$, $\xi(ab|xy\lambda) = \mathrm{Tr}[E^x_a \otimes E^y_b \lambda]$, and replacing them in Eq. 1.19, obtaining $\int_\Lambda d\lambda\mu(\lambda|\rho)\xi(ab|xy\lambda) = \mathrm{Tr}[(E^x_a \otimes E^y_b)\rho] = p_\rho(ab|xy)$.

The interesting feature of introducing ontological models is that we can now talk about properties of the functions μ, ξ, and Λ. Some of these properties have led to the so-called impossibility or no-go theorems. For instance, the properties of *completeness* [2], *ψ-onticity* [2], and *preparation independence* [40, 7], which are properties defined on the function μ, led to the PBR theorem, and similar no-go theorems [40, 7]. The property of *locality* on the other hand, which is a property defined for the function ξ, led to the Bell's theorem [3, 28]. There are more properties that can also be defined for these ontological models, that have whether led to other no-go theorems, or are simply interesting by themselves [7, 41]. However, in this document, we are going to concentrate our efforts on the property of locality, which we detail in what follows.

Local Correlations $L(s)$

The *locality* condition, and therefore local correlations read [3, 28]:

$$\xi(ab|xy\lambda) = \xi_1(a|x\lambda)\xi_2(b|y\lambda), \tag{1.20}$$

$$p_L(ab|xy) = \int_\Lambda d\lambda \mu(\lambda)\xi_1(a|x\lambda)\xi_2(b|y\lambda). \tag{1.21}$$

Eq. 1.20 means that the function ξ is probabilistically independent on the sides a, x and b, y. The structure $\Lambda, \mu, \xi_1, \xi_2$ is now called a *Local Hidden Variable* (LHV) model. In order to give a clear physical interpretation of these local correlations, let us first see how they form a polytope. For the moment, we define the set of local correlations as:

$$L(s) := \{\vec{p}_L \in C(s) | \text{ Eq. 1.21 holds}\}. \tag{1.22}$$

By means of the so-called Fines' theorem [42] (see Appendix A for a derivation of this result), these local correlations (Eq. 1.21) can be written as:

$$p_L(ab|xy) = \sum_{\lambda_2=1}^{O_A^{I_A}} \sum_{\lambda_2=1}^{O_B^{I_B}} \mu(\lambda_1, \lambda_2)\, \delta_a^x(\lambda_1)\delta_b^y(\lambda_2). \tag{1.23}$$

being $\Lambda, \mu, \delta_a^x, \delta_b^y$ a new LHV model, and δ_a^x, δ_b^y the so-called *deterministic correlations*. This result is telling us that local correlations are a finite convex combination of $O_A^{I_A} O_B^{I_B}$ deterministic correlations. Therefore, the set of local correlations form a polytope, being the deterministic correlations its vertices. Let us now address the meaning of these deterministic correlations.

Deterministic Correlations $D(s)$

Given inputs x and y, we are interested in the outcomes a and b. From a classical-deterministic point of view, there should be a way to determine outcomes a, b from inputs x, y. Analysing this for Alice, let us suppose there exists a function (λ_1) such that, given the input of Alice, it predicts the outcome, i. e. $(\lambda(x) = a_x)$. In other words, $p(a|x) = \delta_a^x(\lambda_1)$, where the latter function takes the value 1 only when $a = a_x = \lambda(x)$. Doing similarly for Bob, we have that the joint deterministic correlations are given by:

$$p_D(ab|xy) = \delta_a^x(\lambda_1)\delta_b^y(\lambda_2). \tag{1.24}$$

These correlations are representing the fact of (somehow) knowing about the outcomes, given the inputs, even before performing the experiment. Therefore, in this case, there would not even be the need of a probabilistic theory. However, we have already seen that quantum mechanics only tells us about the probability of obtaining outcomes a, b by means of the Born rule (Eq. 1.9), rather than an explicit way to calculate them using inputs x and y. Therefore, in quantum theory, we do not have access to these functions (λ_1, λ_2), and since they appear as variables in Eq. 1.23 , they have received the name of *hidden variables*. Even though we do not expect these deterministic correlations to be all in quantum mechanics, a natural way to try to generalise this idea, is precisely by considering convex combinations of them, which are ones we have called local correlations (Eq. 1.23).

The fact that these variables are hidden, does not mean that they could not exist, or be derived by, perhaps, a more general deterministic theory. However, as we are going to see in the next section, these LHV theories *cannot* exist (at least as a more general theory including quantum mechanics), because quantum theory does not respect this idea of local-determinism; quantum theory generates correlations that cannot be understood in a local-deterministic fashion (Eq. 1.23).

To wrap up, that local correlations (Eq. 1.23) can be seen as convex combinations of deterministic correlations (Eq. 1.24), which in turn encapsulate the intuitive feature of a deterministic theory, somehow explaining/deriving the outputs of a measurement when given the inputs. For the moment, let us now address some extra geometric properties of the polytopes $NS(s)$ and $L(s)$.

1.2.4 Some Properties of Sets of Correlations

We have already seen that these sets of correlations $NS(s)$ and $L(s)$ form polytopes; additionally, it can be proved that both of them lie in the same dimension (ns) given by $ns < |s|$ [35]. Since we are dealing with polytopes, they can be characterised whether by their *vertices* or their *facets* (also called *Bell Inequalities*) [35, 36]. In particular, we are going to work with the simplest nontrivial scenario $(2, 2, 2, 2)$. It can be proved that in simpler scenarios, the non-signalling correlations become equivalent to the local ones, and therefore they are of no interest [36]. Let us end up the section by extending this nonlocality property from correlations to quantum states.

1.2.5 Nonlocality for Quantum States

We have already introduced the properties of locality and nonlocality for general correlations. Now, in order to extend this idea to the quantum state-measurement domain, we consider the quantum correlations as a bridge. Given $\rho \in D(\mathbb{H})$ and a scenario (s), we have the set of correlations generated by the state in question and by all possible general POVM's as $Q_\rho(s)$ (Eq. 1.11), and when considering only projections as $Q_\rho^P(s)$. On the one hand, we have that if $Q_\rho(s) \subset L(s)$, we say that the quantum state ρ is *local* in the scenario s, if $Q_\rho^P(s) \subset L(s)$, we say that ρ is *local-projective* in the scenario s. On the other hand, if there exists at least a quantum correlation $\vec{p}_\rho \in Q_\rho(s)$ or $\vec{p}_\rho \in Q_\rho^P(s)$ lying outside the polytope $L(s)$, we say that ρ is *nonlocal*. With this, we have extended the property of nonlocality from correlations to quantum states. Let us now address the relationship amongst all of these sets of correlations.

1.3 Relationship Between Sets of Correlations

In this section, we establish the main theorem regarding the relationship between the concepts of states-measurements and correlations which we introduced in the previous sections. The main theorem in question establishes the following: given a scenario $s = (I_A, I_B, O_A, O_B)$ (Eq. 1.8), we have the chain of relations: $S(s) = L(s) \subsetneq Q(s) \subsetneq NS(s)$, featuring: separable $S(s)$, local $L(s)$, quantum $Q(s)$, and non-signaling $NS(s)$ correlations. In Fig. 1.3, we illustrate this result as well as other properties of the sets themselves. We now explicitly point out some remarks about this theorem.

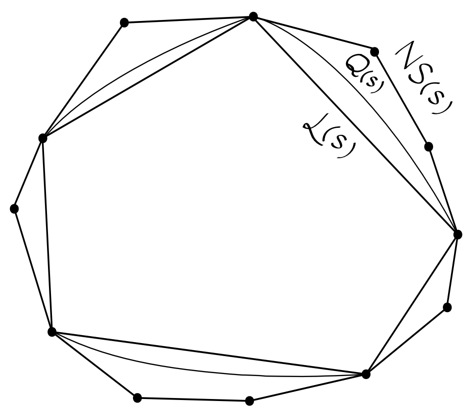

Figure 1.3: Two-dimensional representation of the main theorem $L(s) \subsetneq Q(s) \subsetneq NS(s)$ (the simplest-non-trivial scenario $S = (2,2,2,2)$ deals with these sets in $\mathbb{R}^8$ [28]). Whilst $NS(s)$, $Q(s)$, and $L(s)$ are all convex sets, only $NS(s)$ and $L(s)$ are polytopes. It also illustrates that some facets of que quantum set do coincide with facets of the local one (This has been shown for multipartite setups [43, 44].

On the one hand, correlations generated with separable states can only produce local correlations $S(s) \subseteq L(s)$. Furthermore, we can reproduce local correlations with separable correlations $L(s) \subseteq S(s)$, having $S(s) = L(s)$. However, there exist entangled states that generate nonlocal correlations $L(s) \subsetneq Q(s)$, which is the general conclusion from the Bell's Theorem [3]. One might also have the feeling that every entangled state must generate nonlocal correlations, however, there exist entangled states that can only (as separable states) produce local correlations [26, 27].

On the other hand, all quantum correlations respect the non-signalling principle $Q(s) \subseteq NS(s)$, even the quantum-nonlocal correlations, being therefore, in accordance with the theory of relativity. However, there exist correlations that we cannot achieve with quantum mechanics yet satisfying the non-signalling principle $Q(s) \subsetneq NS(s)$, the so-called superquantum correlations [8].

In Appendix B we provide a proof of this theorem. In Appendix B.1, we write down the proof of the proper subsets part, i. e., that given any scenario (s), we have $S(s) = L(s) \subseteq Q(s) \subseteq NS(s)$, in Appendix B.2 we explicitly show the respective counterexamples for particular scenarios, i. e., that given any scenario (s), we have $L(s) \subsetneq Q(s) \subsetneq NS(s)$. First, regarding $L(s) \subsetneq Q(s)$, we consider the so-called Werner-Isotropic states; entangled two-qubit mixed states which generate only local correlations in the scenario $(I_A, I_B, 2, 2)$ [26]. Second, regarding $Q(s) \subsetneq NS(s)$, we present the so-called PR box: a non-signalling correlation in the scenario $(2, 2, 2, 2)$ that cannot be reproduced quantum-mechanically [8]. Now, for the so-called Bell's theorem, $L(s) \subsetneq Q(s)$, we present a summary in what follows.

We consider the so-called *Bell state*, an entangled two-qubit pure state generating a correlation outside the polytope $L(2, 2, 2, 2)$, and therefore being nonlocal in that scenario [3]. We then also introduce the so-called Horodecki criterion for the violation of the CHSH-Inequality for general two-qubit systems [45], and the so-called Gisin theorem [46, 47, 48], which give us more examples of entangled-nonlocal states for the scenario $(2, 2, 2, 2)$.

1.3.1 Bell's Theorem: $L(2, 2, 2, 2) \subsetneq Q(2, 2, 2, 2)$

In order to show that local correlations form a proper subset of quantum correlations in the scenario $(2, 2, 2, 2)$, we have to provide a quantum correlation $\vec{p}_Q$ such that cannot be reproduced by any LHV model. In other words, we have to generate nonlocality with quantum theory. In this regard, we have already seen that separable states are local for any scenario (s), therefore, we should consider entangled states. We have also seen that the facets of the local polytope (Bell-inequalities) fully characterise the polytope in question [35, 36]. Following this line of thought, it is enough for a quantum correlation to violate a Bell-inequality, in order to show nonlocality. This is essentially what John S. Bell initially proved [3], by showing the existence of quantum nonlocality by generating a quantum correlation with a two-qubit pure state that violated a Bell-inequality. This inequality was later refined to the so-called *Clauser-Horne-Shimony-Holt (CHSH)* inequality [49] which would correctly characterise the polytope $L(2, 2, 2, 2)$ (In Appendix C we further address some features of this inequality). We first introduce the CHSH-inequality [49] for the scenario $s = (2, 2, 2, 2)$, and then we show how the so-called two-qubit Bell state violates it [3]. We then build on more entangled-nonlocal states by introducing the Horodecki criterion for arbitrary two-qubit states [45], which in particular shows us that any two-qubit pure entangled state violates this CHSH inequality [46]. Furthermore, we present the Gisin theorem [46, 47, 48], which establishes that multipartite pure entangled states are nonlocal by means of the CHSH inequality. Let us now address the CHSH inequality in question.

CHSH Inequality

We first consider the simplest-non trivial local scenario, given by the local polytope $L(2, 2, 2, 2)$ [49, 28]. This local polytope has been completely characterised (see a nice review in [36]). It turns out that in this scenario there is only one (up to relabelling transformation [36]) non-trivial Bell-inequality, known as the CHSH-Inequality. Considering the scenario $(2,2,2,2)$, and following the notation for correlators introduced in Appendix B.2.1, the CHSH-inequality takes the form [28]:

$$\left|E^{00}+E^{01}+E^{10}-E^{11}\right| \leq 2, \tag{1.25}$$

(See Appendix C.1 for a derivation). This inequality is then representing the facets of the local polytope $L(2,2,2,2)$. We also remember that these correlators can be produced quantum-mechanically by considering two dichotomic (eigenvalues ± 1) observables per party, namely, $\{A^0, A^1, B^0, B^1\}$, so the correlators read $E^{xy}_\rho := \mathrm{Tr}\left[(A^x \otimes B^y)\,\rho\right]$, $x, y = 0, 1$. Therefore, if we can propose a quantum correlation that violates the CHSH-inequality (Eq. 1.25), we have a nonlocal quantum correlation in the scenario $(2,2,2,2)$. Let us now present how this is indeed possible.

Bell's Theorem: $\vec{p}_{\psi^-} \notin L(2,2,2,2)$

We now present a quantum correlation that violates the aforementioned CHSH-inequality (Eq. 1.25), and therefore, it is a nonlocal quantum correlation. In order to generate this quantum correlation, we consider the pure two-qubit $\psi^- \in D(\mathbb{C}^2 \otimes \mathbb{C}^2)$ state:

$$\psi^- = \left|\psi^-\right\rangle\left\langle\psi^-\right|, \qquad \left|\psi^-\right\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle). \tag{1.26}$$

By using the so-called PPT criterion for entanglement (See details in Eq. B.1), we can check that this is an entangled state. In order to build our quantum correlation in term of the correlators (Eq. B.2), we can also check that $E^{xy}_{\psi^-} := \mathrm{Tr}\left[(A^x \otimes B^y)\rho\right] = -\hat{a}^x \cdot \hat{b}^y$ with $A^x = \vec{\sigma}\cdot\hat{a}^x$, $B^y = \vec{\sigma}\cdot\hat{b}^y$, and $\hat{a}^x, \hat{b}^y$ arbitrary vectors in $\mathbb{R}^3$. Replacing this in the CHSH-inequality (Eq. 1.25), we obtain $\hat{a}^0\cdot(\hat{b}^0+\hat{b}^1)+\hat{a}^1\cdot(\hat{b}^0-\hat{b}^1)$. We now want to maximise this function over these four vectors. If we choose these four vectors as in Fig. 1.4,

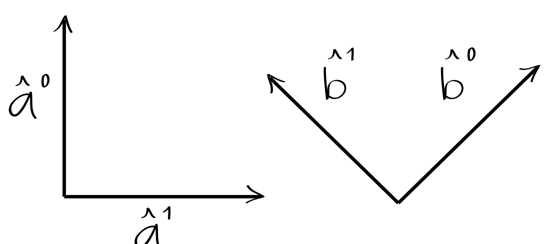

Figure 1.4: Set of possible vectors $\hat{a}^0, \hat{a}^1, \hat{b}^0, \hat{b}^1$ lying in a plane that maximise the CHSH-inequality violation (Eq. 1.25).

we obtain the desired violation since:

$$\hat{a}^0\cdot(\hat{b}^0+\hat{b}^1)+\hat{a}^1\cdot(\hat{b}^0-\hat{b}^1) = 2\sqrt{2} > 2, \tag{1.27}$$

which is beyond the local bound given by 2, and therefore violating the CHSH-inequality (Eq. 1.25). Thus, these correlations generated by the Bell state (Eq. 1.26) cannot be reproduced by any LHV model or $\vec{p}_{\psi^-} = [p_{\psi^-}(ab|xy)] \notin L(2,2,2,2)$ with $p_{\psi^-}(ab|xy)$ in terms of the correlators given by Eq. B.4. Therefore, $\vec{p}_{\psi^-}$ is a nonlocal correlation, making of the Bell state ψ^- (Eq. 1.26) a nonlocal state.□

Relevance of the Bell's Theorem & Generalisations

Bell's theorem is showing us that quantum theory predicts the existence of nonlocal quantum correlations. These predictions were experimentally confirmed [5, 6], and we must then drop locality, and embrace a nonlocal quantum theory. This nonlocality property has been gradually digested, and practical applications have started to come out [28]. As a very fundamental example of these applications, in Appendix C.2 we address the so-called *CHSH game*, which gives us a glimpse on how nonlocality is useful in the field of quantum game theory.

From a technical point of view, Bell's theorem leaves us with several questions regarding the generation of nonlocality with quantum theory. Bell's theorem provides a quantum nonlocal correlation generated with an entangled pure two-qubit state (Eq. 1.26), this, together with certain projections given by the considered vectors in Fig. 1.4. One might then ask questions like: What sort of entangled two-qubit states allow the construction of nonlocal correlations? Could we generate nonlocality with high dimensional entangled states? Given an entangled two-qubit state, could we find (in general) the measurements that maximise the CHSH-inequality violation?

In what follows, we are going to briefly mention partial answers to the aforementioned questions. We first present the so-called Horodecki criterion for the violation of the CHSH-inequality, which explicitly tells us what sort of entangled two-qubit states could possibly violate the CHSH-inequality (by providing the measurements that maximise the violation), which will let us conclude that any entangled pure two-qubit state can violate the CHSH inequality. We then consider a generalisation of the latter result to multipartite systems, called the Gisin theorem. Let us now address these considerations.

Horodecki Criterion, Tsirelson Bound & Gisin Theorem

Given $\rho \in D(\mathbb{C}^2 \otimes \mathbb{C}^2)$, we have the following characterisation for its CHSH-inequality violation by means of the so-called *Horodecki Criterion* [45]:

$$\rho \quad \text{violates the CHSH-inequality (Eq. 1.25)} \quad \overset{[45]}{\Longleftrightarrow} \quad \mu_\rho + \widetilde{\mu}_\rho > 1, \quad (1.28)$$

being $\mu, \widetilde{\mu}$ the two biggest eigenvalues of the matrix $U_\rho := T_\rho^T T_\rho \in M_{3\times 3}(\mathbb{R})$, with T_ρ the matrix with elements given by $[T_\rho]_{ij} = \mathrm{Tr}[(\sigma_i \otimes \sigma_j)\rho]$, and σ_k the Pauli matrices (See Appendix C.3 for a derivation). This result is then completely characterising the CHSH-inequality violation for general two-qubit systems.

There is however, a bound called the *Tsirelson's bound* which establishes that the maximal violation of the CHSH-inequality (Eq. 1.25) allowed by quantum theory is $2\sqrt{2}$ [50]. Interestingly, this was the value achieved by the Bell state in the Bell's theorem (Eq. 1.27).

As a corollary of the Horodecki criterion, we have that every pure entangled two-qubit state violates the CHSH inequality (Eq. 1.25), and therefore, being nonlocal [46, 37]. This particular corollary was previously independently discovered by Nicolas Gisin [46], which was then generalised to bipartite high dimensional pure systems [47], and then to multipartite pure systems [48]. Therefore, nonlocality and entanglement are equivalent for general multipartite pure states, result which has been coined the *Gisin Theorem*. However, as we have already seen, this is not the case for general mixed states, since there exist mixed entangled but local states [26, 27] (as the ones in Appendix B.2.1, Eq. B.5). The moral is then, that there exist nonlocal quantum correlations.

1.4 Final Considerations

We have introduced entanglement as a property of quantum states, and the property of nonlocality also for quantum states, this, by considering whether a quantum state can generate a quantum correlation that cannot be reproduced locally. We also addressed the relationship between quantum mechanics and correlations (See Appendix B for a proof). In particular, we pointed out the existence of entangled local states and the existence of superquantum correlations. We further detailed the existence of nonlocality, mostly in regards of the CHSH-inequality, for which more detailed issues can be found in Appendix C. In the next chapter, we are going to focus on entanglement and nonlocality for two-qubit systems, and further enquire on entangled local states.

Chapter 2

Some Two-qubit Measures & Bipartite Entangled Local States

In this chapter, we first introduce some two-qubit measures, namely, the entanglement of formation, the CHSH-inequality violation, and the usefulness for teleportation. We then consider some entangled local states, as a generalisation of the Werner-Isotropic states given in Eq. B.5, which are going to be of interest when studying the activation of nonlocality in the next chapter. Let us then begin by addressing some two-qubit measures.

2.1 Some two-qubit Measures

In this section, we first introduce the measures of entanglement of formation, and a CHSH-inequality violation measure based on the Horodecki criterion (Eq. 3.6). We then introduce the property of *usefulness for teleportation* which is going to be of relevance when studying the activation of nonlocality in the next chapter. For the moment, this property is going to be of interest because it lies between entanglement and nonlocality, and because it allows us to determine whether a state is useful for the teleportation protocol. Finally, we present a theorem relating these three measures. Let us then address a measure for two-qubit entanglement.

2.1.1 Entanglement of Formation (EoF)

We have already addressed that the PPT criterion (Eq. B.1) which is a necessary and sufficient entanglement *criterion* for two-qubit systems [51]. However, we might also be interested in having a *measure* of entanglement, which in addition to telling us whether a state is entangled, it would also tell us the *amount* of entanglement it has (see a review of entanglement measures in [29]). For pure bipartite states $\psi = |\psi\rangle\langle\psi|$, we can define a measure as $E(\psi) := S(\psi_A) = -\mathrm{Tr}(\psi_A \mathrm{log}_2 \psi_A)$ with $\psi_A := \mathrm{Tr}_B(\psi)$ the partial trace, and S the *von Neumann entropy* [52]. For general mixed states $\rho = \sum_i p_i \psi_i$ with $p_i \geq 0$, and $\sum_i p_i = 1$ (Eq. 1.5), it is natural to consider the possible generalisation $\sum_i p_i E(\psi_i)$. However, since ρ has infinite possible ensemble decompositions (Eq. 1.5), this definition will depend on the chosen ensemble. In order to avoid this, the following measure, known as the *Entanglement of Formation* (EoF), has been introduced [52]:

$$EoF(\rho) := \min_{\{\psi_i, p_i\}} \left[\sum_i p_i E(\psi_i) \right]. \tag{2.1}$$

In 1998, an analytical expression of Eq. 2.1 for the particular case of two-qubit states has been derived [53]. Given $\rho \in D(\mathbb{C}^2 \otimes \mathbb{C}^2)$, its EoF is given by:

$$EoF(\rho) = h\left(\frac{1+\sqrt{1-C(\rho)^2}}{2} \right), \tag{2.2}$$

$$h(x) := -x\mathrm{log}_2 x - (1-x)\mathrm{log}_2(1-x), \tag{2.3}$$

$$C(\rho) := \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}, \tag{2.4}$$

where the λ_i's are the square roots of the eigenvalues of the product matrix $\rho\hat{\rho}$, in decreasing order, with: $\hat{\rho} = (\sigma_y \otimes \sigma_y)\, \rho^* \,(\sigma_y \otimes \sigma_y)$, ρ^* the complex conjugate of ρ and, σ_y the Pauli matrix. The function h is the so-called *binary entropy*, and C is the so-called *concurrence*. Both, C and EoF go from 0 to 1, so as a measure we could use any of them, however, we opt for the EoF, since it has a physical motivated interpretation [52]. There is no known compact analytical expression (as Eq. 2.2) for general high-dimensional systems, except for a couple of classes of two-qudit states, the so-called Werner and Isotropic states [54, 55, 56]. The EoF (Eq. 2.2) is the explicit entanglement measure we are going to use throughout the document. We now continue with a measure for the CHSH-inequality violation for general two-qubit systems.

2.1.2 CHSH-inequality Violation (CHSH)

We have already seen the Horodecki criterion for the violation of the CHSH-inequality (Eq. 1.28), which establishes that it is enough to check $M(\rho) > 1$, being $M(\rho) := \mu_\rho + \hat{\mu}_\rho$ with $\mu_\rho, \hat{\mu}_\rho$ the biggest two eigenvalues of the matrix $T_\rho^T T_\rho$ (Eq. 1.28). This M function goes like $0 \leq M(\rho) \leq 2$, showing nonlocality in the region $1 < M(\rho) \leq 2$. However, instead of $M(\rho)$, we could also work with

$$B(\rho) := \sqrt{\max\{0, M(\rho) - 1\}}, \tag{2.5}$$

showing nonlocality in $0 < M(\rho) \leq 1$. This new function $B(\rho)$ is desirable because it now looks as a measure, and also because for pure states ($\psi = |\psi\rangle\langle\psi|$), it turns out to be equal to the concurrence (Eq. 2.4), i. e., $C(\psi) = B(\psi)$ [57].

Now, in order to analyse nonlocality through the CHSH-inequality and make a direct comparison with the EoF (Eq. 2.2), we will plot:

$$CHSH(\rho) = h\left(\frac{1+\sqrt{1-B(\rho)^2}}{2}\right), \tag{2.6}$$

being h the binary entropy as defined in Eq. 2.3. To wrap up, we search for nonlocality by means of this measure for the CHSH-inequality violation (Eq. 2.6). In order to introduce the two-qubit measure of usefulness for teleportation, let us first briefly review the quantum teleportation protocol.

2.1.3 Teleportation Protocol

The teleportation protocol was first introduced in reference [25]. It works as follows. A quantum pure state $|\phi\rangle \in \mathbb{C}^d$ can be teleported by means of a channel built with a bipartite quantum state $\rho \in D(\mathbb{C}^d \otimes \mathbb{C}^d)$. In order to check how useful for the protocol ρ is, a function $F_d(\rho)$ called the *Fidelity of Teleportation* (FoT) was proposed [58]. A seminal result regarding this function [59] establishes that:

$$F_d(\rho) = \frac{df_d(\rho)+1}{d+1}, \tag{2.7}$$

where f_d is the so-called *Fully Entangled Fraction* (FEF) (sometimes also called *Entanglement Singlet Fraction*) given by:

$$f_d(\rho) := \max_{\psi \in ME} \langle\psi|\rho|\psi\rangle\,, \tag{2.8}$$

where ME stands for the set of maximally entangled states, i. e., the states $\psi := |\psi\rangle\langle\psi|$, defined as: $|\psi\rangle := (U_1 \otimes U_2)\frac{1}{\sqrt{d}}\sum_{i=0}^{d-1}|ii\rangle$, with U_1, U_2 local unitary operations [59]. Taking a look at Eq. 2.8, we have the range $\frac{1}{d+1} < F_d < 1$. If we use separable states in Eq. 2.8, we obtain that $f_d(\rho_{sep}) = \frac{1}{d}$ [59]. Consequently, we have the following characterisation:

$$\rho \;\text{ is useful to teleportation} \quad \Longleftrightarrow \quad f_d(\rho) > \frac{1}{d}. \tag{2.9}$$

In particular, for a two-qubit state ($d = 2$), we first rename these functions as: $F(\rho) := F_2(\rho)$ and $f(\rho) := f_2(\rho)$. We then have the general range $\frac{1}{3} < F(\rho) < 1$, and ρ being useful for teleportation if and only if $f(\rho) > \frac{1}{2}$ or $F(\rho) > \frac{2}{3}$. We now address a full characterisation of the fidelity of teleportation (Eq. 2.7) for general two-qubit systems.

2.1.4 Usefulness for Teleportation (UfT):

Following [60] and [61], we now summarise the fidelity of teleportation (FoT) (Eq. 2.7) characterisation for general two-qubit systems. Given $\rho \in D(\mathbb{C}^2 \otimes \mathbb{C}^2)$, we have that $f(\rho) = \max_i\{\eta_i, 0\}$, where the η_i's are the eigenvalues of the matrix $M = [M_{mn}]$, with elements $M_{mn} = \mathrm{Re}\left(\langle\psi_m|\,\rho\,|\psi_n\rangle\right)$, where $\{|\psi_n\rangle\}$ is the so-called magic basis $|\psi_{ab}\rangle := i^{(a+b)}(|0,b\rangle + (-1)^a\,|1, 1\oplus b\rangle)/\sqrt{2}$. If we consider the function $N(\rho) := \sum_{i=1}^{3}\sqrt{\mu_i}$, being μ_i the eigenvalues of the matrix $T_\rho^T T_\rho$ (which also turns out to be $N(\rho) = -1+4f(\rho)$), we have that the FoT (Eq. 2.7) is given by

$$F(\rho) = \frac{2f(\rho)+1}{3} = \frac{1}{2}\left[\frac{1}{3}N(\rho)+1\right]. \tag{2.10}$$

We have already seen that ρ is useful for teleportation if and only if $F(\rho) > \frac{2}{3}$, which translates to $f > \frac{1}{2}$ and $N > 1$. It should be pointed out that the bound $f = \frac{1}{2}$ also defines usefulness for other two-qubit protocols besides teleportation [60]. We are going to see that usefulness for teleportation lies in between entanglement and nonlocality. Since for pure states we have $C = B$. Then, in order to enquire about this property, we will work with the function

$$D(\rho) := \max\left\{0, \frac{N(\rho)-1}{2}\right\}. \tag{2.11}$$

By calculating whether $f(\rho)$ or $N(\rho)$ we can obtain $D(\rho)$. However, in order to make a direct comparison with the EoF (Eq. 2.2) and our CHSH-inequality violation measure (Eq. 2.6), we define the measure of *Usefulness for Teleportation* (UfT) as:

$$UfT(\rho) = h\left(\frac{1+\sqrt{1-D(\rho)^2}}{2}\right), \tag{2.12}$$

being h the binary entropy as defined in Eq. 2.3. For high-dimensional or multipartite systems, there is no general explicit relation for f (Eq. 2.8), except for the so-called Werner and Isotropic states [62] (which we are going to address in the next section). We then have the fidelity of teleportation as a measure for general two-qubit systems, which gives us information if the state in question is useful for teleportation. We already have these three measures, we now address the main theorem that relates them.

2.1.5 Main Theorem: CHSH $\Rightarrow$ UfT $\Rightarrow$ EoF

We have already seen that we need entanglement in order to generate both CHSH-nonlocality (Eq. 2.6), and usefulness for teleportation (Eq. 2.12). In other words, both nonlocal and useful-for-teleportation states define subsets of the entangled ones. The main result here establishes that the usefulness for teleportation (Eq. 2.12) covers the set of nonlocality (Eq. 2.6), and therefore, it lies in between nonlocality and entanglement [61]. Formally, given a two-qubit state $\rho \in D(\mathbb{C}^2 \otimes \mathbb{C}^2)$, we have the hierarchy of properties depicted by the following chain of implications:

$$\begin{array}{c} \rho \text{ violates} \\ \text{the CHSH} \\ \text{inequality} \end{array} \xRightarrow{[59]} \begin{array}{c} \rho \text{ is useful} \\ \text{for} \\ \text{teleportation} \end{array} \xRightarrow{[61]} \begin{array}{c} \rho \\ \text{is} \\ \text{Entangled} \end{array} \tag{2.13}$$

Additionally, we have already seen that for pure states we get $C(\psi) = B(\psi) = D(\psi)$, then also for pure states we will get $EoF(\psi) = CHSH(\psi) = UfT(\psi)$, though they become different when considering mixed states. The first implication is restrictive in the sense that there exist entangled local states (states that do not violate any Bell inequality, in particular, the CHSH-inequality), although useful for teleportation [58, 61]. The second implication states that usefulness for teleportation does not cover all the entangled states set, i. e., there exist entangled states not useful for teleportation [61, 59]. Putting together what we have seen in this chapter, in Fig. 2.1 we consider a diagram for two qubits and we can also extend it for two-qudit states.

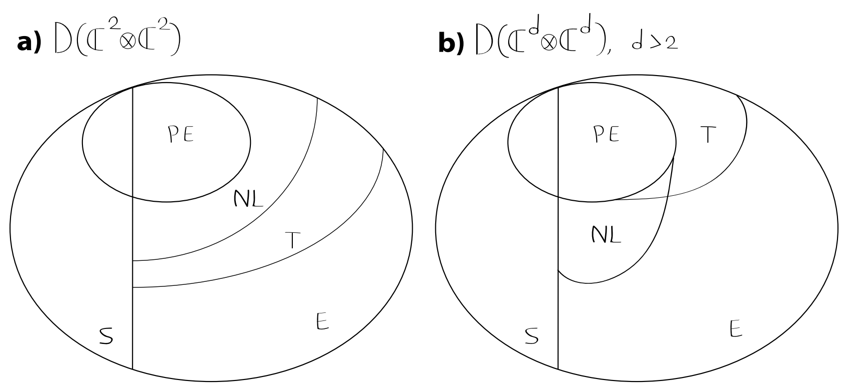

Figure 2.1: Venn-like representation of several results concerning the properties of entanglement, nonlocality, and usefulness for teleportation for: **a)** two-qubit states, and **b)** two-qudit ($d > 2$) states. Featuring: S as Separable states, E as entangled states, PE as pure-entangled states, NL as nonlocal states, and T as states useful for teleportation. We have depicted the following results. First, $NL \subset E$ (Appendix B.1.1). Second, $PE \subset NL$ (by the Gisin theorem section 1.3.1 which holds true even for multipartite states). Third, for two-qubit states **a)** we have $NL \subset T$ (Eq. 2.13); this is not longer true when $d > 2$ **b)**. The latter is due to the existence of bound entangled (and therefore not useful for teleportation) nonlocal states [63] (these states disproved the so-called Peres Conjecture [63]), so now NL and T become independent. Fourth, $PE \subset T$ holds for two-qubit states **a)** as an immediate consequence of $PE \subset NL$ and $NL \subset T$, whilst for two-qudit states ($d > 2$) **b)**, it holds true if we first extract nonlocality by means of the Gisin theorem.

We have already seen three measures for general two-qubit states and the relationship between each other. Let us now address some entangled local states that are going to be of interest when studying the activation of nonlocality.

2.2 Some Bipartite entangled local States

We have already addressed the existence of some entangled local states, namely, the two-qubit Werner-Isotropic (WI) states in Eq. B.5. In this section, we start by further enquiring about these WI states, and then listing some additional bipartite entangled local states, which can be seen generalisations of the WI states. First, we present two generalisations toward two-qudit systems: Werner [26], and Isotropic States [64], and then, two to two-qubit states, the so-called Hirsch states [65], and some noisy states which we will call Game states. We now depict the notation we will use and the relationship between these states in Fig. 2.2. We are going to check for entanglement, locality, and the fully entangled fraction (Eq. 2.8). It should be pointed out, that there are not that many entangled local states, since locality is not an easy property to guarantee (see a recent review in [27])[1].

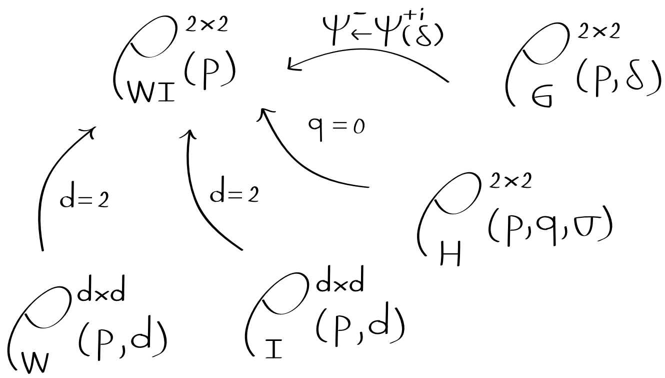

Figure 2.2: Some bipartite entangled local states, namely; Werner states $\rho_W(p)$, Isotropic states $\rho_I(p,d)$, Hirsch states $\rho_H(p,q,\sigma)$, and Game states $\rho_G(p,q)$. These states can be seen as generalisations of the Werner-Isotropic (WI) states $\rho_{WI}(p)$ (Eq. B.5). Starting from any of these generalisations, we can recover the ρ_{WI} by assigning the values depicted in the arrows.

2.2.1 Werner-Isotropic (WI) States

We start by further addressing the Werner-Isotropic (WI) states (see also Eq. B.5) [26], which are defined as:

$$\rho_{WI}(p) = p\,\psi^- + \frac{(1-p)}{4}\mathbb{1}, \tag{2.14}$$

with $0 \leq p \leq 1$, and $\psi^- := |\psi^-\rangle\langle\psi^-|$, $|\psi^-\rangle := \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$. By using the PPT criterion (Appendix B.2.1), these states are entangled if and only if $p \geq$

[1] There has been a recent breakthrough toward numerical constructions of LHV models for entangled states [66, 67].

1/3, and local (projective) if $p \leq 1/2$. One can also check that these WI states are useful for teleportation if and only if $p \geq 1/3$ (same than the entanglement region). We summarise further results regarding locality in Fig. 2.3.

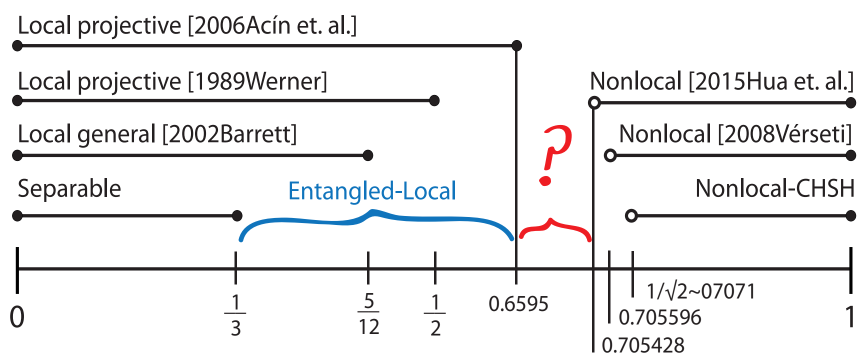

Figure 2.3: Properties of entanglement, nonlocality, and locality for the Werner-Isotropic (WI) states (Eq. 2.14). These states are entangled if and only if $1/3 < p$. These states were first proved to be local (projective) if $p \leq 1/2$ [26]. Then, they were proven to be local (general) if $p \leq 5/12$ [68]. Then, it was proven that these states are local (projective) *if and only if* $p \leq 1/K_G(3)$, being $K_G(3)$ the Grothendieck constant of order three [69]. Even though this latter result would fully characterise the local(projective) region for these states, the exact value of $K_G(3)$ is not known, and there are only calculations for its lower and upper bounds. The best known upper bound reads $K_G(3) \leq 1.5163$, and therefore, the WI states are local(projective) if $p \leq 0.6595$. The CHSH-inequality imposes the lower bound $\sqrt{2} \leq K_G(3)$, or that the WI states are nonlocal if $0.7071 < p$. In 2008, this lower bound was improved to $1.417241 \leq K_G(3)$, or that the WI states are nonlocal if $0.705596 < p$ [70]. So far, the best known lower bound reads $1.41758 \leq K_G(3)$, or that the WI states are nonlocal if $0.705428 < p$ [71]. Therefore, leaving the gap $1.41758 \leq K_G(3) \leq 1.5163$ or $0.6595 < p \leq 0.705428$ where it is not known whether these states are local (projective).

2.2.2 Werner States

We now consider the two-qudit Werner states $\rho_W^d \in D(\mathbb{C}^d \otimes \mathbb{C}^d)$ which were actually introduced at the same time than the Werner-Isotropic states [26], and in terms of a parameter p, they are defined as:

$$\rho_W^d(p) = \frac{p}{d(d-1)} 2P^- + \frac{(1-p)}{d^2}\mathbb{1}, \tag{2.15}$$

$$1 - \frac{2d}{d+1} \leq p \leq 1, \qquad P^- := \frac{1}{2}\left(\mathbb{1} - \sum_{ij}^{d} |i\rangle\langle j| \otimes |j\rangle\langle i|\right).$$

We now consider another parametrisation as:

$$\rho_W^d(f) = \frac{d-f}{d^3-d}\mathbb{1} + \frac{df-1}{d^3-d}V, \tag{2.16}$$

$$f = \mathrm{Tr}(\rho_W V), \qquad -1 \leq f \leq 1, \qquad V = \sum_{i,j=1}^{d} |ij\rangle\langle ji|,$$

with $|\psi_d\rangle := \frac{1}{\sqrt{d}}\sum_{i=0}^{d-1}|ii\rangle$ the maximally entangled state of dimension d. The relation with the previous parameter is given by $p = \frac{1-df}{d+1}$. We now list the properties of entanglement, nonlocality and fully entanglement fraction for these Werner states. In terms of the parameters p and f we have:

$$p > \frac{1}{d+1} \qquad \longleftrightarrow \qquad f < 0, \qquad \xleftrightarrow{[26]} \qquad \text{Entangled}, \tag{2.17}$$

$$p \leq \frac{d-1}{d} \qquad \longleftrightarrow \qquad f \geq \frac{-d^2+d+1}{d^2} \qquad \xrightarrow{[26]} \qquad \text{Local (Projective)}, \tag{2.18}$$

$$p \leq \frac{(3d-1)(d-1)^{(d-1)}}{(d+1)d^d} \longleftrightarrow f \geq \frac{d^d-(d-1)^{(d-1)}(3d-1)}{d^{d+1}} \xrightarrow{[68]} \text{Local (General)}. \tag{2.19}$$

It should be pointed out that there is also a multipartite generalisation to three-qubit entangled local states $\rho_I \in D(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2)$ [27]. We now consider the property of fully entangled fraction (Eq. 2.8) for these Werner states which reads [62]: (the first column represents d even, whilst the second represents d odd)

$$f_d[\rho_W^d(f)] = \begin{cases} \frac{1-f}{d(d-1)}, \frac{d^2-d^2f+df+d-2}{d^2(d^2-1)} & if \ -1 \leq f < \frac{1}{d}, \\ \\ \frac{f+1}{d(d+1)} & if \ \frac{1}{d} \leq f \leq 1. \end{cases} \tag{2.20}$$

We can check that this function is always less than $1/d$, so we cannot guarantee (by Eq. 3.1) that these Werner states (Eq. 2.15) are useful for teleportation. Let us now consider the so-called Isotropic states.

2.2.3 Isotropic States

The two-qudit Isotropic states $\rho_I^d \in D(\mathbb{C}^d \otimes \mathbb{C}^d)$ [64] which in terms of a parameter p they are defined as:

$$\rho_I^d(p) = p\,|\psi_d\rangle\langle\psi_d| + \frac{(1-p)}{d^2}\mathbb{1}, \tag{2.21}$$

$$0 \leq p \leq 1, \qquad |\psi_d\rangle = \frac{1}{\sqrt{d}}\sum_{i}^{d}|ii\rangle.$$

We also introduce another parametrisation as:

$$\rho_I^d(f) = \frac{1-f}{d^2-1}\mathbb{1} + \frac{d^2f-1}{d^2-1}|\psi_d\rangle\langle\psi_d|, \tag{2.22}$$

$$f = \langle\psi_d|\rho_I^d(f)|\psi_d\rangle\,, \qquad \frac{1}{d^2} \leq f \leq 1, \qquad |\psi_d\rangle = \frac{1}{\sqrt{d}}\sum_i^d |ii\rangle\,,$$

being $|\psi_d\rangle$ the maximally entangled state of dimension d. The relation between these two parameters is $f = p + \frac{(1-p)}{d^2}$. We now consider the properties of entanglement, locality, and fully entanglement fraction for these Isotropic states in terms of the parameters p and f [64]:

$$p > \frac{1}{d+1}, \qquad \longleftrightarrow \qquad f > \frac{1}{d}, \qquad \longleftrightarrow \qquad \text{Entangled}, \tag{2.23}$$

$$p \leq \frac{-1+\sum_{k=1}^{d}\frac{1}{k}}{d-1} \longleftrightarrow f \leq \frac{-1}{d} + \frac{(d+1)}{d^2}\sum_{k=1}^{d}\frac{1}{k} \longrightarrow \text{Local (Projective)}, \tag{2.24}$$

$$p \leq \frac{(3d-1)(d-1)^{(d-1)}}{(d+1)d^d} \longleftrightarrow f \leq \frac{(d-1)^d(3d-1)+d^d}{d^{d+2}} \longrightarrow \quad \text{Local (General)}. \tag{2.25}$$

Let us now address the fully entangled fraction FEF (Eq. 2.8), which for these Isotropic states reads: [62]

$$f_d[\rho_I^d(f)] = \begin{cases} \frac{1-f}{d^2-1} & if\ 0 \leq f < \frac{1}{d^2}, \\ \\ f & if\ \frac{1}{d^2} \leq f \leq 1. \end{cases} \tag{2.26}$$

We have already seen all entangled Isotropic states are entangled if and only if $f > 1/d$ (Eq. 2.23), and therefore (because of Eq. 2.9), all of them are useful for teleportation. Let us now come back to consider some entangled local two-qubit states.

2.2.4 Hirsch States

We now analyse the two-qubit states studied by Hirsch, Quintino, Bowles and Brunner [65] which for simplicity we will call Hirsch states and are defined as:

$$\rho_H(p,q,\sigma) = \ p\,\psi^- + [1-p]\left[q\sigma + (1-q)\frac{\mathbb{1}}{2}\right] \otimes \frac{\mathbb{1}}{2}, \tag{2.27}$$

with $0 \leq p \leq 1$, $0 \leq q \leq 1$, $\psi^- := |\psi^-\rangle\langle\psi^-|\,, |\psi^-\rangle := \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$, and σ an arbitrary one-qubit state. These states can also be thought as a generalisation of the two-qubit WI states (Eq. 2.14); in fact, we recover them by putting $q = 0$. Putting $\sigma = |0\rangle\langle 0|$, and $q = 1$, these states become

$$\rho_H(p) = p\,\psi^- + (1-p)\,|0\rangle\langle 0| \otimes \frac{\mathbb{1}}{2}, \tag{2.28}$$

which are entangled for all $0 \leq p$. Interestingly, the authors of Ref. [65] were able to prove locality (projective) for $p \leq 1/2$. Even more interestingly (though we are not going to cover these subjects here) they derived a method to transform local (projective) states into local (general) states [65].

2.2.5 Game States

We now consider another two-qubit generalisation of the WI states (Eq. 2.14) which we will call game states (for reasons we are about to explain) and are defined as:

$$\rho_G(p,\delta) = p\,\psi^{+i}(\delta) + \frac{(1-p)}{4}\mathbb{1}, \tag{2.29}$$

with $0 \leq p \leq 1$, $\psi^{+i}(\delta) := \left|\psi^{+i}(\delta)\right\rangle \left\langle\psi^{+i}(\delta)\right|$, with $\left|\psi^{+i}(\delta)\right\rangle = \cos\frac{\delta}{2}\left|00\right\rangle + i\sin\frac{\delta}{2}\left|11\right\rangle$, $\delta \in [0, \frac{\pi}{2}]$. We are interested in these states for two reasons: First, there is a locality bound for the joint correlation (as defined in Eq. B.4), $p_L \approx 0.6009$ [69]; which means that these game states with $p \leq p_L$ are local (for the joint correlator). Actually, this locality bound holds for general two-qubit states of the form $\rho := p\rho' + \frac{(1-p)}{4}\mathbb{I}$, $0 \leq p \leq 1$, being ρ' an arbitrary two-qubit state [69]. Second, these states have been studied in games' strategies[2], and in particular, it has been proven that regardless of the value of p, whenever $\delta \geq \delta_2 := \sin^{-1}\left(\sqrt{\frac{2}{5}}\right) \sim 0.6847$, these states turn out to be useful for the quantum version of the so-called *prisoner-dilemma* game.

2.3 Final Considerations

We have already seen some measures for two-qubit states for the properties of entanglement, nonlocality and usefulness for teleportation, and we have also addressed the relationship between them. We have also already seen some bipartite entangled local states. There are a few multipartite generalisations of these LHV models (see a recent review [27]), and there has also been a recent breakthrough toward the numerical generation of LHV models for entangled local states [66, 67]. Having introduced these entangled local states, we have now gathered all the needed ingredients in order to address main subject of this work, the activation of nonlocality scenarios which we are going to address in the next chapter.

[2] Preprint titled: *Quantum locality in game strategy* by C. Susa, C. A. Melo, A. F. Ducuara, A. Barreiro and J. H. Reina.

Chapter 3

Activation of Nonlocality Scenarios

In this chapter, we address the activation of quantum nonlocality scenarios. We have already seen that whilst for pure states, entanglement and nonlocality are equivalent (by means of the Gisin theorem [46, 47, 48]), this is not the case for general mixed states (particular examples from the last chapter). This, in principle, would settle the question about the relationship between entanglement and nonlocality within their original definitions. However, in the mid nineties, Sandu Popescu was able to transform an entangled local Werner state (Eq. 2.15) into a nonlocal state by means of local filters [72], this particular phenomenon has been called *hidden nonlocality* [28]. This triggered similar results by means of other procedures, which were coined *activation-of-quantum nonlocality scenarios*. Let us address their relevance.

From a fundamental point of view, these phenomena raised again the question about the relationship between entanglement and what we could call new definitions (generalisations) of nonlocality inspired by these activation scenarios. The question is whether these generalisations may be equivalent to entanglement. We are going to see some instances for which the latter is indeed possible. Therefore, it is of interest to explore the aforementioned new relationships.

From a practical point of view, even though entanglement has been regarded as a resource for many tasks [29], it is also well known that there are protocols that explicitly require either nonlocality or the violation of certain Bell inequalities [28, 73]. Let us consider the following situation: suppose that an experimentalist is able to prepare entangled local states only; however, she needs to implement protocols that require nonlocality, therefore; she should address the activation of nonlocality scenarios.

We are going to focus on the activation of nonlocality scenarios based on the original *Local Filtering* (LF) procedure, and the so-called *Tensoring* (T) procedure, as well as combinations of them (there are more activation of nonlocality procedures [28], however, we are only going to focus on some of them, in particular, those which we can numerically approach). We now briefly describe these two general procedures.

Local Filtering (LF): S. Popescu in Ref. [72] took Werner entangled local states (Eq. 2.15), and after applying a *local filter*, an operation that could be thought as a projection onto a two-qubit system, namely, $P \otimes Q$ with $P := |0\rangle\langle 0| + |1\rangle\langle 1|$, and $Q := |0\rangle\langle 0| + |1\rangle\langle 1|$, he found that the final state violated the CHSH-inequality (Eq. 2.6), whenever $d > 5$. We can see a sketch of this scenario in Fig. 3.1.

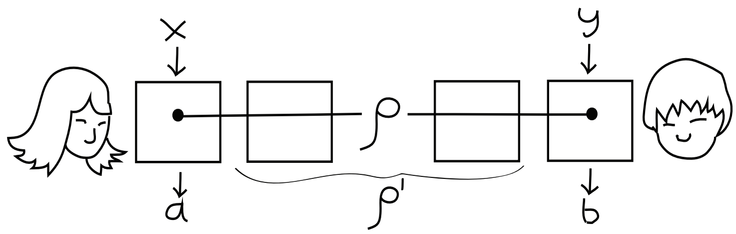

Figure 3.1: Activation of nonlocality through the *Local Filtering* procedure [72]. Experimentalists Alice and Bob receive a part of a bipartite quantum state ρ. They both first apply locals filters each, obtaining the new state ρ' with certain probability, if ρ' is the desired state, they then proceed with an standard Bell test. If they do not obtain the desired state, they repeat the experiment until they get so.

Soon after this result, N. Gisin [74] proposed another example, but this time with states that, even though it was not known whether they were local, at least they did not violate the CHSH-inequality. Recently, a complete characterisation of the hidden nonlocality for two-qubit states has been derived [75]. Let us now address the procedure known as tensoring.

Tensoring (T): Here, the question is: is it possible to find entangled local states ρ_1 and ρ_2 such that $\rho_1 \otimes \rho_2$ is entangled nonlocal? If $\rho_2 = \rho_1$, or in general, if $\rho\otimes^k$ is nonlocal, the phenomenon is called *superactivation of nonlocality*, similarly, we could also say that the state ρ is *k-copy nonlocal* or that we have activation through the *k-tensoring* procedure. We can see this scenario depicted in Fig. 3.2. It is also natural to think about the combination of tensoring with local filtering. Historically, the first tensoring procedure appeared as a combination of k-tensoring with LF [76, 77]. Then, general tensoring and LF arose [78, 79]. The activation through tensoring alone (without LF) was reported in 2008 [80], and three years later (2011) superactivation was put forward [81].

Remark: We are going to see that these procedures are interesting to be applied only on entangled local states. This is because, if we apply either of these two procedures on separable states, they would generate only separable states again (and therefore local states). If we want to activate nonlocality on separa-

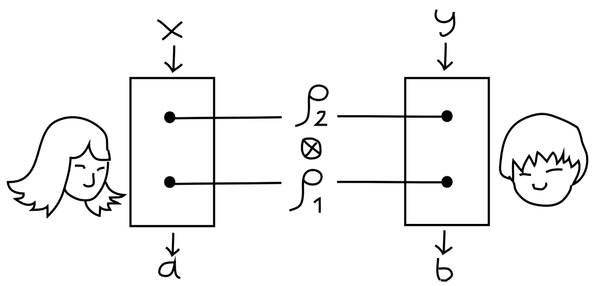

Figure 3.2: Activation of nonlocality through the *Tensoring* procedure. Experimentalists Alice and Bob prepare the state $\rho_1 \otimes \rho_2$. They then make a standard Bell test.

ble states, we would first have to consider procedures for entanglement creation before thinking in nonlocality activation [29]. We then focus our efforts on entangled local states, and in particular, the bipartite states addressed in the previous chapter.

This chapter is organised as follows. First, we consider the tensoring procedure alone, in particular, the so-called *k-tensoring* procedure, which works for two-qudit states [81, 82], and will also let us consider a two-qubit measure in order to look for *k-copy Nonlocality* (kNL). Second, we consider hidden nonlocality, in particular, the full characterisation for two-qubit states [75], which would let us define the two-qubit measure of *Hidden CHSH-inequality Violation* (HCHSH). Third, we consider a composed scenario where we first apply a local filter and then k-tensoring [83]. Here, we will consider the two-qubit measure of *Hidden k-copy Nonlocality* (HkNL). Fourth, we consider the scenario where we first apply general tensoring, and then local filtering [78, 79]. This procedure works for general multipartite systems, though we are only going to be interested into bipartite ones, for which we will consider the two-qudit measure of *Tensor/Filter activation of Hidden Nonlocality* (TFHN). We have chosen these particular scenarios because they can be numerically approached, in particular, for arbitrary two-qubit states (bipartite states in the case of TFHN). Let us now start by addressing the k-tensoring procedure.

3.1 k-Tensoring (kT)

In this section, we focus on the case of the activation of nonlocality through tensor product only; in particular, when the state is capable of activating nonlocality by itself, i. e., superactivating [81, 82]. We start by considering the main theorem regarding this scenario which will let us consider a measure to look for k-copy nonlocality of general two-qubit states. We then address the extraction of the *superactivation tensoring factor* (k).

3.1.1 Main Theorem

Given $\rho \in D(\mathbb{C}^d \otimes \mathbb{C}^d)$, we have already addressed the fully entangled fraction $f_d(\rho)$ given by Eq. 2.8 which determines the usefulness of ρ for the teleportation protocol (Eq. 2.9). The main theorem regarding k-copy nonlocality establishes the second implication of the following chain of relationships:

$$f_d(\rho) > \frac{1}{d} \overset{[59]}{\Longleftrightarrow} \begin{array}{c}\rho \text{ is useful}\\ \text{for}\\ \text{teleportation}\end{array} \overset{[81,\ 82]}{\Longrightarrow} \begin{array}{c}\rho \text{ is}\\ k\text{-copy}\\ \text{nonlocal}\end{array} \tag{3.1}$$

In particular, for a two-qubit state $\rho \in D(\mathbb{C}^2 \otimes \mathbb{C}^2)$, we have the range $0 < f(\rho) < 1$, and that ρ is useful for teleportation if and only if $f > 1/2$. Therefore, ρ is k-copy nonlocal for the same region (Eq. 3.1). If we now combine this result (Eq. 3.1) with the one we already had for two-qubit states in relation with the CHSH-inequality violation (Eq. 2.13), we now have the chain of relationships:

$$\begin{array}{c}\rho \text{ violates}\\ \text{the CHSH}\\ \text{inequality}\end{array} \overset{[61]}{\Longrightarrow} \begin{array}{c}\rho \text{ is useful}\\ \text{for}\\ \text{teleportation}\end{array} \overset{[81,\ 82]}{\Longrightarrow} \begin{array}{c}\rho \text{ is}\\ \text{k-copy}\\ \text{nonlocal}\end{array} \overset{[81,\ 82]}{\Longrightarrow} \begin{array}{c}\rho\\ \text{is}\\ \text{Entangled}\end{array} \tag{3.2}$$

We had already remarked that usefulness for teleportation does not cover all the entangled states set, i. e., there exist entangled states not useful for teleportation [61]. However, with the introduction of k-copy nonlocality lying in between usefulness for teleportation and entanglement, it is natural to wonder about the following two instances; first, regarding the existence of an entangled *never* k-copy nonlocal state, and second, regarding the existence of a k-copy nonlocal, not useful to teleportation state. Both instances are, up to our knowledge, currently open questions.

Since usefulness for teleportation is a property which is possible to numerically search for, by means of the measure UfT (Eq. 2.12), it would consequently also allows us to enquire about k-copy nonlocality, in a necessary, though not sufficient fashion.

It turns out that from the second implication of Eq. 3.1, it is also possible to extract the k-value necessary for the superactivation [81, 82]. This k value is a minimum, since it is only necessary to achieve a nonlocal state once, in order to activate nonlocality through k-tensoring [48]. We call this minimum k-value the *superactivation tensoring factor*, which we now show how to calculate.

3.1.2 The Superactivation Tensoring Factor $k(d, f_d)$:

By means of the above-mentioned theorem (second implication in Eq. 3.1), it is possible to extract the explicit value of k, in terms of the dimension d of the two-qudit states, and their fully entangled fraction f_d. From the second implication in Eq. 3.1, those k's must satisfy the following inequality:

$$\left[\frac{C'}{C(\ln d)^2}\right]\frac{(f_d d)^k}{k^2} > 1, \tag{3.3}$$

with constants $C := e^4$ and $C' := 4$. Since in Eq. 3.3, we have a function to the power of k over a quadratic polynomial function of k, we have that the activation goes from $f > 1/d$. In Fig. 3.3 (a), we numerically see the behaviour of $k(d, f_d)$. For most of the $d - f$ region, $k < 10$ is enough (as it is depicted by the white curve in the left); just when f_d is close to the boundary $\frac{1}{d}$, it becomes asymptotically more difficult to superactivate it. In Fig. 3.3 (b), we see cuts for $d = 2, 3, 4$ and 5.

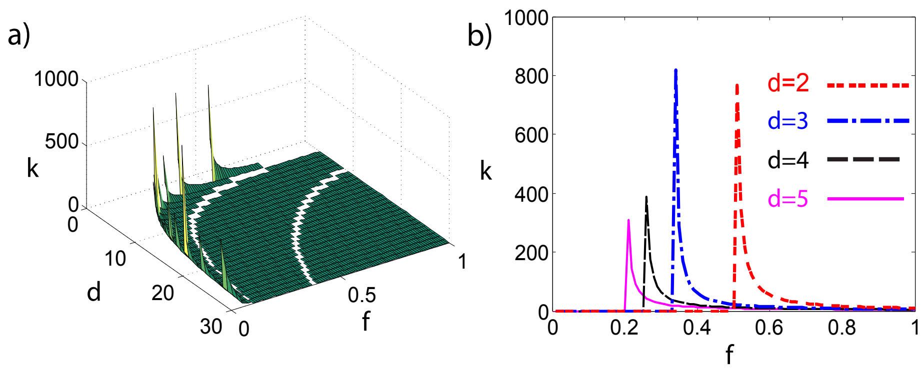

Figure 3.3: **(a)** Superactivation tensoring factor $k(d, f_d)$ in terms of the fully entangled fraction (FEF) (Eq. 2.8) $0 \leq f_d \leq 1$ and the dimension $d \geq 2$ of the two-qudit system. The white curves represent the threshold for $k < 10$ and $k < 3$ from left to right respectively. **(b)** Transversal cuts of Fig 3 (a) for $d = 2, 3, 4$ and 5.

We have seen that we can look for k-copy nonlocality by means of the usefulness for teleportation measure. In order to do so, we first calculate the value of f_d as in Eq. 2.8, and we then come to Fig. 3.3, in order to extract the k-value necessary for the superactivation. Let us now move on to the local filtering scenario.

3.2 Local Filtering (LF)

In this section, we address the complete characterisation of hidden nonlocality for two-qubit systems, from which we define the measure of *Hidden CHSH-inequality violation* for general two-qubit states.

3.2.1 Hidden Nonlocality for Two Qubits (HCHSH)

A complete characterisation of hidden nonlocality for two-qubit systems has been recently derived [75]. Given $\rho \in D(\mathbb{C}^2 \otimes \mathbb{C}^2)$, ρ possesses hidden nonlocality (or ρ can be transformed into a CHSH-violating state ρ' by means of local filters) if and only if: $\lambda_\rho^1 + \lambda_\rho^2 > \lambda_\rho^0$, where the λ_ρ^i's are the eigenvalues of the matrix $C_\rho := \eta T_\rho \eta T_\rho^T$ organised in decreasing order, with $\eta := \mathrm{diag}(1, -1, -1, -1)$

and $T_\rho := [t_{nm}]$ a matrix with elements $t_{nm} := \mathrm{Tr}[(\sigma_m \otimes \sigma_n)\rho]$, as defined for the CHSH-inequality violation measure (Eq. 2.6). The CHSH-inequality violation of that new state ρ' is given by the Horodecki criterion (Eq. 1.28) as $HM(\rho) := M(\rho') = \frac{\lambda_\rho^1+\lambda_\rho^2}{\lambda_\rho^0}$. Instead of $HM(\rho)$, we could work with

$$HB(\rho) := \sqrt{\max\{0, HM(\rho) - 1\}}, \tag{3.4}$$

because this function will let us make a direct comparison with the CHSH-inequality violation (Eq. 2.6), and the EoF (Eq. 2.6) we have already introduced. Then, in order to analyse hidden nonlocality, we will plot:

$$HCHSH(\rho) = h\left(\frac{1+\sqrt{1-HB(\rho)^2}}{2}\right), \tag{3.5}$$

being h the binary entropy (Eq. 2.3). This generalisation of nonlocality does not cover the whole set of entangled states, and therefore, it is not equivalent to entanglement [75]. We now consider some combinations of the already mentioned scenarios. In a first instance, we consider the scenario where we first apply a local filter, and then k-tensoring.

3.3 Hidden k-copy Nonlocality (HkNL)

In this section, we are going to combine the two procedures we have already introduced: tensoring and local filtering. Particularly, by first considering local filtering, and only then, k-tensoring. We will define the measure of *Hidden Usefulness for Teleportation* (HUfT) for general two-qubit states, which will let us look for hidden k-copy nonlocality. Finally, we discuss the behaviour of the EoF (Eq. 2.2) after local filters, which has been coined the *Maximum Extractable Entanglement of Formation* (MEEoF), showing that this MEEoF turns out to be equivalent to the HUfT. Let us then start by defining the HUfT.

3.3.1 Hidden Usefulness for Teleportation (HUfT $\rightarrow$ HkNL)

Given $\rho \in D(\mathbb{C}^2 \otimes \mathbb{C}^2)$, we have already seen that if ρ is useful for teleportation, then it is k-copy nonlocal (Eq. 3.1), i.e., ρ admits superactivation of nonlocality [81, 82], and that usefulness for teleportation can be numerically tested by considering the UfT measure (Eq. 2.12). This UfT measure is written in terms of the D function (Eq. 2.11) which can consequently be written in terms of whether the $f(\rho)$ function (Eq. 2.8) or the $N(\rho)$ function (Eq. 2.10). As in the case of hidden nonlocality (Eq. 3.5), we now wonder what happens with these functions when trying to maximise them with local filters. It turns out that the filters that maximise the CHSH-inequality violation, also maximise these teleportation-related functions [83]. The D function of the new state ρ' can be written as $HD(\rho) := D(\rho') = \max\{0, (HN(\rho) - 1)/2\}$ with

$HN = -1 + 4(1/2(1 + HC(\rho)))$, where the HC function is the maximised concurrence which reads

$$HC(\rho) = \frac{C(\rho)}{\lambda_4 + \lambda_3 + \lambda_2 + \lambda_1},$$

being C the concurrence (Eq. 2.4), and the λ_i's referring to the square root of the eigenvalues belonging to the auxiliary operator $\rho\tilde{\rho}$ arranged in decreasing order, and $\tilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)$, as defined for the EoF (Eq. 2.2). The HD function could have also be written in terms of $Hf := \frac{1}{2}[1 + HC(\rho)]$. However, for our purposes, in terms of HN is enough. If we replace HN into HD we obtain $HD = HC$, a result that we are going to address later. In order to make a direct comparison with the measures of EoF (Eq. 2.2), CHSH (Eq. 2.6), UfT (Eq. 2.12), and HCHSH (Eq. 3.6), we now define the *Hidden Usefulness for Teleportation* (HUfT) measure as:

$$HUfT(\rho) = h\left(\frac{1}{2}\left[1 + \sqrt{1 - HD(\rho)^2}\right]\right), \tag{3.6}$$

being h the binary entropy (Eq. 2.3). This measure is characterising the scenario where we first apply a local filter, and then considering the k-tensoring. We sketch this procedure in Fig. 3.4.

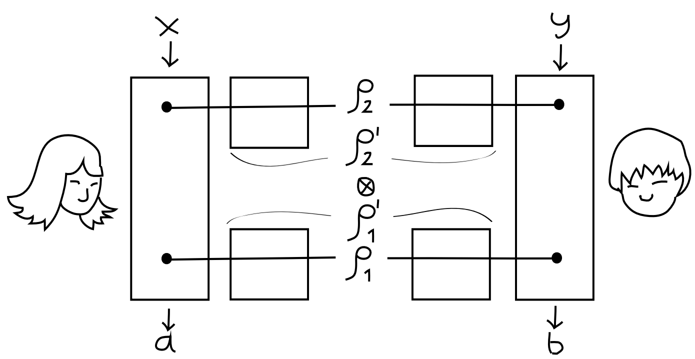

Figure 3.4: Activation of nonlocality through the combination of local filtering and k-tensoring. Experimentalist Alice and Bob start by preparing states ρ_1 and ρ_2. They then apply local filters in order to get the states ρ_1' and ρ_2'. They then consider the tensor product $\rho_1' \otimes \rho_2'$, and proceed with a standard Bell test.

We have already seen the behaviours of the CHSH (Eq. 2.6) and UfT (Eq. 2.12) under local filters, which led us to define their hidden counterparts HCHSH (Eq. 2.6) and HUfT (Eq. 3.6). It is then also natural to consider the maximisation of the EoF (Eq. 2.2) through local filters. This new function has been called the maximum extractable entanglement.

3.3.2 On the Maximum Extractable Entanglement

We now address the maximisation of the EoF (Eq. 2.2) under local filters as we have already done for the CHSH and UfT. One might be tempted to term this new function as *Hidden EoF*; however, one can check that local filters cannot create entanglement (they cannot transform a separable state into an entangled one), but rather they could increase the amount of it [83]. Because of this reason, this new measure has received the name of *Maximum Extractable Entanglement of Formation* (MEEoF), instead of Hidden EoF. It turned out that the local filters that maximise the CHSH-inequality violation and the usefulness for teleportation, also maximise the entanglement of formation [83, 84]. Given $\rho \in D(\mathbb{C}^2 \otimes \mathbb{C}^2)$, its MEEoF is given by:

$$MEEoF(\rho) = h\left(\frac{1}{2}\left[1 + \sqrt{1 - HC(\rho)^2}\right]\right), \tag{3.7}$$

being HC the maximum extractable concurrence. We have already seen that $HD = HC$, and therefore by directly comparing Eq. 3.6 and Eq. 3.7, we obtain

$$HUfT(\rho) = MEEoF(\rho). \tag{3.8}$$

From this result, we can see that every two-qubit entangled state is hidden k-copy nonlocal, and therefore making of *hidden k-copy nonlocality* (unlike hidden nonlocality alone) an equivalent property to entanglement.

We have considered some activation of nonlocality scenarios, and defined some two-qubit measures to look for them. First, the usefulness for teleportation (Eq. 2.12) let us explore for k-copy nonlocality. Second, hidden-CHSH nonlocality (Eq. 3.5). Third, we have considered the hidden k-copy nonlocality (Eq. 3.6). However, from another perspective, we could say that we have analysed the three properties from last chapter (EoF, UfT, and CHSH) under local filtering, though maximum extractable entanglement turned out to be equal to hidden k-copy nonlocality, so effectively considering two new properties (HUfT=MEEoF, HCHSH).

In the hidden k-copy nonlocality scenario (Eq. 3.6), we have performed first the local filter and only then the k-tensoring procedure. It is natural to wonder about the opposite procedure, namely, when first considering tensoring and only then the local filter. Let us address this scenario.

3.4 Tensor/Filter Activation of Hidden Nonlocality (TFHN)

In this section, we present the activation scenario as the result of first considering general tensoring (not necessarily with itself) and then local filtering. We start by defining the set of states that do not violate the CHSH inequality even after any possible local filter, in other words, states that are not hidden nonlocal. Second, we present the main theorem regarding this scenario. Third, we present a numerical approach that will let us define the measure of *Tensor/Filter activation of Hidden Nonlocality* (TFHN) which works for general multipartite systems, though we are only going to focus on bipartite systems.

3.4.1 P_{CHSH} & P'_{CHSH} sets

Let us start by defining the set $P_{\text{CHSH}} \subset D(\mathbb{H})$ formed by the states that do not violate the CHSH-inequality (Eq. 1.25) even after any possible local filter (onto two qubits) [78]. Formally, $\rho \in P_{\text{CHSH}} \iff \forall \Omega : D(\mathbb{H}) \longrightarrow D\left(\mathbb{C}^2 \otimes \mathbb{C}^2\right)$, $\Omega(\rho)$ does not violate the CHSH inequality (Eq. 1.25), being Ω a *Local Filter* which should be understood as a *separable map* of the form $\Omega(\rho) = \sum_i (A_i \otimes B_i)\rho(A_i \otimes B_i)^\dagger$, with A_i, B_i Kraus operators [78]. This P_{CHSH} set has been characterised for two-qubit systems [78, 75].

Let us now define the set $P'_{\text{CHSH}} \subset D(\mathbb{H})$ as the set of states that do not violate the CHSH-inequality for any local filter Ω, and not even after first k-tensoring the state with itself and then applying a local filter. If a state ρ does not belong to P'_{CHSH}, it is said that ρ *asymptotically violate* the CHSH-inequality [77]. By the definition of P'_{CHSH}, we have the relationship $P'_{\text{CHSH}} \subset P_{\text{CHSH}}$. There is an equivalence between this asymptotically violation of CHSH [77] and the property called *distillability* [85], which will let us search numerically for states in P_{CHSH}. For the moment, let us address the main theorem regarding the activation of the states in P_{CHSH}.

3.4.2 Main Theorem

The main theorem regarding the activation of nonlocality for states in P_{CHSH} works for general multipartite systems [79]. However, since we are focused on two-qubit systems, we highlight this main theorem for bipartite systems [78]. The main theorem establishes the following:

$$\begin{array}{ccc} \rho \in D\left(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}\right) & & \exists \tau_\rho \in D\left[\bigotimes_{i=1}^{2} \left(\mathbb{C}^{d_i} \otimes \mathbb{C}^2\right)\right] \\ \text{is} & \Longleftrightarrow & \text{s.t. } \tau_\rho \in P_{\text{CHSH}}, \\ \text{entangled} & & \rho \otimes \tau_\rho \notin P_{\text{CHSH}}. \end{array} \tag{3.9}$$

On the one hand, since the result holds for any entangled state ρ, it also applies for entangled states $\rho \in P_{\text{CHSH}}$. In this case, the theorem guarantees the existence of another state $\tau_\rho \in P_{\text{CHSH}}$ such that $\rho \otimes \tau_\rho \notin P_{\text{CHSH}}$. In other words, for this new state $\rho \otimes \tau_\rho$, there exist an integer k and a local filter lf (onto two-qubit states), such that the two-qubit state $lf[(\rho \otimes \tau_\rho)^k]$ is a CHSH-inequality violating state. Because of this reason, we address this procedure as *Tensor/Filter activation of Hidden Nonlocality*. Therefore, this theorem is characterising the activation scenario depicted in Fig. 3.5.

On the other hand, the theorem provides the local filter (lf) in question, and even though it also guarantees the existence of the ancillary matrix τ_ρ, it

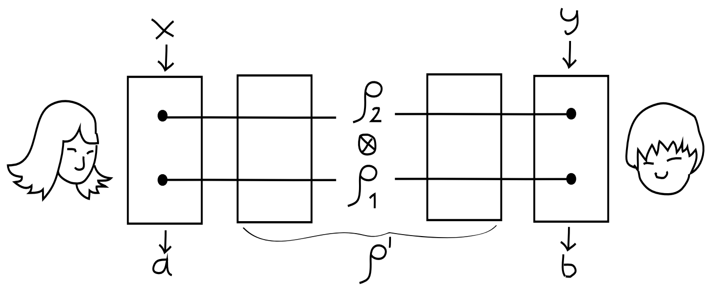

Figure 3.5: Activation of nonlocality through the combination of first general tensoring and then local filtering. Experimentalists Alice and Bob prepare the state $\rho_1 \otimes \rho_2$. They then apply a particular local filter that takes the bipartite state to a two-qubit state. They then proceed with an standard Bell test.

does not explicitly tell us how to calculate it. In this regard, let us address a numerical approach to try to find this auxiliary matrix.

3.4.3 Numerical Approach

Given $\rho \in D(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2})$ an entangled state, we would like to find the aforementioned ancillary density matrix τ_ρ. In order to do so, we follow the approach reported in [79]. From the proof of the main theorem [79], it is enough to look for a density matrix $\tau_\rho \in D\left[(\mathbb{C}^{d_A} \otimes \mathbb{C}^2) \otimes (\mathbb{C}^{d_B} \otimes \mathbb{C}^2)\right]$ with the following characteristics. First, in order to have the condition $\rho \otimes \tau_\rho \notin P_{\mathrm{CHSH}}$, the τ_ρ matrix must satisfy the inequality

$$\mathrm{Tr}\left[\tau_\rho \left(\rho^{\mathrm{T}} \otimes \mathrm{H}_{\pi/4}\right)\right] < 0, \tag{3.10}$$

$$\text{with:} \quad H_\theta := \mathbb{1} \otimes \mathbb{1} - \cos\theta \sigma_{\mathrm{x}} \otimes \sigma_{\mathrm{x}} - \sin\theta \sigma_{\mathrm{z}} \otimes \sigma_{\mathrm{z}}. \tag{3.11}$$

Second, we have to check that $\tau_\rho \in P_{\mathrm{CHSH}}$ which, in principle, is not numerically possible (with the exception of two-qubit systems [75]). Fortunately, we have some results that partially allow us to search for this condition. First, we have already pointed out that $P'_{\mathrm{CHSH}} \subset P_{\mathrm{CHSH}}$. Second, P'_{CHSH} is equivalent to *Bound Entanglement* (BE) [77]. Third, we have that the Positive Partial Transpose (PPT) (Appendix B.2.1) are bound entangled states [51, 86]. Therefore, putting all of this together, we have the following hierarchy of properties:

$$PPT \Longrightarrow BE \Longleftrightarrow P'_{\mathrm{CHSH}} \Longrightarrow P_{\mathrm{CHSH}}.$$

Hence, we could look for a matrix τ_ρ with a positive partial transpose (PPT) respect to the first subsystem $\mathbb{C}^{d_A} \otimes \mathbb{C}^2$ in order to guarantee that it belongs to P_{CHSH}. We then could consider the following minimisation problem:

$$\underset{\text{over}\{\rho_\tau\}}{\text{minimise:}} \quad \sigma(\rho) := \mathrm{Tr}\left[\tau_\rho \left(\rho^{T} \otimes \mathrm{H}_{\pi/4}\right)\right], \tag{3.12}$$

$$\text{and constraints:} \qquad \tau_\rho \geq 0, \qquad \tau_\rho^{T_1} \geq 0.$$

A problem with these characteristics is a *Semidefinite Programming* (SDP) problem that is numerically solvable [79, 87]. We implemented it by using MATLAB [88] with the YALMIP toolbox [89], and the SeDuMi solver [90]. In what follows, we define a measure for this activation scenario.

3.4.4 Bipartite Measure

Given $\rho \in D(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2})$, the main theorem (Eq. 3.9) provides the necessary filter (lf) for the activation, and even though it does not explicitly provide the ancillary matrix τ_ρ as a function of ρ, we can numerically search for this matrix by means of the numerical procedure we have just addressed (Eq. 3.12). We want to define a measure for this scenario. In order to do so, let us write the final two-qubit state state as $\rho' := (U \otimes V)(\tau_\rho \otimes \rho)(U \otimes V)$ being U, V the local filters used in the protocol, explicitly given in Ref [78]. We can now apply the Horodecki criterion (Eq. 3.6) for the CHSH-inequality violation which reads $M'(\rho) := M(\rho') = {\mu'_1}^2 + {\mu'_2}^2 > 1$. Because of the form of the filters U, V [78], these two biggest eigenvalues can be calculated as:

$$\begin{aligned} \mu'_1 &= \mathrm{Tr}[\rho'(\sigma_{\mathrm{x}} \otimes \sigma_{\mathrm{x}})] = \gamma \mathrm{Tr}\left\{\tau_\rho \left[\rho^{\mathrm{T}} \otimes (\sigma_{\mathrm{x}} \otimes \sigma_{\mathrm{x}})\right]\right\}, \\ \mu'_2 &= \mathrm{Tr}[\rho'(\sigma_{\mathrm{z}} \otimes \sigma_{\mathrm{z}})] = \gamma \mathrm{Tr}\left\{\tau_\rho \left[\rho^{\mathrm{T}} \otimes (\sigma_{\mathrm{z}} \otimes \sigma_{\mathrm{z}})\right]\right\}, \\ \gamma &:= 1/\mathrm{Tr}\left\{\tau_\rho \left[\rho^{T} \otimes (\mathbb{1} \otimes \mathbb{1})\right]\right\}, \end{aligned}$$

being σ_x, σ_z the Pauli matrices. Instead of M', we could consider the function:

$$F(\rho) = \sqrt{\max\{0, M'(\rho) - 1\}}. \tag{3.13}$$

In order to make a direct comparison of this activation scenario with all of the measures we have already introduced, we define the *Tensor/Filter activation of Hidden Nonlocality* (TFHN) measure as:

$$\mathrm{TFHN}(\rho) = h\left(\frac{1 + \sqrt{1 - F(\rho)^2}}{2}\right), \tag{3.14}$$

being h the binary entropy (Eq. 2.3). This activation scenario works for general bipartite systems, and furthermore, for general multipartite systems [79].

3.5 Final Considerations

We have already seen that entanglement and nonlocality are not equivalent for general mixed states (particular examples of entangled local states from the previous chapter). However, we could now ask the same with the generalisations of nonlocality we have considered here, for which we have the following results. First, hidden nonlocality is not equivalent to entanglement. Second, hidden

k-copy nonlocality and tensor/filter activation of hidden nonlocality are, on the other hand, equivalent to entanglement. Third, k-copy nonlocality is not known whether it is equivalent to entanglement or not. Finally, putting all of these results together, and taking into account a previous diagram from the previous chapter (Fig. 2.1), we have the following diagram (Fig. 3.6).

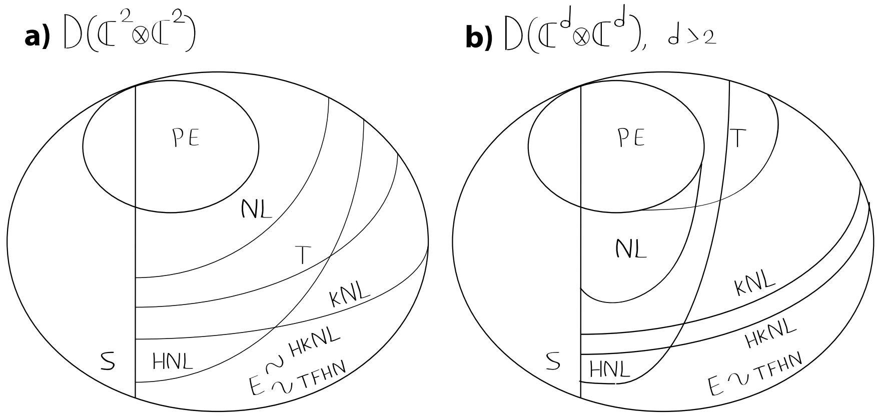

Figure 3.6: Venn-like representation of several results concerning some generalisations of nonlocality for: **a)** two-qubit states, and **b)** two-qudit ($d > 2$) states. This diagram should be seen as an extension of the diagram in Fig. 2.1 (where the definition of E, S, PE, NL, T and their relation with each other is established). We now focus on generalisations of nonlocality and their relationship with entanglement. First, hidden nonlocality HNL (Eq. 3.5) is not equivalent to entanglement. Second, hidden k-copy nonlocality HkNL (Eq. 3.6) is equivalent to entanglement for two-qubit **a)** states. Third, tensor/filter activation of hidden nonlocality TFHN (Eq. 3.14) is equivalent to entanglement for two-qubit **a)** and two-qudit ($d > 2$) states **b)** and furthermore, for multipartite systems. Fourth, k-copy nonlocality kNL (Eq. 2.12), for which it is not known whether it covers the whole set of entangled states.

To wrap up, from a practical point of view, we have defined measures for these generalisations of nonlocality for general two-qubit systems, (bipartite systems for the case of TFHN (Eq. 3.14). In comparison with the previous chapter, in addition of the EoF (Eq. 2.2), UfT (Eq. 2.12), and CHSH (Eq. 2.6), we now have three extra properties to calculate, namely, HCHSH (Eq. 3.5), HUfT (Eq. 3.6), and TFHN (Eq. 3.14), making a total of six non-locality related properties up to study for general two-qubit systems. We now point out a couple of things: First, from a numerical point of view, all of these measures are straightforward to calculate, with the exception of the TFHN (Eq. 3.14), which requires an extra effort, since it involves a semidefinite programming minimisation. Second, for the activation scenarios that involve local filtering, we have addressed the way that certain measures get transformed when transforming the state through the local filters in question, rather than explicitly calculating the filters, obtaining the state, and then calculating again the measure (with the exception of TFHN for which we have the particular filter (lf) and the ancillary matrix, and therefore the final state). In this regard, it should be said that there exist general procedures for explicitly calculating the filters in question and therefore, the final state [91, 92]. We have not implemented these procedures, because it was not our main interest. That being said, in the next chapter, we are going to study these properties for particular states of interest.

Chapter 4

Activation of Nonlocality for Some Bipartite Entangled States

In this chapter, we shall use the formalism and tools already described in the previous chapters, in order to analyse the quantum nonlocality-related properties that we depict in Fig. 4.1, for some particular bipartite entangled states of interest.

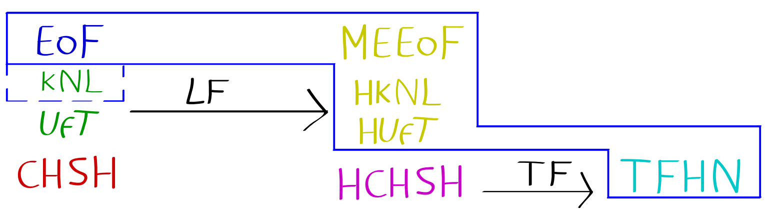

Figure 4.1: Quantum nonlocality-related properties. In the first column we present the properties of: Entanglement of Formation (EoF Eq. 2.2), CHSH-inequality Violation (CHSH Eq. 2.6), and Usefulness for Teleportation (UfT Eq. 2.12), which allows us to look for k-copy Nonlocality (kNL Eq. 3.1). In the second column, we present these very same properties after they have gone through the local filter that maximises them (which turns out to be the same for the three properties [83, 84]), namely; Maximum Extractable Entanglement of Formation (MEEoF Eq. 3.7) which becomes equivalent to the Hidden Usefulness for Teleportation (HUfTEq. 3.6), and Hidden CHSH-Nonlocality (HCHSH Eq. 3.5). In the third column, we present the Tensor/Filter activation of Hidden Nonlocality (TFHN Eq. 3.14). The properties inside the blue frame are equivalent to entanglement, whilst for kNL it is not known.

We start by analysing these properties (Fig. 4.1) for the Werner-Isotropic states. We then report TFHN activation regions for Werner states, reproducing the values reported in [79]. We then report TFHN activation regions for Isotropic states. We then come back to two-qubit systems by analysing the Hirsch, and Game states. Finally, we consider some not CHSH-inequality violating states that arise from the dynamics of an open quantum system. We end up with some final considerations.

Remark on the notation: We will deal with quantum states in terms of a parameter p, i. e., $\rho = \rho(p)$. In these cases, we will consider the following notation: if $p > p_E$, then the state is entangled; if $p > p_{NL}$, the state is nonlocal; if $p > p_{UfT}$, the state is useful for teleportation (an therefore k-copy nonlocal); if $p > p_{HN}$, the state contains hidden nonlocality; if $p > p_{TFHN}$, the minimisation procedure described in the previous section (Eq. 3.14) has found an ancillary state that helps to activate the initial state by means of TFHN. Finally, if $p \leq p_L$, the state is local.

4.1 Werner-Isotropic (WI) States

We first study the Werner-Isotropic states [26], which we first introduced in Eq. B.5, and then further addressed in Eq. 2.14:

$$\rho_{WI}(p) = p\,\psi^- + \frac{(1-p)}{4}\mathbb{1}, \tag{4.1}$$

with $0 \leq p \leq 1$ and $\psi^- := |\psi^-\rangle\langle\psi^-|$, $|\psi^-\rangle := \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$. In Fig. 4.2, we plot the nonlocality-related properties discussed throughout the manuscript and summarised in Fig. 4.1, for these WI states. Regarding the TFHN activation scenario (Eq. 3.14), we extracted the ancillary matrix $\tau_\rho(p)$ for $p = 0.6569$, which turns out to be useful for all the $0.6569 < p < 1$ region, as we show in the $R(p)$ function plotted in Fig. 4.2. Therefore, $\tau_\rho \otimes \rho_{WI}(p)$ is hidden CHSH-nonlocal after the local filter from [78], with

$$\tau_\rho = \frac{1}{16}\sum_{i,j=0}^{3} R_{ij}\sigma_i \otimes \sigma_i \otimes \sigma_j \otimes \sigma_j, \tag{4.2}$$

$$R := \frac{1}{9}\begin{pmatrix} 9 & 3 & 3 & 3 \\ 1 & -1 & 3 & -1 \\ 1 & -1 & 3 & -1 \\ 1 & -1 & 3 & -1 \end{pmatrix}.$$

Remarks: First, even though the nonlocality limit obtained from the CHSH-inequality is $p_{NL} = \frac{1}{\sqrt{2}} \approx 0.7071$, there exist a better bound which reads $p_{NL} = 0.705428$ [71] as we already discussed in Fig. 2.3. Second, EoF coincides with UfT, because these WI states satisfy the criterion detailed in Ref. [82]. Third, CHSH-nonlocality coincides with hidden nonlocality because these WI states are already in Bell-diagonal form (see [93] for a detailed explanation). We see that

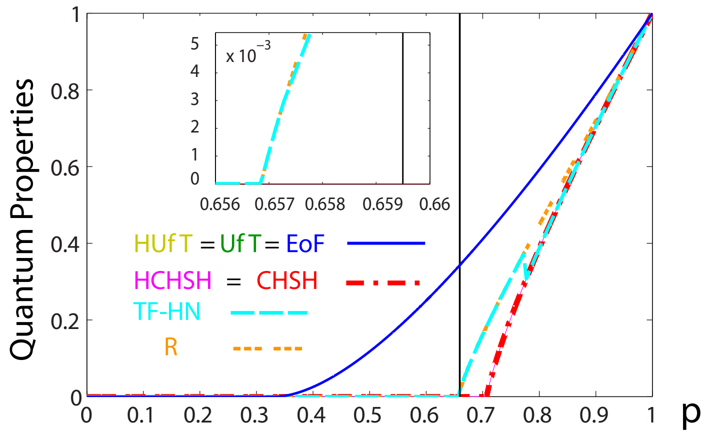

Figure 4.2: Quantum properties depicted in Fig. 4.1 for the WI states (Eq. 4.1). Entanglement of Formation (EoF, blue solid thick curve Eq. 2.2), which turns out to be equal to Usefulness for Teleportation (UfT, Eq. 2.12), and consequently also equal to Hidden Usefulness for Teleportation (HUfT, Eq. 3.6) which is always equal to the Maximum Extractable Entanglement (MEEoF, Eq. 3.7). Nonlocality (CHSH, red dashed-dotted curve Eq. 2.6), which turns out to be equal to Hidden CHSH-Nonlocality (HCHSH, Eq. 3.5). Tensor/Filter activation of Hidden Nonlocality (TFHN, cyan dashed curve Eq. 3.14). R (orange triply dashed curve), activation TFHN with the ancillary matrix (Eq. 4.2) obtained at $p = 0.6569$. Finally, Projective-Locality (black vertical solid line) at $p \leq p_L = 0.6595$ according to the best known bound derived in [69] and already discussed in Fig. 2.3. We remark that all entangled states are k-copy nonlocal, that HCHSH could *not* do better that CHSH, and that TFHN slightly improved HCHSH.

even though the states in the entangled local region $0.3333 < p < 0.6596$ do not present hidden nonlocality, they do present UfT and TFHN. To complete the results reported in Fig. 4.2, Table 4.1 gives the limit bounds of the nonlocality-related properties we have addressed for these WI states. Next, we consider the two-qudit generalisations of these WI states, the Werner and Isotropic states.

p_E	p_{SA}	p_{TFHN}	p_L	p_{HN}	p_{NL}
0.3333	0.3333	0.6569	0.6595	0.7054	0.7054

Table 4.1: Limit values of some quantum nonlocality-related properties for the WI states (Eq. 4.1), extracted form Fig. 4.2.

4.2 Werner States

We now address the so-called two-qudit Werner states [26] with the particular parametrisation given in Eq. 2.15:

$$\rho_W^d(p) = \frac{p}{d(d-1)} 2P_{\text{anti}} + \frac{(1-p)}{d^2}\mathbb{1}, \tag{4.3}$$

$$1 - \frac{2d}{d+1} \leq p \leq 1, \qquad P_{\text{anti}} := \frac{1}{2}\left(\mathbb{1} - \sum_{ij}^{d} |i\rangle\langle j| \otimes |j\rangle\langle i|\right).$$

In Fig. 4.3, we have plotted the activation TFHN by means of the minimisation of the $\sigma[\rho_W^d(p)]$ function, according to Eq. 3.12, and up to qudits of dimension $d = 6$. In Table 4.2, we report the limit values for the nonlocality-related properties discussed throughout the paper, also up to $d = 6$. The entanglement and locality limits come from Eq. 2.17 and Eq. 2.18, respectively [26]. The UfT limit comes from Eq. 2.20 [62], from which it turns out they (as soon as $d > 2$) are not useful for teleportation. Therefore, it is unknown if they are superactivable or not, so we have marked them in the table with an X. The activation column follows the plots shown in Fig. 4.3 and also reported in Ref. [79]. Finally, in the last column, we present hidden nonlocality from Ref. [72].

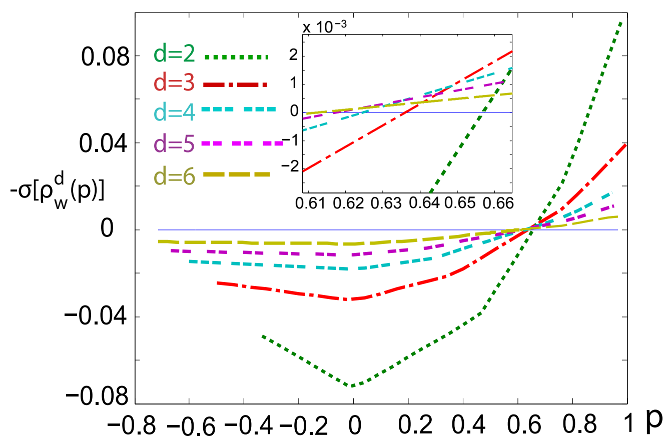

Figure 4.3: Activation of nonlocality in the scenario TFHN by means of the minimisation procedure (Eq. 3.12) of the function $\sigma[\rho_W^d(p)]$ vs parameter p from the two-qudit Werner states $\rho_W^d(p)$ (Eq. 4.3). We have the following notation: $d = 2$ (green dotted curve), $d = 3$ (red dashed-dotted curve), $d = 4$ (cyan triply dashed curve), $d = 5$ (magenta doubly dashed curve), $d = 6$ (yellow dashed curve). The inset shows a zoom of the functions close to the zero value. We reproduce the values reported in Ref. [79], which we explicitly report in Table 4.2.

d	p_E	p_{UfT}	p_{TFHN}	p_L	p_{HNL}
2	0.3333	0.3333	0.6569	0.6595	0.7054
3	0.2500	X	0.6360	0.6667	0.7630
4	0.2000	X	0.6247	0.7500	0.7837
5	0.1429	X	0.6174	0.8000	0.7944
6	0.1667	X	0.6127	0.8333	0.8009

Table 4.2: Limit values of some quantum nonlocality-related properties for the two-qudit Werner states (Eq. 4.3).

4.3 Isotropic States

We now consider the Isotropic states [64] with the particular parametrisation given by Eq. 2.21:

$$\rho_I^d(p) = p\,|\psi_d\rangle\langle\psi_d| + \frac{(1-p)}{d^2}\mathbb{1}, \tag{4.4}$$

with $0 \leq p \leq 1$, and $|\psi_d\rangle := \frac{1}{\sqrt{d}}\sum_{i=1}^{d}|ii\rangle$. In Fig. 4.4, we have plotted the activation TFHN by means of the minimisation of the $\sigma[\rho_I^d(p)]$ function, according to Eq. 3.14 and up to qudits of dimension $d = 6$. In Table 4.3 we report the limit values for the nonlocality-related properties discussed throughout the paper, also up to $d = 6$. The entanglement and locality limits are taken from Eq. 2.23 and Eq. 2.24, respectively [64]. The UfT limit is obtained from Eq. 2.26 [62], all of them are useful for teleportation, and therefore all them are superactivable. Activation TFHN values come from the plots shown in Fig. 4.4. Finally, nonlocality limit values are obtained from the Collins-Gisin–Linden-Massar-Popescu (CGLMP) inequalities [94].

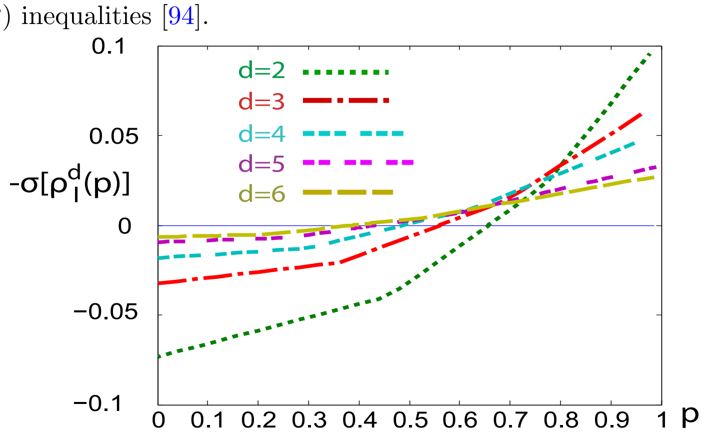

Figure 4.4: Activation TFHN for Isotropic states. Minimisation procedure Eq. 3.14 of the function $\sigma[\tau_I^d(p)]$ vs parameter p for $\tau_I^d(p)$ high dimensional Isotropic states Eq. 4.4. We have the following notation: $d = 2$ (green dotted curve), $d = 3$ (red dahsed-dotted curve), $d = 4$ (cyan triply dashed curve), $d = 5$ (magenta doubly dashed curve), $d = 6$ (yellow dashed curve). We report in Table 4.3 the values we are interested in.

d	p_E	p_{UfT}	p_{TFHN}	p_L	p_{NL}
2	0.3333	0.3333	0.6569	0.6595	0.7054
3	0.2500	0.2500	0.5606	0.4167	0.6961
4	0.2000	0.2000	0.4890	0.3611	0.6905
5	0.1429	0.1429	0.4337	0.3208	0.6872
6	0.1667	0.1667	0.3895	0.2900	0.6849

Table 4.3: Limit values of some quantum nonlocality-related properties for the two-qudit isotropic states (Eq. 4.4).

For the two-qudit Isotropic states, even though nonlocality-related properties have been studied, no bounds for activation TFHN have been reported yet. In Fig. 4.4, we have calculated these bounds, which we report in Table 4.3 column 4 (p_{TFHN}) in terms of the dimension d of the qudits. From these results one can observe the following: Unlike the Werner states' bounds, these Isotropic states' bounds do not cover the local states (given $d > 2$, values in Table 4.3 column 4 (p_{TFHN}) are still greater than values in Table 4.3 column 5 (p_L)). However, these new bounds now do extend the known nonlocality region (given $d > 2$, values in Table 4.3 column 6 (p_{NL}) are greater than values in Table 4.3 column 4 (p_{TFHN})). Unfortunately, there is no two-qudit characterisation of the set P_{CHSH} (unlike two-qubit systems [75]), so we cannot say anything in this regard[1]. Let us now come back to other two-qubit states.

4.4 Hirsch States

We now analyse the two-qubit states studied by Hirsch, Quintino, Bowles and Brunner [65] which we call Hirsch states (Eq. 2.27), defined as:

$$\rho_H(p, q, \sigma) = \; p\left|\psi_-\right\rangle\left\langle\psi_-\right| + [1-p]\left[q\sigma + (1-q)\frac{\mathbb{1}}{2}\right] \otimes \frac{\mathbb{1}}{2}, \tag{4.5}$$

$0 \leq p \leq 1$, $0 \leq q \leq 1$, $|\psi\rangle := \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$, and σ and arbitrary one-qubit state. These states can also be thought of as a generalisation of the two-qubit WI states (Eq. 4.1) (as we saw in Fig. 2.2) which we recover by putting $q = 0$. Interestingly, these states for $\sigma = |0\rangle\langle 0|$, and $q = 1$ become:

$$\rho_H(p) = p\left|\psi_-\right\rangle\left\langle\psi_-\right| + (1-p)\left|0\right\rangle\left\langle 0\right| \otimes \frac{\mathbb{1}}{2}, \tag{4.6}$$

for which the authors of [65] were able to build a local (projective) model for all $p \leq 1/2$. Even though nonlocality-related properties have been reported for these two-qubit Hirsch states, bounds for TFHN have not been reported yet. In Fig. 4.5 (a), (b) we have calculated these bounds.

[1] The results concerning Isotropic states are summarised in the article titled: *On the activation of quantum nonlocality* by Ducuara AF, Madroñero J, Reina JH, Universitas Scientiarum, **21** (2): 129-158, 2016.

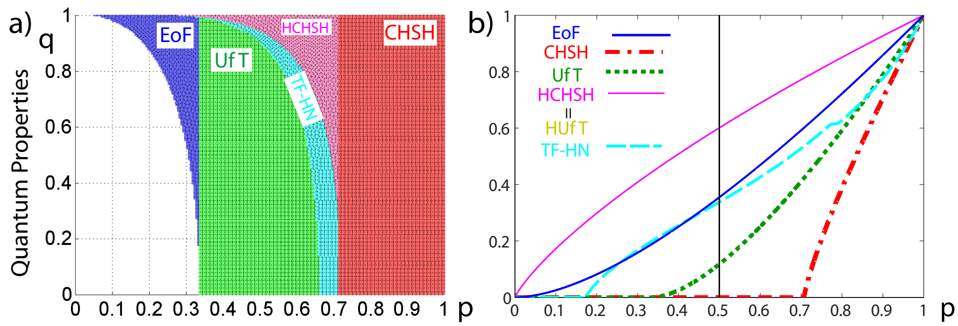

Figure 4.5: Quantum nonlocality-related properties for: **(a)** the two-parameter Hirsch states from Eq. 4.5 with $\sigma = |0\rangle\langle 0|$, **(b)** the one-parameter Hirsch states Eq. 4.6 (or two-parameter Hirsch states with $\sigma = |0\rangle\langle 0|$ and $q = 1$, previous plot **(a)** at $q = 1$). Entanglement of Formation (EoF, blue solid thick curve 2.2), CHSH-inequality violation (CHSH, red dashed-dotted curve Eq. 2.6), Usefulness for Teleportation (UfT, green dotted curve Eq. 2.12), Hidden CHSH-inequality violation (HCHSH, magenta solid thin curve Eq. 3.5), Activation TFHN (cyan dashed curve Eq. 3.14), and Projective-Locality (black vertical solid line).

Regarding Fig. 4.5, we first remark the hierarchy amongst these properties. All of these properties are inside entanglement and all of them cover the standard definition of nonlocality represented by the CHSH-inequality violation. Second, we are able to report a P_{CHSH} activation region for these qubits, here depicted by the cyan region (TFHN) that is not covered by the magenta region (HCHSH) in Fig. 4.5 (a).

We have depicted the particular case of Fig. 4.5 (a) for $q = 1$ in Fig. 4.5 (b) and Table 4.4, because there is a locality bound for these states [65]. First, the states within the locality region ($p \leq 0.5$) are usually considered useless for quantum protocols based on nonlocality. However, they are now displaying a nice variety of generalised nonlocality-related properties. Actually, all of the three generalisations we are considering here, unlike Werner states Fig. 4.3 which present UfT (all of the local region) and TFHN (a small region) but no HCHSH. Second, comparing this again with the Werner states (Fig. 4.2), we can numerically see that the relation between these generalisations of nonlocality is not trivial. In particular, there are Werner states with UfT but no HCHSH whilst there are Hirsch states with HCHSH but no UfT[2].

p_E	p_{HN}	p_{TFHN}	p_{UfT}	p_L	p_{NL}
0	0	0.1716	0.333	0.5000	0.7071

Table 4.4: Limit values of some quantum nonlocality-related properties for the Hirsch states (Eq. 4.6), extracted from Fig. 4.5 (b).

[2]The results concerning Hirsch states are summarised in the article titled: *On the activation of quantum nonlocality* by Ducuara AF, Madroñero J, Reina JH, Universitas Scientiarum, **21** (2): 129-158, 2016.

4.5 Game States

We now consider another two-qubit generalisation of the WI states, the ones we have called Game states (Eq. 2.29), and read:

$$\rho_G(p, \delta) = p\, \psi^{+i}(\delta) + \frac{(1-p)}{4}\mathbb{1}, \tag{4.7}$$

with $0 \leq p \leq 1$, $\psi^{+i}(\delta) := \left|\psi^{+i}(\delta)\right\rangle \left\langle \psi^{+i}(\delta)\right|$, with $\left|\psi^{+i}(\delta)\right\rangle = \cos\frac{\delta}{2}\left|00\right\rangle + i\sin\frac{\delta}{2}\left|11\right\rangle$, $\delta \in [0, \frac{\pi}{2}]$. In Fig. 4.6, in addition of all the information already addressed in subsection 2.2.5, we also depict the calculation of the nonlocality-related properties (Fig. 4.1).

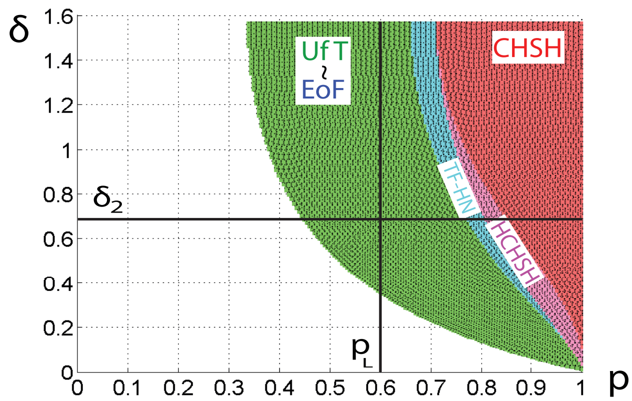

Figure 4.6: Quantum nonlocality-related properties for the two-qubit states given by Eq. 4.7, and their usefulness for the prisoners-dilemma game. Regardless the value of p, whenever $\delta \geq \delta_2 := \sin^{-1}\left(\sqrt{\frac{2}{5}}\right) \sim 0.6847$ (horizontal solid black line), these states $\rho(p, \delta \geq \delta_2)$ turn out to be useful in the prisoners-dilemma game. The locality (for the joint correlator) bound (vertical solid black line) [69] defines a region of local (for the joint correlator) states $\rho(p \leq p_L, \delta)$. These two regions define the set of local states $\rho(p \leq p_L, \delta \geq \delta_2)$ which becomes useful and advantageous in the prisoners-dilemma game.

From Fig. 4.6, we first point out that the non-locality properties follow the expected hierarchy, namely, that all of them lie inside entanglement, whilst covering the standard definition of nonlocality. Second, that all entangled states are useful for teleportation (which was expected [82]). Third, we define the set of local states given by $\rho(p \leq p_L, \delta \geq \delta_2)$ which becomes useful and advantageous in the prisoners-dilemma game[3].

Let us now address some states for which we cannot argue whether they are local, though they do not violate the CHSH-inequality, and therefore, are of interest when considering the activation of nonlocality.

[3] Preprint titled: *Quantum locality in game strategy* by C. Susa, C. A. Melo, A. F. Ducuara, A. Barreiro and J. H. Reina, up to be submitted to Nature Communications.

4.6 Some Not CHSH-inequality Violating States from an open quantum system

We have already seen that all of these non-locality related properties (Fig. 4.1) first require entanglement in order to show up. We have also seen that pure states are desirable, since entanglement and nonlocality are equivalent for them (Gisin Theorem), and therefore, no activation of nonlocality is needed. However, in a realistic set-up, quantum systems do not remain pure-entangled, but rather become mixed when interacting with an environment. Furthermore, they tend to lose their entanglement, becoming separable, and consequently, also losing these nonlocality-related properties. However, sometimes there are interesting phenomena during this loss of entanglement, featuring several dynamics such as; exponential decay, sudden death, sudden birth, and death and revival [95, 96]. Therefore, it is natural to wonder about what happens to these nonlocality-related properties when considering a dissipative entanglement dynamics.

Let us now address this, when considering different dynamics of entanglement for particular evolutions[4], namely; sudden death, sudden birth, and death and revival of entanglement. Thus, we next refer to the dynamics of a physical setup addressed in [95, 96], which comprises a set of quantum emitters in interactions with a coherent laser field.

4.6.1 Sudden Death

In Fig. 4.7 we present the system's evolution when it displays the dynamics of entanglement sudden death. First, since all of the nonlocality measures are ultimately limited by entanglement, here measured by the EoF, therefore, as one might expect, all of these nonlocality-related properties also present a sudden death. Second, we can identify two general behaviours as follows: Whilst the measures of CHSH, HCHSH, and TFHN go much faster to zero than the EoF, the measures of UfT and HUfT support more entangled states, and suddenly disappear at the same time than the EoF. We could interpret this by stating that the entanglement is dying in an *effective* way, since no relevant entanglement can be extracted by local filters (HUfT=MEEoF).

4.6.2 Sudden Birth

In Fig. 4.8 we present the system's evolution when it displays the dynamics of entanglement sudden birth. For initial separable states, the dissipative dynamics induces entanglement after $t \sim 30$. The measures given by UfT and HUfT start evolving different from zero just at the same time than the EoF. However, the measures of CHSH, HCHSH, and TFHN remain zero for much longer. The original CHSH nonlocality, actually, never appears in this evolution, which

[4] I warmly thank Cristian E. Susa, who kindly introduced me to these examples of entanglement dynamics, which also make part of a preprint titled: *Dynamics of nonlocality-related properties* by A. F. Ducuara, C. E. Susa, and J. H. Reina.

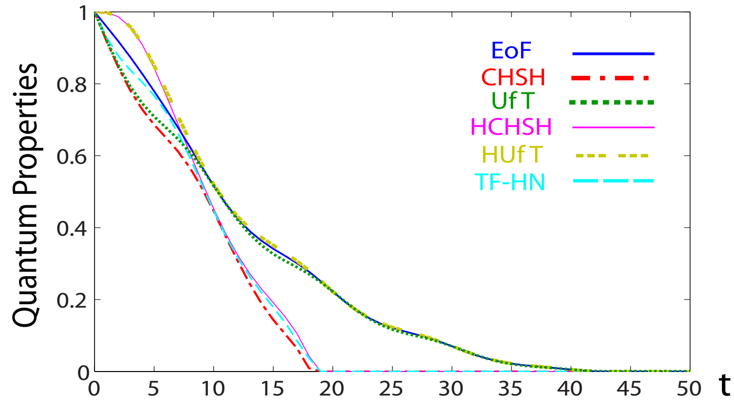

Figure 4.7: Quantum nonlocality-related properties (Fig. 4.1) for states that come from the dynamics of open quantum states [95, 96]. This particular evolution is depicting the so-called phenomenon of *Sudden Death of Entanglement*. EoF (blue thick solid curve), CHSH (red dashed dotted curve), UfT (green dotted curve), HCHSH (magenta thin solid curve), MEEoF=HUfT (yellow triply dashed curve), TFHN (cyan dashed curve).

means that the quantum state reached by the qubits system at every time does not violate the CHSH inequality. The hidden nonlocality, on the other hand, is born at $t \sim 40$, recovering a considerably high amount of nonlocality (~ 0.5). The TFHN births slightly before than the HCHSH, but achieving an amount of ~ 0.4. It is also worth noting that whilst the EoF after the birth is relatively low (~ 0.1), local filters are able to extract a considerably high amount of it (~ 0.5) HUfT=MEEoF.

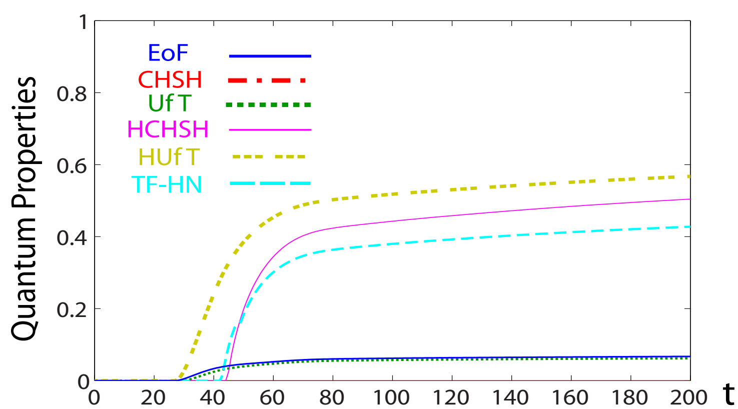

Figure 4.8: Quantum nonlocality-related properties (Fig. 4.1) for states that come from the dynamics of open quantum states [95, 96]. This particular evolution is depicting the so-called *Sudden Birth of Entanglement* phenomenon. The notation of the properties as in Fig. 4.7.

4.6.3 Death and Revival

In Fig. 4.9, we present the system's evolution when it displays the dynamics of death and revival. In the first stage of the dynamics ($0 \leq t \leq 25$), all of the nonlocality-related properties are present. In the second stage ($25 \leq t \leq 75$), however, with the exception of the CHSH, all of the properties revive. In a third stage ($85 \leq t \leq 125$), there only remain UfT and HUfT. Finally, in the fourth stage ($150 \leq t \leq 175$), only HkNL is present (which is always present when there is entanglement).

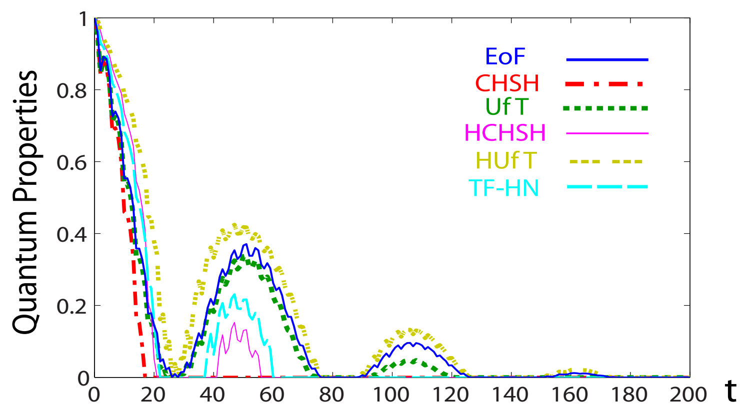

Figure 4.9: Quantum nonlocality-related properties (Fig. 4.1) for states that arise from the dynamics of open quantum states [95, 96]. This particular evolution is depicting the so-called *Death and Revival of Entanglement* phenomenon. The notation of the properties as in Fig. 4.7.

4.7 Final Considerations

We have studied the nonlocality-related properties (Fig. 4.1) upon bipartite entangled states of interest. We first addressed some entangled local states. We then considered some not CHSH-inequality violating states that arise from quantum open systems, addressing some preliminary results regarding the dynamics of the nonlocality-properties when considering different dynamics of entanglement. The latter is of interest, due to the fact that even though these states did not present the standard CHSH-nonlocality, nonlocality now can be recovered by considering generalisations of nonlocality by means of some of the activation scenarios.

Conclusions

From a fundamental point of view, we have reviewed some essential aspects of quantum nonlocality which are connected to the mathematical structure underneath it. We also reviewed some generalisations of nonlocality by means of the activation of nonlocality scenarios. We believe that this work has let us acquire some fundamentals tools, which would potentially help us to tackle other advanced topics.

From a practical point of view, we left several nonlocality-related properties ready to be calculated on arbitrary two-qubit systems, namely; entanglement of formation (EoF), usefulness for teleportation (UfT), CHSH-inequality violation (CHSH), hidden usefulness for teleportation (HUfT=MEEoF), hidden CHSH-inequality violation (HCHSH), and tensor/filter activation of hidden nonlocality (TFHN). This allowed us to make a couple of direct applications by calculating these properties upon some states of interest for which some of these properties had not been calculated before. We first studied the two-qudit Isotropic and two-qubit Hirsch states. We then moved onto two-qubit states that come from the study of both games' strategies and the dynamics of an open quantum system. We highlight our findings in what follows.

- In a first instance, for the two-qudit Isotropic and two-qubit Hirsch states, the TFHN had not been reported before, (this had only been done for the two-qudit Werner states). For the Isotropic states, even though the calculation of TFHN did not cover known locality regions, it did extend the known nonlocality one. For the Hirsch states, the calculation of the TFHN is of relevance, since we could also calculate the HCHSH, which in turn allows us to define a region of effective TFHN.

- In a second instance, we worked with two-qubit states that come from the study of games' strategies. It has been proven that there exist separable states that are useful for the particular quantum game of the prisoner dilemma, as well as entangled states that are not useful for this game, showing that entanglement does not fully characterise the efficiency in this case. In this regard, it is clear that neither nonlocality nor any of its generalisations could possibly have a role in this efficiency, this, since they are limited by the presence of entanglement. However, in this regard, we were able to point out some useful (for the game) entangled local states, thus, extending the region where these states are expected to be useless for these particular quantum information protocols.

- Finally, we studied these nonlocality-related properties on states that come from an open quantum system. We remark that these kind of states are of importance since they may represent real physical processes of systems in interaction with a *decoherent* environment, where we usually lose all of these interesting quantum properties. Here, our findings are the following.
  - First, within the parameter range we have worked with so far, we have identified the following interesting common behaviour. Throughout the dynamics of entanglement, there has not been an interesting evolution of the standard notion of nonlocality (no births nor revivals when entanglement does it). However, when considering its generalisations, they now present an interesting behaviour, which explicitly evidence the usefulness of considering these activation of nonlocality scenarios. In other words, that the systems here considered do not naturally evidence nonlocality during their evolution, though if we work a bit more (this, by implementing local filters or/and tensor product), we are able to extract/recover/activate nonlocality.
  - Second, we remember that the role of the measures we are working with is two-fold: whilst they first act as a *criterion* (by identifying whether the property in question is present or not) they also tell us about the *amount* of it. In this regard, it is therefore interesting to achieve the highest possible values. In the dynamics of sudden birth of entanglement, some of these measures achieve around $\sim 50\%$ of their maximum possible amount. We are looking forward to identifying whether these activations could become maximal ($\sim 100\%$), and the respective parameters range where this happens.

Appendices

Appendix A

Fine's Theorem

In this appendix we present a proof of the so-called Fine's theorem which, colloquially speaking, says that the set of local correlations (Eq. 1.22) forms a polytope. Given a scenario $s = (I_A, I_B, O_A, O_B)$, and a local correlation $p_L \in L(s)$ with components given by:

$$p_L(ab|xy) = \int d\lambda \mu(\lambda) \xi_A(a|x\lambda) \xi_B(b|y\lambda),$$

with $\mu(\lambda)$, $\xi_A(a|x\lambda)$, and $\xi_B(b|y\lambda)$ probability functions, the Fine's theorem establishes that this local correlation can be written as the convex combination of $O_A^{I_A} O_B^{I_B}$ deterministic correlations as in Eq. 1.23. Let us explicitly address the transformation to this new LHV model. Introducing a new hidden variable $\lambda' := (\lambda, \mu_1, \mu_2)$ with $\mu_1, \mu_2 \in [0, 1]$, $\mu(\lambda') = \mu(\lambda)$, and functions ξ'_A, ξ'_B defined by:

$$\xi'_A(a|x\lambda') := \begin{cases} 1 & \text{if } F_A(a-1|x\lambda) \le \mu_1 \le F_A(a|x\lambda) \\ \\ 0 & \text{otherwise} \end{cases} \qquad F_A(a|x\lambda) := \sum_{\tilde{a} \le a} \xi_A(\tilde{a}|x\lambda),$$

and equivalently for Bob, we consider the correlations given by these functions as:

$$\begin{aligned} p'_L(ab|xy) &= \int d\lambda' \mu(\lambda') \xi'_A(a|x\lambda') \xi'_B(b|y\lambda') && \text{(A.1)} \\ &= \int \int_0^1 \int_0^1 d\lambda \; d\mu_1 \; d\mu_2 \; \mu(\lambda) \xi_A(a|x\lambda) \xi_B(b|y\lambda), \\ &= \int d\lambda \mu(\lambda) \int_0^1 d\mu_1 \; \xi_A(a|x\lambda\mu_1) \int_0^1 d\mu_2 \; \xi_B(b|y\lambda\mu_2), \end{aligned}$$

The first integral gives us:

$$\begin{aligned}\int_0^1 d\mu_1\ \xi_A(a|x\lambda\mu_1) &= \int_0^{F_A(a-1|x\lambda)} d\mu_1\ \xi_A(a|x\lambda\mu_1)\\ &+ \int_{F_A(a-1|x\lambda)}^{F_A(a|x\lambda)} d\mu_1\ \xi_A(a|x\lambda\mu_1)\\ &+ \int_{F_A(a|x\lambda)}^{1} d\mu_1\ \xi_A(a|x\lambda\mu_1),\end{aligned}$$

from which only the one in the middle survives, giving $F_A(a-1|x\lambda)-F_A(a|x\lambda) = \xi_A(a|x\lambda)$. Analogously with the other part of the integral, we have:

$$\begin{aligned}p'_L(ab|xy) &= \int d\lambda' \mu(\lambda')\xi'_A(a|x\lambda')\xi'_B(b|y\lambda')\\ &= \int d\lambda\ \mu(\lambda)\xi_A(a|x\lambda)\xi_B(b|y\lambda) = p_L(ab|xy).\end{aligned}$$

We have then proven that it is possible to recover the initial local correlations with this new LHV model. Writing $\lambda' = (\lambda'_1, \lambda'_2)$, then $p_D(ab|xy\lambda') = \xi'_A(a|x\lambda'_1)\xi'_B(b|y\lambda'_2)$, we have that any local correlation can be written as a convex combination of them. We now argue that the amount of correlations $\xi'_A(a|x\lambda'_1)$ and $\xi'_B(b|y\lambda'_2)$ correlations is $O_A^{I_A}$ and $O_B^{I_B}$, repectively. Therefore, the amount of correlations $p_D(ab|xy\lambda')$ is $O_A^{I_A}O_B^{I_B}$.

Let us start by analysing the new LHV model for Alice and the correlations $\xi'_A(a|x\lambda'_1)$. Given x, this correlation tells us that for certain value (say $a = a_x$), we have $\xi'_1(a = a_x|x\lambda'_1) = 1$, so amongst the possible λ'_1, there is one (say λ'_{a_x}, which can also be interpreted as the function $\lambda'_{a_x}(x) = a_x$), so that $\xi'_1(a = a_x|x\lambda'_{a_x}) = 1$, and the correlation can also be written as $\xi'_A(a|x\lambda'_1) = \delta_a^x(\lambda'_1)$. Writing these all possible hidden variables as: $\lambda'_1 = (\lambda_{a_1}, ..., \lambda_{a_i}, ..., \lambda_{a_{I_A}})$, we can check that there are only $O_A^{I_A}$ different variables. Similarly doing it for bob, we have that any local correlation can be written as:

$$p_L(ab|xy) = \sum_{\lambda_2=1}^{O_A^{I_A}} \sum_{\lambda_2=1}^{O_B^{I_B}} \mu(\lambda'_1, \lambda'_2)\ \delta_a^x(\lambda'_1)\delta_b^y(\lambda'_2). \tag{A.2}$$

being $\Lambda, \mu, \delta_a^x, \delta_b^y$ a new LHV model, and δ_a^x, δ_b^y the so-called *deterministic correlations.*

Appendix B

Relationship Between Sets of Correlations

In this appendix, we present a proof of the main theorem that relates the two worlds of state-measurement and correlations. In the first section, we present a proof of the proper inclusions. In the second section. we provide the simplest counterexamples, this, with the exception of $L(s) \subsetneq Q(s)$, which we present in the main body of this document (this counterexample is of unavoidable relevance, since it basically encompasses the Bell's theorem). Let us start with the proper inclusions.

B.1 Proper Inclusions: $L(s) \subset Q(s) \subset NS(s)$

In this section, we use the definitions introduced in the first chapter in order to prove the proper inclusions from the main theorem, i. e., we show $S(s) = L(s) \subseteq Q(s) \subseteq NS(s)$. It should be pointed out that these three proofs hold for any given scenario $s = (I_A, I_B, O_A, O_B)$.

B.1.1 $S(s) \subseteq L(s)$: Separable are Local Correlations

Let us consider the correlations generated by separable states. Given a separable state $\rho = \sum_i p_i \rho_i^1 \otimes \rho_i^2 \in D(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2})$, and local POVM's, $M^x = \{E_a^x\}$, and $M^y = \{F_b^y\}$ (and therefore defining a scenario s), we now analyse the quantum correlations $\vec{p}_\rho = [p_\rho(ab|xy)]$ generated by this state. From the Born rule (Eq. 1.9), we have:

$$\begin{aligned} p_\rho(ab|xy) &= \mathrm{Tr}\left[(E_a^x \otimes F_b^y)\,\rho\right], \\ &= \mathrm{Tr}\left[(E_a^x \otimes F_b^y)\left(\sum_i p_i \rho_i^1 \otimes \rho_i^2\right)\right], \\ &= \sum_i p_i \mathrm{Tr}\left[E_a^x \rho_i^1\right] \mathrm{Tr}\left[F_b^y \rho_i^2\right]. \end{aligned}$$

In order to show that these correlations are local, we should propose a LHV model to reproduce these correlations as in Eq. 1.22. In order to do so, let us consider the LHV model given by: $\Lambda := \{i\}, \lambda = i$, $\xi_1(a|xi) := \mathrm{Tr}\left[E_a^x \rho_i^1\right]$, $\xi_2(b|yi) := \mathrm{Tr}\left[F_b^y \rho_i^2\right]$ and $\mu(\lambda) = \mu(i) := p_i$. We can check that the functions ξ_1, ξ_2 and μ are well-defined probability functions, in the sense of Eq. 1.22. Considering the correlations generated by this LHV model, i. e., $p_L(ab|xy) := \int_\Lambda \mu(i)\xi_1(a|xi)\xi_2(b|yi)$, we can check that they indeed reproduce the considered quantum correlations, i. e., $p_L(ab|xy) = p_\rho(ab|xy)$. Therefore, the quantum correlations generated by separable states are local correlations $S(s) \subseteq L(s)$ $\square$. Let us continue with the next inclusion.

B.1.2 $L(s) \subseteq S(s) \subseteq Q(s)$: Local are Quantum Correlations

In order to show that local are quantum correlations, we should take a local correlation $\vec{p}_L = (p_L(ab|xy))$, and show that there exist a Hilbert space with a density matrix and local POVM's such that the local correlation can be reproduced quantum mechanically, $p_L(ab|xy) = p_Q(ab|xy)$. Given a scenario $s = (I_A, I_B, O_A, O_B)$, and a local correlation $\vec{p}_L = [p(ab|xy)]$ taking values

$$p_L(ab|xy) = \sum_{\lambda_1=1}^{O_A^{I_A}} \sum_{\lambda_2=1}^{O_B^{I_B}} \mu(\lambda_1, \lambda_2)\xi_1(a|x\lambda_1)\xi_2(b|y\lambda_2),$$

with $\mu(\lambda_1, \lambda_2)$, $\xi_1(a|x\lambda_1)$, and $\xi_2(b|y\lambda_2)$ some well-defined probability functions. We now propose the set of density matrices $D(\mathbb{H})$ over the Hilbert space $\mathbb{H} := \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$ with $d_1 = O_A^{I_A}$ and $d_2 = O_B^{I_B}$. We consider an ortonormal basis for the first subsystem and we label their elements with the hidden variables as $\{|\lambda_1\rangle\}$, we do the same with the second susbsytem $\{|\lambda_2\rangle\}$. We then consider the separable state and two local POVM's $M^x = \{E_a^x\}$, $N^y = \{F_b^y\}$ given by:

$$\rho := \sum_{\lambda_1=1}^{d_1} \sum_{\lambda_2=1}^{d_2} \mu(\lambda_1, \lambda_2)|\lambda_1\rangle\langle\lambda_1| \otimes |\lambda_2\rangle\langle\lambda_2|,$$

$$E_a^x := \sum_{\lambda_1'=1}^{d_1} \xi_1(a|x\lambda_1')|\lambda_1'\rangle\langle\lambda_1'|, \qquad F_b^y := \sum_{\lambda_2'=1}^{d_2} \xi_2(b|y\lambda_2')|\lambda_2'\rangle\langle\lambda_2'|.$$

We first note that these definitions establish a well-defined density matrix and POVM's, since they satisfy the conditions given by Eq. 1.5 and Eq. 1.6, respectively. Putting all of this together by means of the Born rule (Eq. 1.9), we can check that $p_\rho(ab|xy) = \mathrm{Tr}[(E_a^x \otimes F_b^y)\rho] = p_L(ab|xy)$, quantum-mechanically reproducing the desired local correlation. We have then reproduced local correlations with separable states, so $L(s) \subseteq S(s) \subseteq Q(s)$ $\square$.

Remark: We have already seen that separable states produce only local correlations, and that local correlations can be reproduced by separable states. One might be tempted to assume that entangled states must generate only nonlocal correlations. However, it turns out that some of them, as separable states, can only generate local correlations [26, 27]. This means that there is not a one-to-one states-correlations relationship, i. e., a local correlation in addition of being generated by a separable state (as we just saw), it can also, possibly, be generated by an entangled state. We are going to see an example of this phenomenon in the counterexamples section. For the moment, let us finish with the last inclusion of the main theorem.

B.1.3 $Q(s) \subseteq NS(s)$: Quantum are Non-Signalling Correlations

In order to prove that quantum are non-signalling correlations, we should verify that quantum correlations satisfy the non-signalling conditions given by Eq. 1.16 and Eq. 1.17 [8]. Given a quantum correlation $\vec{p}_\rho = [p_\rho(ab|xy)]$ generated by some quantum state ρ, and local POVM's $M^x = \{E_a^x\}$, and $N^y = \{F_b^y\}$ (and therefore defining a scenario s), we now check the non-signalling condition Eq. 1.16, we have:

$$
\begin{aligned}
\sum_a \operatorname{Tr}\left[\left(E_a^x \otimes F_b^y\right)\rho\right] \qquad &= \operatorname{Tr}\left[\sum_a \left(E_a^x \otimes F_b^y\right)\rho\right],\\
&= \operatorname{Tr}\left[\left(\sum_a E_a^x \otimes F_b^y\right)\rho\right],\\
&= \operatorname{Tr}\left[\left(\mathbb{1} \otimes F_b^y\right)\rho\right] = p(b|y).
\end{aligned}
$$

We have that the last term does not depend on the x choice, satisfying the non-signalling condition. The same can be done for the other part, $\sum_b \operatorname{Tr}\left[(E_a^x \otimes F_b^y)\rho\right] = p(a|x)$. We then have that any quantum correlation satisfies the non-signalling condition, so $Q(s) \subseteq NS(s)$ $\square$. This result is telling us that all quantum correlations, even the nonlocal quantum correlations, respect the non-signalling principle, and therefore, cannot be used to generate faster-than-light communication, in accordance with general relativity. Let us now address the counterexamples, this, again, without $L(s) \subsetneq Q(s)$ which we present in the main body of the document.

B.2 Counterexamples: $L(s) \subsetneq Q(s) \subsetneq NS(s)$

In this section, we provide counterexamples. First, we address the existence of the so-called Werner-Isotropic states (ρ_{WI}), entangled states that are local. Second, we address a quantum nonsignalling correlation (the so-called PR Box) that cannot be built with quantum mechanics, evidencing the existence of the so-called superquantum correlations. Let us start by addressing entangled local states.

B.2.1 $Q^P_{\rho_{WI}}(I_A, I_B, 2, 2) \subseteq L(I_A, I_B, 2, 2)$

In order to prove this, we need to provide a entangled local state. We are going to do so in the scenario $(I_A, I_B, 2, 2)$ with a two-qubit state. We first introduce a necessary and sufficient entanglement criterion for two-qubit systems called the *Positive Partial Transpose* (PPT) criterion. We then present the concept of *correlators*. Finally, we address the so-called Werner-Isotropic state (ρ_{WI}), an entangled two-qubit state that only generates local correlations in the scenario $(I_A, I_B, 2, 2)$, having $Q^P_{\rho_{WI}}(I_A, I_B, 2, 2) \subseteq L(I_A, I_B, 2, 2)$.

PPT criterion

Given $\rho \in D(\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B})$, which we can always write as $\rho := \sum_{ijkl} \rho_{ijkl} |i\rangle\langle j| \otimes |k\rangle\langle l|$ with ρ_{ijkl} complex coefficients, the *partial transpose respect to the first subsystem* is defined as:

$$\rho^{T_1} := \sum_{ijkl} \rho_{ijkl} (|i\rangle\langle j|)^T \otimes |k\rangle\langle l| = \sum_{ijkl} \rho_{ijkl} |j\rangle\langle i| \otimes |k\rangle\langle l|,$$

being T the standard transposition. The state ρ is said to be a *Positive Partial Transpose* (PPT) state or that it has PPT, if its partial transpose ρ^{T_1} remains positive ($\rho_\tau^{T_1} \geq 0$). We can check that a separable is then a PPT state, which allows us to define the following entanglement criterion:

$$\text{If a state is not PPT then it is entangled,} \tag{B.1}$$

This criterion is a necessary and sufficient criterion for two-qubit systems and qubit-qutrit systems [51], however it does not hold for higher dimensions [97]. This is enough to study entanglement for the two-qubit state that we will see later. For the moment, let us address the concept of correlators.

Correlators

We have already introduced and worked with the concept of correlation as a vector which components are given by joint conditional probability functions $\vec{p} = [p(ab|xy)]$, we now introduce the concept of *correlator*, which is going to extend the concept of "expected value", and it will come in handy when giving the counterexample. First, we start by restricting the scenario (s) to $(I_A, I_B, 2, 2)$, i. e., outcomes a and b taking only two values each, which we label as $a, b \in \{1, -1\}$, whilst x, y are left open. The *joint correlator* E^{xy}, and *marginal correlators* E^x, E^y become [28]:

$$E^{xy} := \sum_{a,b\in\{1,-1\}} abp(ab|xy), \tag{B.2}$$

$$E^x := \sum_{a\in\{1,-1\}} abp(a|x), \qquad E^y := \sum_{b\in\{1,-1\}} bp(b|y). \tag{B.3}$$

These correlators are then generalising the concept of expected value. We can check that we can recover the probabilities in terms of the correlations as [28]:

$$p(ab|xy) = \frac{1}{4}\left(1 + aE^x + bE^y + abE^{xy}\right). \tag{B.4}$$

Second, we can quantum mechanically build these correlators by considering observables and defining $E_\rho^{xy} = \mathrm{Tr}[(A^x \otimes B^y)\rho]$, $E_\rho^x = \mathrm{Tr}[(A^x \otimes \mathbb{1})\rho]$, and $E_\rho^y = \mathrm{Tr}[(\mathbb{1} \otimes B^y)\rho]$. The observables A^x and B^y should be dichotomic observables, in order to fulfil our two outcomes condition. Third, since we are going to work with two-qubit states, these observables have to be traceless. Therefore, we can check that our two-qubit observables must have the form: $A^x := \vec{\sigma} \cdot \hat{a}^x$ and $B^y := \vec{\sigma} \cdot \hat{b}^y$, being $\vec{\sigma} := \sum_{i=1,2,3} \sigma_i \hat{e}_i$ with σ_i the Pauli matrices, and $\hat{a}^x$, $\hat{b}^y \in \mathbb{R}^3$ arbitrary normalised vectors. We take advantage of this definitions to define *locality for the joint correlator*, when we can locally reproduce a given quantum joint correlator E_ρ^{xy}, regardless of the marginal correlators E_ρ^x or E_ρ^y. It is clear that standard locality implies locality for the join correlator, though not vice-versa [69]. Let us now present the first counterexample.

Two-qubit Werner-Isotropic (WI) States

We present the Werner-Isotropic (WI) states which were first introduced in [26] as the two-qubit version of the so-called two-qudit Werner states [26]. We have opted to address them as "Werner-Isotropic", since they also happen to be the two-qubit version of other two-qudit states called Isotropic states [64]. We further detail these relationships in Chapter 3. In the meanwhile, let us then introduce these two-qubit Werner-Isotropic states as:

$$\rho_{WI}(p) = p\psi^- + \frac{(1-p)}{4}\mathbb{1}, \tag{B.5}$$

with $\psi^- := |\psi^-\rangle\langle\psi^-|$, $|\psi^-\rangle := \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$, and $0 \leq p \leq 1$. Using the PPT criterion (Eq. B.1), we can check that these states are entangled for $\frac{1}{3} < p$. We are now going to show that these sates are local-projective in $s = (I_A, I_B, 2, 2)$ for $p \leq \frac{1}{2}$, and therefore, being entangled local-projective in the region $\frac{1}{3} < p \leq \frac{1}{2}$ [26, 37]. In order to prove so, we first prove that the state with $p = 1/2$ is a local state. Then, taking into account that the state $\mathbb{1}$ is a separable (and therefore local state) and that the linear combination of local correlations is a local correlation, we will have proven what we want.
There are more examples of entangled local states, in particular, in Chapter 3 we are going to take a look at some more examples (a nice review could be found in [27]). Recently, there have been breakthroughs in the numerical construction of local models for entangled states, by means of a property called *quantum steering* [66, 67]. Let us then prove that WI the state $\rho_{WI}(p = 1/2)$ is local. Let us redefine the state of interest as:

$$\rho := \rho_{WI}\,(p = 1/2) = \frac{1}{2}\left|\psi^-\right\rangle\left\langle\psi^-\right| + \frac{1}{8}\mathbb{1}. \tag{B.6}$$

We now consider the local scenario $s = (I_A, I_B, 2, 2)$, i. e., two outcomes which we label as $a, b \in \{1, -1\}$, whilst leaving x and y open. We now consider the correlations generated by this state in that scenario $Q^P_\rho(s)$. We are going to show that $Q^P_\rho(s) \subset L(s)$. From Eq. B.4, we have that the quantum correlations generated by this state $\vec{p}_\rho = [p_\rho(ab|xy)] \in Q^P_\rho(s)$ can be written in terms of the correlators E^x_ρ, E^y_ρ, and E^{xy}_ρ with certain observables $A^x = \vec{\sigma}\cdot\hat{a}^x$, and $B^x = \vec{\sigma}\cdot\hat{b}^y$. We can check that with the state in Eq. B.6 we achieve: $E^x_\rho = E^y_\rho = 0$, and $E^{xy}_\rho = E^{xy}_{\psi^-} = -\hat{a}^x\cdot\hat{b}^y$. Replacing all of this in Eq. B.4, the quantum correlations can be written as:

$$p_\rho(ab|xy) = \frac{1}{4}\left[1 + ab\frac{1}{2}(-\hat{a}^x \cdot \hat{b}^y)\right]. \tag{B.7}$$

In order to prove that this state is local, we have to reproduce these correlations with a LHV model. Let us consider the local correlation $\vec{p}_L$ with components $p_L(ab|xy) = \int_\Lambda d\vec{\lambda}\mu(\vec{\lambda})\xi_1(a|x\vec{\lambda})\xi_2(b|y\vec{\lambda})$ given by the following LHV model. The hidden variable set is the 2-sphere $\Lambda = S^2$, with hidden variables parametrised as $\vec{\lambda} := (\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta)$, probability function $\mu(\vec{\lambda}) := \frac{1}{4\pi}\sin\theta d\theta d\phi$, and probabilities given by:

$$\xi_1(a|x\vec{\lambda}) := \frac{1}{2}\left(1 + a\hat{a}^x\cdot\vec{\lambda}\right), \qquad \xi_2(b|y\vec{\lambda}) := \begin{cases} 1 & \text{if } \hat{b}^y\cdot\vec{\lambda} \leq 1, \\ \\ 0 & \text{if } \hat{b}^y\cdot\lambda > 1. \end{cases}$$

We now check that this LHV model does indeed reproduce the quantum correlations generated by the state in Eq. B.6. Let us first calculate $p_L(a1|xy)$. Choosing the spherical coordinate system such that: $\vec{y} = \vec{z}$, we have:

$$\begin{aligned} p_L(a\,1|xy) &= \int_{\vec{b}^y\cdot\lambda\leq 1} d\vec{\lambda}\mu(\vec{\lambda})\frac{1}{2}\left(1 + a\vec{a}^x\cdot\lambda\right), \\ &= \int_{\vec{b}^y\cdot\lambda\leq 1} d\vec{\lambda}\mu(\vec{\lambda})\frac{1}{2}\left(1 + a\vec{a}^x\cdot\lambda\right), \\ &= \frac{1}{4} + \frac{1}{2}a\int_{\vec{b}^y\cdot\lambda\leq 1} d\vec{\lambda}\mu(\vec{\lambda})\left(\vec{a}^x\cdot\lambda\right), \\ &= \frac{1}{4} - \frac{1}{2}aa^x_z = \frac{1}{4} - \frac{1}{2}a(1)\hat{a}^x\cdot\hat{b}^y, \\ &= p_\rho(a\,1|xy). \end{aligned}$$

It is also possible to do this for $p_{LM}(a\ -1|\hat{a}^x\hat{b}^y)$. We therefore have that $p_\rho(ab|\hat{a}^x\hat{b}^y) = p_L(a,b|\hat{a}^x,\hat{b}^y)$, showing that the state given by Eq. B.6 is local (projective) in the scenario $s = (I_A, I_B, 2, 2)$. As we mentioned earlier, we then also have that all states in Eq. B.5 with $p < 1/2$ are local, since their correlations can be though as a convex combination of the correlations from $\mathbb{1}\otimes\mathbb{1}$ and $\rho_{WI}(1/2)$ which are both local. In other words, that $Q^P_{\rho_{WI}}(s) \subseteq L(s)$, with $\rho_{WI}(0 < p \leq 1/2)$, and therefore the states $\rho_{WI}(1/3 < p \leq 1/2)$ are entangled local (projective) in the scenario $s = (I_A, I_B, 2, 2)$ $\square$. Let us now address the next counterexample.

B.2.2 $Q(2,2,2,2) \subsetneq NS(2,2,2,2)$

In order to show this, we have to provide a non-signalling correlation that we cannot reproduce quantum-mechanically. We have already seen that quantum mechanics allows a maximal violation of $2\sqrt{2}$ for the CHSH-Inequality (Eq. 1.25), which is the Tsirelson's Bound. Therefore, we need a nonsignalling correlation violating the CHSH-Inequality beyond this bound. This correlation was first introduced by Sandu Popescu and Daniel Rorhlich [8] (though the correlation itself had been already introduced [9, 10]) working in the scenario $s = (2,2,2,2)$, and it has been called the *PR-box* or non-signalling superquantum correlation. We write it as $\vec{p}_{PR} = [p_{PR}(ab|xy)] \in C(s)$ with components given by the following function. For inputs $(x=0, y=0)$, $(x=0, y=1)$ or $(x=1, y=0)$ we define:

$$p_{PR}(ab|xy) := \begin{cases} 1/2 & \text{if } \ a=b=0, \\ 1/2 & \text{if } \ a=b=1, \\ 0 & \quad \text{otherwise}, \end{cases} \quad p_{PR}(ab|11) := \begin{cases} 1/2 & \text{if } \ a=0, b=1, \\ 1/2 & \text{if } \ a=1, b=0, \\ 0 & \quad \text{otherwise}, \end{cases} \tag{B.8}$$

We can check that this correlation is non-signalling, by checking the conditions Eq. 1.16 and Eq. 1.17. We can also check that violates the CHSH-Inequality Eq. 1.25 by an amount of 4, which is greater than the Tsirelson's bound which reads $2\sqrt{2}$. Therefore, we cannot reproduce this correlation with quantum mechanics, and that is why, it has been termed, a non-signalling superquantum correlation $\square$.

The PR box (Eq. B.8) leaves us with several questions. Why quantum theory cannot generate them, since they are allowed by relativity? Why quantum theory is not more nonlocal of what it already is? (quantum theory only achieves a violation of the CHSH-inequality of $2\sqrt{2}$). We have two options here, we whether find more principles to further constrain quantum theory [11, 12, 13, 14] or we generalise quantum mechanics to a theory that allows us to implement/understand these superquantum correlations [15, 16]. This is an active area of research, and hopefully, it is going to shed light on the unification of quantum theory and the theory relativity.

Appendix C

CHSH-inequality

In this appendix we address a couple of matters related with the CHSH inequality. First, we provide a derivation of the inequality itself. Second, we describe a quantum game which has been coined the CHSH game, because it explicitly uses this inequality. Finally, we present the so-called Horodecki CHSH criterion for the violation of the CHSH inequality for general two-qubit systems. Let us start by addressing the derivation of the CHSH inequality.

C.1 CHSH-inequality Derivation

Let us start with considering local correlations $\vec{p}_L \in L(s)$ with elements given by

$$p_L(ab|xy) = \sum_{\lambda_1}\sum_{\lambda_2}\mu(\lambda_1,\lambda_2)\xi_A(a|x\lambda_1)\xi_B(b|y\lambda_2).$$

Considering the scenario $s = (I_A, I_B, 2, 2)$, these probabilities can be written in terms of the correlators (Eq. B.3) as in Eq. B.4. After replacing local correlations, these correlators read:

$$E^{xy} = \sum_{a,b\in\{-1,1\}} abp_L(ab|xy) = \sum_{\lambda}\mu(\lambda)E^x_A(\lambda)E^y_B(\lambda), \tag{C.1}$$

with $E^x_A(\lambda) = \sum_a a\xi_A(a|x\lambda)$, and $E^y_B(\lambda) = \sum_b b\xi_B(b|y\lambda)$. Let us now see that $\Big|E_A(a|x\lambda)\Big| \leq 1$ and $\Big|E_B(b|y\lambda)\Big| \leq 1$. Taking absolute value of the definition, we have:

$$\Big|E_A(a|x\lambda)\Big| = \Big|\sum_a a\xi_A(a|x\lambda)\Big| \leq \sum_a \Big|a\Big|\Big|\xi_A(a|x\lambda)\Big| = \sum_a \xi_A(a|x\lambda) = 1, \tag{C.2}$$

this, since the probability ξ_A is a positive function, and that $a = \pm 1$. We have a similar result for Bob. We now consider the scenario with two inputs per party, and therefore narrowing our scenario to $(2,2,2,2)$. We now consider the magnitude[5]:

$$S := E^{00} + E^{01} + E^{10} - E^{11} = \sum_{\lambda} \mu(\lambda) S_{\lambda}, \tag{C.3}$$

replacing Eq. C.1 and organising, we have:

$$S_{\lambda} = E_A^0(\lambda)\Big[E_B^0(\lambda) + E_B^1(\lambda)\Big] + E_A^1(\lambda)\Big[E_B^0(\lambda) - E_B^0(\lambda)\Big].$$

We also have that $S \leq \Big|S\Big| \leq \sum_{\lambda} \mu(\lambda)\Big|S_{\lambda}\Big|$ ($\mu(\lambda)$ positive function). Let us see that $\Big|S_{\lambda}\Big| \leq 2$ and therefore that $\Big|S\Big| \leq 2$, which is the so-called CHSH-inequality. We have:

$$\begin{aligned}
S_{\lambda} &\leq |S_{\lambda}| = \Bigg|E_A^0(\lambda)\Big[E_B^0(\lambda) + E_B^1(\lambda)\Big] + E_A^1(\lambda)\Big[E_B^0(\lambda) - E_B^0(\lambda)\Big]\Bigg|, \\
&\leq \Bigg|E_A^0(\lambda)\Big[E_B^0(\lambda) + E_B^1(\lambda)\Big]\Bigg| + \Bigg|E_A^1(\lambda)\Big[E_B^0(\lambda) - E_B^0(\lambda)\Big]\Bigg|, \\
&= \Bigg|E_A^0(\lambda)\Bigg|\Bigg|E_B^0(\lambda) + E_B^1(\lambda)\Bigg| + \Bigg|E_A^1(\lambda)\Bigg|\Bigg|E_B^0(\lambda) - E_B^0(\lambda)\Bigg|,
\end{aligned}$$

Assuming $\Big|E_A^0(\lambda)\Big| = \Big|E_A^1(\lambda)\Big| = 1$ then:

$$S_{\lambda} \leq \Big|E_B^0(\lambda) + E_B^1(\lambda)\Big| + \Big|E_B^0(\lambda) - E_B^0(\lambda)\Big|.$$

Now, without loss of generality we can assume that $\Big|E_B^0(\lambda)\Big| \geq \Big|E_B^1(\lambda)\Big| \geq 0$, and furthermore, that $\Big|E_B^1(\lambda)\Big| = 0$, then:

$$\Big|E_B^0(\lambda) + E_B^1(\lambda)\Big| + \Big|E_B^0(\lambda) - E_B^1(\lambda)\Big| \leq 2\Big|E_B^0(\lambda)\Big| \leq 2.$$

Because of Eq. C.3, we then have the so-called CHSH-inequality:

$$\Big|S\Big| = \Big|E^{00} + E^{01} + E^{10} - E^{11}\Big| \leq 2,$$

In his original paper [3], J. S. Bell used another inequality, which he derived by means of imposing an extra assumption. However, this inequality was later refined as the CHSH-inequality [49].

[5]The motivation to introduce this magnitude might seem artificial, though in general, the problem of finding Bell inequalities can be systematically addressed.

C.2 CHSH Game

The so-called *CHSH Game* is composed by two players, A for Alice, B for Bob, and a *Referee*. The Referee chooses bits $x, y \in \{0,1\}$ according to certain joint probability distribution $\pi(x,y)$, and give them to Alice and Bob, respectively. The Referee then explains to Alice and Bob that they have to produce outcomes bits $a, b \in \{0,1\}$ and that they will win the game *if and only if* they fulfil certain *predicate* $V(ab|xy)$ which is imposed by the Referee itself (the players are not allowed to communicate between them after they have received the input bits x and y). The players decide what is going to be their outcome based on a prestablished *strategy*, which is here represented by a joint conditional probability distribution $p(ab|xy)$. All of these probability distributions satisfy the standard probability properties. The *probability of winning* the game can be defined as the number.

$$P^{\text{win}} := \sum_{x,y\in\{0,1\}} \pi(x,y) \sum_{a,b\in\{0,1\}} V(ab|xy)p(ab|xy), \tag{C.4}$$

which we can check, it is indeed a number in $[0,1]$. The definition of the functions $\pi(x,y)$ and $V(ab|xy)$ define the game in question, so different functions lead to different games. In particular, the CHSH game is established as follows:

$$\pi(x,y) := \frac{1}{4}, \qquad V(ab|xy) := \begin{cases} 1 & \text{if } a \oplus b = x \cdot y \\ 0 & \text{otherwise} \end{cases} . \tag{C.5}$$

The function $p(ab|xy)$ is left open in order to model the different strategies that the players can opt to play with. We can see an sketch of this game in Fig. C.1.

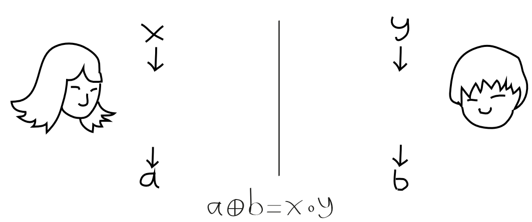

Figure C.1: CHSH Game

We are going to see classical, quantum, and non-signalling strategies $p(ab|xy)$. Let us first start by addressing an intuitive classical strategy.

$$p_C(ab|xy) := \begin{cases} 1 & \text{if } a,b = 1, \\ 0 & \text{otherwise,} \end{cases} \qquad \text{replacing in Eq. C.4: } P_C^{\text{win}} = \frac{3}{4} = 0.75.$$

In other words, Alice and Bob agree to play always with $a, b = 1$ regardless of the received inputs x, y. We are going to prove that this is the best classical strategy, so that $p_{\text{win}}^C \leq \frac{3}{4}$. Let us now rewrite this probabilities in terms of the correlators as depending on the sum $c := a \oplus b$ (since this is the value that the players should aim for)

$$p_Q(a \oplus b =: c|xy) := \frac{1}{2}\Big[1 + (-1)^c E^{xy}\Big]. \tag{C.6}$$

We can see this from the following. The marginal and joint correlators are defined as in Eq. B.3, and the probabilities can be written in terms of these correlators as in Eq. B.4. Then, if we calculate $p_Q(a \oplus b =: c|xy)$ we obtain Eq. C.6. Now that everything is in terms of correlators E^{xy}. Replacing all of this in Eq. C.4, we obtain:

$$P_Q^{\text{win}} = \frac{1}{2}\left(1 + \frac{S}{4}\right), \tag{C.7}$$

$$S = E^{00} + E^{01} + E^{10} - E^{11}. \tag{C.8}$$

We have already seen that this S function defines the so-called CHSH-inequality $|S| \leq 2$ (Eq. 1.25), and that its violation signals the presence of nonlocality. We have bounds for the violation of the CHSH-inequality by local, quantum, and nosignalling correlations $p(ab|xy)$ (and therefore winning probabilities) given by:

$$S_C \leq 2, \qquad S_Q \leq 2\sqrt{2}, \qquad S_{NS} \leq 4,$$

$$P_C^{\text{win}} \leq \frac{3}{4} = 0.75, \qquad P_Q^{\text{win}} \leq 0.85, \qquad P_{NS}^{\text{win}} \leq 1.$$

The winning probability (P^{win}) depends on $p(ab|xy)$ which represents the pre-established strategy. If the players are allowed to implement quantum-mechanical protocols, specifically, if they are allowed to share a two-qubit CHSH-violating state, they then can outperform classical strategies. If they are allowed superquantum nonsignalling correlations, they then can even win the game with probability one! Even though general non-signalling correlations have not been completely rule out, the particular extremal nonsignalling PR box (Eq. B.8) has been ruled out by means of other information-inspired principles [11, 12, 13, 14].

C.3 The Horodeckis' CHSH Criterion

Let us now address the criterion for the violation of the CHSH-inequality for general two-qubit states. We have to maximise the violation of the CHSH-inequality function (1.25) which can be rewritten as:

$$|E_{00} + E_{01} + E_{10} - E_{11}| =: \mathrm{Tr}\left[B_{CHSH}\rho\right], \tag{C.9}$$

$$\begin{aligned} B_{CHSH} : \quad &= \quad A_0 \otimes B_0 + A_0 \otimes B_1 + A_1 \otimes B_0 - A_1 \otimes B_1, \\ &= \quad A_0 \otimes (B_0 + B_1) + A_1 \otimes (B_0 - B_1), \end{aligned}$$

being B_{CHSH} the so-called *Bell operator*, where each A_i and B_i is actiong upon qubit systems. We are interested in observables like $A_i := \vec{\sigma} \cdot \hat{a}_i$ and $B_i := \vec{\sigma} \cdot \hat{b}_i$ with $i = 0, 1$, $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ and $\hat{a}_i, \hat{b}_i$ three-dimensional unit vectors. So we have: $B_{CHSH} = \vec{\sigma} \cdot \hat{a}_0 \otimes \vec{\sigma} \cdot (\hat{b}_0 + \hat{b}_1) + \vec{\sigma} \cdot \hat{a}_1 \otimes \vec{\sigma} \cdot (\hat{b}_0 - \hat{b}_1)$. On the other hand, a general two-qubit system can be written as:

$$\rho = \frac{1}{4} \left[\mathbb{1}_2 \otimes \mathbb{1}_2 + \mathbb{1}_2 \otimes (\vec{s}_\rho \cdot \vec{\sigma}) + (\vec{r}_\rho \cdot \vec{\sigma}) \otimes \mathbb{1}_2 + \sum_{i,j=1}^{3} T_{ij} \sigma_i \otimes \sigma_j \right], \quad \text{(C.10)}$$

with $\vec{\sigma} = [\sigma_i]$, σ_i, $i = 1, 2, 3$, the Pauli matrices, $\vec{r} = [r_i]$ with $r_i := \mathrm{Tr}[(\sigma_i \otimes \mathbb{1})\rho]$, $\vec{s} = [s_i]$ with $s_i := \mathrm{Tr}[(\mathbb{1} \otimes \sigma_i)\rho]$, and $T_{ij} := \mathrm{Tr}[\rho(\sigma_i \otimes \sigma_j)]$ making the matrix $T_\rho := [T_{ij}] \in M_{3\times3}(\mathbb{R})$. Now, the idea is to maximise the function (C.9) over all of those four observables, or equivalently, over those four vectors. Replacing the general form of the two-qubit state (C.10) into C.9 and after a bit of algebra, we obtain:

$$\mathrm{Tr}\left[B_{\mathrm{CHSH}}\rho\right] = \hat{a}_0 \cdot T_\rho \left(\hat{b}_0 + \hat{b}_1\right) + \hat{a}_1 \cdot T_\rho \left(\hat{b}_0 - \hat{b}_1\right).$$

We are interested in to maximise this function, so we consider the expression:

$$\mathrm{Tr}\left[B^{\max}_{\mathrm{CHSH}}\rho\right] = \max_{\hat{a}_0, \hat{a}_1, \hat{b}_0, \hat{b}_1} \left[\hat{a}_0 \cdot T_\rho \left(\hat{b}_0 + \hat{b}_1\right) + \hat{a}_1 \cdot T_\rho \left(\hat{b}_0 - \hat{b}_1\right)\right].$$

Introducing two ortonormal vectors $\hat{c}_0, \hat{c}_1$ in terms of $\hat{b}_0$, $\hat{b}_1$ and $\cos(2\theta) = \hat{b}_0 \cdot \hat{b}_1$ as:

$$\hat{c}_0 = \frac{\hat{b}_0 + \hat{b}_1}{2\cos\theta}, \qquad \hat{c}_1 = \frac{\hat{b}_0 - \hat{b}_1}{2\sin\theta}, \quad \text{(C.11)}$$

we have:

$$\mathrm{Tr}\left[B^{\max}_{\mathrm{CHSH}}\rho\right] = 2 \max_{\hat{a}_0, \hat{a}_1, \theta, \hat{c}_0, \hat{c}_1} \left(\cos\theta\, \hat{a}_0 \cdot T_\rho \hat{c}_0 + \sin\theta\, \hat{a}_1 \cdot T_\rho \hat{c}_1\right).$$

Maximizing over $\hat{a}_0$ and $\hat{a}_1$, we can see the maximum is achieved when $\hat{a}_0 = \frac{T_\rho \hat{c}_0}{||T_\rho \hat{c}_0||}$ and $\hat{a}_1 = \frac{T_\rho \hat{c}_1}{||T_\rho \hat{c}_1||}$, then:

$$\mathrm{Tr}\left[B^{\max}_{\mathrm{CHSH}}\rho\right] = 2 \max_{\theta, \hat{c}_0, \hat{c}_1} \left(\cos\theta\, ||T_\rho \hat{c}_0|| + \sin\theta\, ||T_\rho \hat{c}_1||\right).$$

Using the trigonometric relation $\max_\theta (a\cos\theta + b\sin\theta) = \sqrt{a^2 + b^2}$ (or $\theta = \tan^{-1}\left(\frac{b}{a}\right)$), we have:

$$\begin{aligned} \mathrm{Tr}\left[B^{\max}_{\mathrm{CHSH}}\rho\right] &= 2\sqrt{\max_{\hat{c}_0, \hat{c}_1} (||T_\rho \hat{c}_0||^2 + ||T_\rho \hat{c}_1||^2)}, \\ &= 2\sqrt{\max_{\hat{c}_0} ||T_\rho \hat{c}_0||^2 + \max_{\hat{c}_1} ||T_\rho \hat{c}_1||^2}. \end{aligned}$$

T_ρ is real so $T^\dagger_\rho = T^T_\rho$, and defining $U_\rho := T^T_\rho T_\rho$, with its two biggest eigenvalues as μ_0 and μ_1, then:

$$\mathrm{Tr}\left[B^{\max}_{\mathrm{CHSH}}\rho\right] = 2\sqrt{\mu_0 + \mu_1} = 2\sqrt{M(\rho)}.$$

being $M(\rho) := \mu_0 + \mu_1 \square$.

Publications & Preprints

- *On the Activation of Quantum Nonlocality*
  Ducuara AF, Madroñero J, and Reina JH
  Universitas Scientiarum, **21** (2): 129-158, 2016
  http://revistas.javeriana.edu.co/index.php/scientarium/article/view/12834
  https://arxiv.org/abs/1609.08025
  This work is based on the results shown in section 4.3, and section 4.4.

- *Quantum Locality in Game Strategies*
  Carlos A. Melo, Cristian E. Susa, Andrés F. Ducuara, Astrid Barreiro, and John H. Reina.
  https://arxiv.org/abs/1609.06389
  Submitted to: Scientific Reports (Nature Publication Group)
  This work uses part of the results shown in section 4.5.

- *Dynamics of Nonlocality-Related Properties*
  Andrés F. Ducuara, Cristian E. Susa, and John H. Reina.
  This ongoing work is based on the preliminary results shown in section 4.6.

Bibliography

[1] A. Einstein, B. Podolsky, and N. Rosen, "Can quantum-mechanical description of physical reality be considered complete?" *Phys. Rev.*, vol. **47**, no. 10, pp. 777–780, May 1935. [Online]. Available: http://journals.aps.org/pr/abstract/10.1103/PhysRev.47.777

[2] N. Harrigan and R. W. Spekkens, "Einstein, incompleteness, and the epistemic view of quantum states," *Foundations of Physics*, vol. **40**, no. 2, pp. 125–157, 2010. [Online]. Available: http://dx.doi.org/10.1007/s10701-009-9347-0

[3] J. S. Bell, "On the einstein-podolsky-rosen paradox." *Physics*, vol. **1**, pp. 195–200, 1964. [Online]. Available: http://ebooks.cambridge.org/ebook.jsf?bid=CBO9780511815676

[4] S. J. Freedman and J. F. Clauser, "Experimental test of local hidden-variable theories," *Phys. Rev. Lett.*, vol. 28, pp. 938–941, Apr 1972. [Online]. Available: http://link.aps.org/doi/10.1103/PhysRevLett.28.938

[5] A. Aspect, J. Dalibard, and G. Roger, "Experimental test of bell's inequalities using time- varying analyzers," *Phys. Rev. Lett.*, vol. **49**, pp. 1804–1807, Dec 1982. [Online]. Available: http://link.aps.org/doi/10.1103/PhysRevLett.49.1804

[6] B. Hensen and et. al., "Loophole-free bell inequality violation using electron spins separated by 1.3 kilometres," *Nature*, vol. **526**, p. 682–686, 2015. [Online]. Available: http://www.nature.com/nature/journal/v526/n7575/full/nature15759.html

[7] M. Leifer, "Is the quantum state real? an extended review of ψ-ontology theorems," *Quanta*, vol. **3**, no. 1, pp. 67–155, 2014. [Online]. Available: http://quanta.ws/ojs/index.php/quanta/article/view/22

[8] S. Popescu and D. Rohrlich, "Quantum nonlocality as an axiom," *Foundations of Physics*, vol. **24**, no. 3, pp. 379–385, 1994. [Online]. Available: http://dx.doi.org/10.1007/BF02058098

[9] P. Rastall, "Locality, bell's theorem, and quantum mechanics," *Foundations of Physics*, vol. **15**, no. 9, pp. 963–972, 1985. [Online]. Available: http://dx.doi.org/10.1007/BF00739036

[10] L. Khalfin and B. Tsirelson, "Quantum and quasi-classical analogs of bell inequalities." *Symposium on the Foundations of Modern Physics*, pp. 441–460, 1985. [Online]. Available: http://www.tau.ac.il/~tsirel/download/khts85.html

[11] M. P. Pawowski, T. Paterek, D. Kaszlikowski, V. Scarani, and W. A. . Żukowski M., "Information causality as a physical principle," *Nature*, vol. **461**, pp. 1101–1104, 2009. [Online]. Available: http://www.nature.com/nature/journal/v461/n7267/full/nature08400.html

[12] T. Fritz, A. Sainz, R. Augusiak, J. B. Brask, R. Chaves, A. Leverrier, and A. Acín, "Local orthogonality as a multipartite principle for quantum correlations," *Nature Communications*, vol. **4**, p. 2263, 2013. [Online]. Available: http://www.nature.com/ncomms/2013/130816/ncomms3263/full/ncomms3263.html

[13] M. Navascués, Y. Guryanova, and A. Hoban, M. J. & Acín, "Almost quantum correlations," *Nature Communications*, vol. **6**, p. 6288, 2014. [Online]. Available: http://www.nature.com/ncomms/2015/150220/ncomms7288/full/ncomms7288.html

[14] W. van Dam, "Implausible consequences of superstrong nonlocality," *Natural Computing*, vol. **12**, no. 1, pp. 9–12, 2013. [Online]. Available: http://dx.doi.org/10.1007/s11047-012-9353-6

[15] A. Acín, R. Augusiak, D. Cavalcanti, C. Hadley, J. K. Korbicz, M. Lewenstein, L. Masanes, and M. Piani, "Unified framework for correlations in terms of local quantum observables," *Phys. Rev. Lett.*, vol. **104**, p. 140404, Apr 2010. [Online]. Available: http://link.aps.org/doi/10.1103/PhysRevLett.104.140404

[16] S. W. Al-Safi and A. J. Short, "Simulating all nonsignaling correlations via classical or quantum theory with negative probabilities," *Phys. Rev. Lett.*, vol. **111**, p. 170403, Oct 2013. [Online]. Available: http://link.aps.org/doi/10.1103/PhysRevLett.111.170403

[17] R. P. Feynman, "Simulating physics with computers," *Int. J. Theor. Phys.*, vol. 21, no. **6**, pp. 467–488, 1982. [Online]. Available: http://dx.doi.org/10.1007/BF02650179

[18] D. Deutsch, "Quantum theory, the church-turing principle and the universal quantum computer," *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, vol. **400**, no. 1818, pp. 97–117, 1985. [Online]. Available: http://rspa.royalsocietypublishing.org/content/400/1818/97

[19] P. Benioff, “The computer as a physical system: A microscopic quantum mechanical hamiltonian model of computers as represented by turing machines,” *J. Stat. Phys.*, vol. **22**, no. 5, pp. 563–591, 1980. [Online]. Available: http://dx.doi.org/10.1007/BF01011339

[20] Y. I. Manin, “Computable and noncomputable.” *Sov.Radio.*, p. 13–15, 1980. [Online]. Available: http://arxiv.org/abs/quant-ph/0005003v2

[21] C. H. Bennett and B. G., “Quantum cryptography: Public key distribution and coin tossing,” in *Proceedings of IEEE International Conference on Computers, Systems and Signal Processing*, 1984, pp. 175–179. [Online]. Available: http://researcher.watson.ibm.com/researcher/view_person_pubs.php?person=us-bennetc&t=1

[22] A. K. Ekert, “Quantum cryptography based on bell’s theorem,” *Phys. Rev. Lett.*, vol. **67**, pp. 661–663, Aug 1991. [Online]. Available: http://link.aps.org/doi/10.1103/PhysRevLett.67.661

[23] P. Shor, “Algorithms for quantum computation: discrete logarithms and factoring,” in *Foundations of Computer Science, 1994 Proceedings., 35th Annual Symposium on*, Nov 1994, pp. 124–134. [Online]. Available: http://ieeexplore.ieee.org/xpl/login.jsp?tp=&arnumber=365700&url=http%3A%2F%2Fieeexplore.ieee.org%2Fxpls%2Fabs_all.jsp%3Farnumber%3D365700

[24] L. K. Grover, “A fast quantum mechanical algorithm for database search,” in *Proceedings of the Twenty-eighth Annual ACM Symposium on Theory of Computing*, ser. STOC ’96. New York, NY, USA: ACM, 1996, pp. 212–219. [Online]. Available: http://doi.acm.org/10.1145/237814.237866

[25] C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W. K. Wootters, “Teleporting an unknown quantum state via dual classical and einstein-podolsky-rosen channels.” *Phys. Rev. Lett.*, vol. **70**, pp. 1895–1899, Mar 1993. [Online]. Available: http://link.aps.org/doi/10.1103/PhysRevLett.70.1895

[26] R. F. Werner, “Quantum states with einstein-podolsky-rosen correlations admitting a hidden-variable model.” *Phys. Rev. A*, vol. **40**, pp. 4277–4281, Oct 1989. [Online]. Available: http://link.aps.org/doi/10.1103/PhysRevA.40.4277

[27] R. Augusiak, M. Demianowicz, and A. Acín, “Local hidden–variable models for entangled quantum states,” *J. Phys. A: Mathematical and Theoretical*, vol. **47**, no. 42, p. 424002, 2014. [Online]. Available: http://stacks.iop.org/1751-8121/47/i=42/a=424002

[28] N. Brunner, D. Cavalcanti, S. Pironio, V. Scarani, and S. Wehner, “Bell nonlocality.” *Rev. Mod. Phys.*, vol. **86**, pp. 419–478, Apr 2014. [Online]. Available: http://link.aps.org/doi/10.1103/RevModPhys.86.419

[29] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, "Quantum entanglement." *Rev. Mod. Phys.*, vol. **81**, pp. 865–942, Jun 2009. [Online]. Available: http://link.aps.org/doi/10.1103/RevModPhys.81.865

[30] H. M. Wiseman, "The two bell's theorems of john bell," *J. Phys A: Mathematical and Theoretical*, vol. **47**, no. 42, p. 424001, 2014. [Online]. Available: http://stacks.iop.org/1751-8121/47/i=42/a=424001

[31] B. C. Hall, *Quantum Theory for Mathematicians.* Springer New York, 2013. [Online]. Available: http://link.springer.com/book/10.1007%2F978-1-4614-7116-5

[32] J. Watrous, "Lecture notes: Theory of quantum information," 2014. [Online]. Available: https://cs.uwaterloo.ca/~watrous/LectureNotes.html

[33] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information.* Cambridge University Press, Cambridge, UK, 2010. [Online]. Available: https://books.google.com.co/books/about/Quantum_Computation_and_Quantum_Informat.html?id=65FqEKQOfP8C&redir_esc=y

[34] J. Preskill, "Lecture notes at caltech: Quantum computation," 2014. [Online]. Available: http://www.theory.caltech.edu/people/preskill/ph229/

[35] S. Pironio, "Aspects of quantum nonlocality," Ph.D. dissertation, Université Libre de Bruxelles, 2004. [Online]. Available: http://quic.ulb.ac.be/_media/publications/2004-thesis-stefano.pdf

[36] M. T. Quintino, "Black box correlations: Locality, noncontextuality, and convex polytopes," Master's thesis, Universidade Federal de Minas Gerais, 2012. [Online]. Available: http://www.mat.ufmg.br/~tcunha/Disserta_MTQ.pdf

[37] V. Scarani, *The device-independent outlook on quantum physics*, ser. Acta physica Slovaca. Inst. of Physics, Slovak Acad. of Sciences, 2012. [Online]. Available: https://books.google.com.co/books?id=b3GPoAEACAAJ

[38] Y.-C. Liang, "Correlations, bell inequality violation & quantum entanglement," Ph.D. dissertation, University of Queensland, 2008. [Online]. Available: http://arxiv.org/abs/0810.5400v1

[39] G. M. Guimarães, "Não-localidade em sistemas quânticos," Master's thesis, Universidade Federal de Minas Gerais, 2012. [Online]. Available: https://docs.google.com/viewer?url=http%3A%2F%2Fwww.mat.ufmg.br%2F~tcunha%2FDissGlaucia.pdf

[40] J. Pusey, M. F. Barrett and T. Rudolph, "On the reality of the quantum state," *Nature Physics*, vol. **8**, pp. 475–78, 2012. [Online]. Available: http://www.nature.com/nphys/journal/v8/n6/full/nphys2309.html

[41] D. Jennings and M. Leifer, "No return to classical reality," *Contemporary Physics*, pp. 1–23, 2015. [Online]. Available: http://www.tandfonline.com/doi/abs/10.1080/00107514.2015.1063233?journalCode=tcph20

[42] A. Fine, "Hidden variables, joint probability, and the bell inequalities," *Phys. Rev. Lett.*, vol. **48**, pp. 291–295, Feb 1982. [Online]. Available: http://link.aps.org/doi/10.1103/PhysRevLett.48.291

[43] M. L. Almeida, J.-D. Bancal, N. Brunner, A. Acín, N. Gisin, and S. Pironio, "Guess your neighbor's input: A multipartite nonlocal game with no quantum advantage," *Phys. Rev. Lett.*, vol. **104**, p. 230404, Jun 2010. [Online]. Available: http://link.aps.org/doi/10.1103/PhysRevLett.104.230404

[44] R. Augusiak, T. Fritz, M. Kotowski, M. Kotowski, M. Pawłowski, M. Lewenstein, and A. Acín, "Tight bell inequalities with no quantum violation from qubit unextendible product bases," *Phys. Rev. A*, vol. **85**, p. 042113, Apr 2012. [Online]. Available: http://link.aps.org/doi/10.1103/PhysRevA.85.042113

[45] R. Horodecki, P. Horodecki, and M. Horodecki, "Violating bell inequality by mixed spin-12 states: necessary and sufficient condition." *Physics Letters A*, vol. **200**, no. 5, pp. 340 – 344, 1995. [Online]. Available: http://www.sciencedirect.com/science/article/pii/037596019500214N

[46] N. Gisin, "Bell's inequality holds for all non-product states." *Physics Letters A*, vol. **154**, no. 5-6, pp. 201 – 202, 1991. [Online]. Available: http://www.sciencedirect.com/science/article/B6TVM-46T4M8S-14M/2/2da119bcc5ba644094925ebdeffc8de1

[47] N. Gisin and A. Peres, "Maximal violation of bell's inequality for arbitrarily large spin." *Physics Letters A*, vol. **162**, no. 1, pp. 15 – 17, 1992. [Online]. Available: http://www.sciencedirect.com/science/article/pii/037596019290949M

[48] S. Popescu and D. Rohrlich, "Generic quantum nonlocality." *Physics Letters A*, vol. **166**, no. 5-6, pp. 293 – 297, 1992. [Online]. Available: http://www.sciencedirect.com/science/article/pii/037596019290711T

[49] J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, "Proposed experiment to test local hidden-variable theories." *Phys. Rev. Lett.*, vol. **23**, pp. 880–884, Oct 1969. [Online]. Available: http://link.aps.org/doi/10.1103/PhysRevLett.23.880

[50] B. S. Cirel'son, "Quantum generalizations of bell's inequality," *Letters in Mathematical Physics*, vol. **4**, no. 2, pp. 93–100, 1980. [Online]. Available: http://dx.doi.org/10.1007/BF00417500

[51] A. Peres, "Separability criterion for density matrices." *Phys. Rev. Lett.*, vol. **77**, pp. 1413–1415, Aug 1996. [Online]. Available: http://link.aps.org/doi/10.1103/PhysRevLett.77.1413

[52] C. H. Bennett, D. P. DiVincenzo, J. A. Smolin, and W. K. Wootters, "Mixed-state entanglement and quantum error correction." *Phys. Rev. A*, vol. **54**, pp. 3824–3851, Nov 1996. [Online]. Available: http://link.aps.org/doi/10.1103/PhysRevA.54.3824

[53] W. K. Wootters, "Entanglement of formation of an arbitrary state of two qubits." *Phys. Rev. Lett.*, vol. **80**, pp. 2245–2248, Mar 1998. [Online]. Available: http://link.aps.org/doi/10.1103/PhysRevLett.80.2245

[54] K. G. H. Vollbrecht and R. F. Werner, "Entanglement measures under symmetry." *Phys. Rev. A*, vol. **64**, p. 062307, Nov 2001. [Online]. Available: http://link.aps.org/doi/10.1103/PhysRevA.64.062307

[55] B. M. Terhal and K. G. H. Vollbrecht, "Entanglement of formation for isotropic states." *Phys. Rev. Lett.*, vol. **85**, pp. 2625–2628, Sep 2000. [Online]. Available: http://link.aps.org/doi/10.1103/PhysRevLett.85.2625

[56] W. K. Wootters, "Entanglement of formation and concurrence." *Quantum Info. Comput.*, vol. **1**, no. 1, pp. 27–44, Jan. 2001. [Online]. Available: http://dl.acm.org/citation.cfm?id=2011326.2011329

[57] A. Miranowicz, "Violation of bell inequality and entanglement of decaying werner states." *Physics Letters A*, vol. **327**, no. 4, pp. 272 – 283, 2004. [Online]. Available: http://www.sciencedirect.com/science/article/pii/S0375960104006322

[58] S. Popescu, "Bell's inequalities versus teleportation: What is nonlocality?" *Phys. Rev. Lett.*, vol. **72**, pp. 797–799, Feb 1994. [Online]. Available: http://link.aps.org/doi/10.1103/PhysRevLett.72.797

[59] M. Horodecki, P. Horodecki, and R. Horodecki, "General teleportation channel, singlet fraction, and quasidistillation." *Phys. Rev. A*, vol. **60**, pp. 1888–1898, Sep 1999. [Online]. Available: http://link.aps.org/doi/10.1103/PhysRevA.60.1888

[60] J. Grondalski, D. Etlinger, and D. James, "The fully entangled fraction as an inclusive measure of entanglement applications." *Physics Letters A*, vol. 300, no. **6**, pp. 573 – 580, 2002. [Online]. Available: http://www.sciencedirect.com/science/article/pii/S0375960102008848

[61] R. Horodecki, M. Horodecki, and P. Horodecki, "Teleportation, bell's inequalities and inseparability." *Physics Letters A*, vol. **222**, no. 1-2, pp. 21–25, 1996. [Online]. Available: http://www.sciencedirect.com/science/article/pii/0375960196006391

[62] M.-J. Zhao, Z.-G. Li, S.-M. Fei, and Z.-X. Wang, "A note on fully entangled fraction." *J. Phys. A: Mathematical and Theoretical*, vol. **43**, no. 27, p. 275203, 2010. [Online]. Available: http://stacks.iop.org/1751-8121/43/i=27/a=275203

[63] T. Vérseti and N. Brunner, "Disproving the peres conjecture by showing bell nonlocality from bound entanglement," *Nature Communications*, vol. **5**, p. 5297, 2014. [Online]. Available: http://www.nature.com/ncomms/2014/141105/ncomms6297/full/ncomms6297.html

[64] M. Horodecki and P. Horodecki, "Reduction criterion of separability and limits for a class of distillation protocols." *Phys. Rev. A*, vol. **59**, pp. 4206–4216, Jun 1999. [Online]. Available: http://link.aps.org/doi/10.1103/PhysRevA.59.4206

[65] F. Hirsch, M. T. Quintino, J. Bowles, and N. Brunner, "Genuine hidden quantum nonlocality." *Phys. Rev. Lett.*, vol. **111**, p. 160402, Oct 2013. [Online]. Available: http://link.aps.org/doi/10.1103/PhysRevLett.111.160402

[66] D. Cavalcanti, L. Guerini, R. Rabelo, and P. Skrzypczyk, "General method for constructing local-hidden-state (and -variable) models for multiqubit entangled states," 2015. [Online]. Available: http://arxiv.org/abs/1512.00277

[67] F. Hirsch, M. T. Quintino, T. Vértesi, M. F. Pusey, and N. Brunner, "Algorithmic construction of local hidden variable models for entangled quantum states," 2015. [Online]. Available: http://arxiv.org/abs/1512.00262

[68] J. Barrett, "Nonsequential positive-operator-valued measurements on entangled mixed states do not always violate a bell inequality," *Phys. Rev. A*, vol. **65**, p. 042302, Mar 2002. [Online]. Available: http://link.aps.org/doi/10.1103/PhysRevA.65.042302

[69] A. Acín, N. Gisin, and B. Toner, "Grothendieck's constant and local models for noisy entangled quantum states," *Phys. Rev. A*, vol. **73**, p. 062105, Jun 2006. [Online]. Available: http://link.aps.org/doi/10.1103/PhysRevA.73.062105

[70] T. Vértesi, "More efficient bell inequalities for werner states," *Phys. Rev. A*, vol. **78**, p. 032112, Sep 2008. [Online]. Available: http://link.aps.org/doi/10.1103/PhysRevA.78.032112

[71] B. Hua, M. Li, T. Zhang, C. Zhou, X. Li-Jost, and S.-M. Fei, "Towards grothendieck constants and lhv models in quantum mechanics." *J. Phys. A: Mathematical and Theoretical*, vol. **48**, no. 6, p. 065302, 2015. [Online]. Available: http://stacks.iop.org/1751-8121/48/i=6/a=065302

[72] S. Popescu, “Bell’s inequalities and density matrices: Revealing “hidden” nonlocality.” *Phys. Rev. Lett.*, vol. **74**, pp. 2619–2622, Apr 1995. [Online]. Available: http://link.aps.org/doi/10.1103/PhysRevLett.74.2619

[73] A. Acín, N. Gisin, and L. Masanes, “From bell’s theorem to secure quantum key distribution.” *Phys. Rev. Lett.*, vol. **97**, p. 120405, Sep 2006. [Online]. Available: http://link.aps.org/doi/10.1103/PhysRevLett.97.120405

[74] N. Gisin, “Hidden quantum nonlocality revealed by local filters.” *Physics Letters A*, vol. **210**, no. 3, pp. 151 – 156, 1996. [Online]. Available: http://www.sciencedirect.com/science/article/pii/S0375960196800016

[75] R. Pal and S. Ghosh, “Non-locality breaking qubit channels: the case for chsh inequality.” *J. Phys. A: Mathematical and Theoretical*, vol. **48**, no. 15, p. 155302, 2015. [Online]. Available: http://stacks.iop.org/1751-8121/48/i=15/a=155302

[76] A. Peres, “Collective tests for quantum nonlocality.” *Phys. Rev. A*, vol. **54**, pp. 2685–2689, Oct 1996. [Online]. Available: http://link.aps.org/doi/10.1103/PhysRevA.54.2685

[77] L. Masanes, “Asymptotic violation of bell inequalities and distillability.” *Phys. Rev. Lett.*, vol. **97**, p. 050503, Aug 2006. [Online]. Available: http://link.aps.org/doi/10.1103/PhysRevLett.97.050503

[78] L. Masanes, Y.-C. Liang, and A. C. Doherty, “All bipartite entangled states display some hidden nonlocality.” *Phys. Rev. Lett.*, vol. **100**, p. 090403, Mar 2008. [Online]. Available: http://link.aps.org/doi/10.1103/PhysRevLett.100.090403

[79] Y.-C. Liang, L. Masanes, and D. Rosset, “All entangled states display some hidden nonlocality.” *Phys. Rev. A*, vol. **86**, p. 052115, Nov 2012. [Online]. Available: http://link.aps.org/doi/10.1103/PhysRevA.86.052115

[80] M. Navascués and T. Vértesi, “Activation of nonlocal quantum resources.” *Phys. Rev. Lett.*, vol. **106**, p. 060403, Feb 2011. [Online]. Available: http://link.aps.org/doi/10.1103/PhysRevLett.106.060403

[81] C. Palazuelos, “Superactivation of quantum nonlocality.” *Phys. Rev. Lett.*, vol. **109**, p. 190401, Nov 2012. [Online]. Available: http://link.aps.org/doi/10.1103/PhysRevLett.109.190401

[82] D. Cavalcanti, A. Acín, N. Brunner, and T. Vértesi, “All quantum states useful for teleportation are nonlocal resources.” *Phys. Rev. A*, vol. **87**, p. 042104, Apr 2013. [Online]. Available: http://link.aps.org/doi/10.1103/PhysRevA.87.042104

[83] L.-X. Cen, F.-L. Li, and S.-Y. Zhu, “Optimal entanglement manipulation for an arbitrary state of two qubits,” *Physics Letters A*, vol. **275**, no. 5–6, pp. 368 – 372, 2000. [Online]. Available: http://www.sciencedirect.com/science/article/pii/S0375960100006113

[84] F. Verstraete, "A study of entanglement in quantum information theory," Ph.D. dissertation, Katholieke Universiteit Leuven, 2002. [Online]. Available: http://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.6.924

[85] C. H. Bennett, G. Brassard, S. Popescu, B. Schumacher, J. A. Smolin, and W. K. Wootters, "Purification of noisy entanglement and faithful teleportation via noisy channels." *Phys. Rev. Lett.*, vol. **76**, pp. 722–725, Jan 1996. [Online]. Available: http://link.aps.org/doi/10.1103/PhysRevLett.76.722

[86] M. Horodecki, P. Horodecki, and R. Horodecki, "Separability of mixed states: necessary and sufficient conditions." *Physics Letters A*, vol. **223**, no. 1-2, pp. 1 – 8, 1996. [Online]. Available: http://www.sciencedirect.com/science/article/pii/S0375960196007062

[87] L. Vandenberghe and S. Boyd, "Semidefinite programming." *SIAM Review*, vol. **38**, no. 1, pp. pp. 49–95, 1996. [Online]. Available: http://www.jstor.org/stable/2132974

[88] MATLAB, *version 7.13.0.564 (R2011b).* Natick, Massachusetts: The MathWorks Inc., 2011. [Online]. Available: http://uk.mathworks.com/products/matlab/index.html?s_tid=gn_loc_drop

[89] J. Löfberg, "YALMIP : A toolbox for modeling and optimization in MATLAB." in *CCA/ISIC/CACSD*, Sep. 2004. [Online]. Available: http://control.ee.ethz.ch/index.cgi?action=details;id=2088;page=publications

[90] J. Sturm, "Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones." *Optimization Methods and Software*, vol. **11–12**, pp. 625–653, 1999. [Online]. Available: http://www.tandfonline.com/doi/abs/10.1080/10556789908805766

[91] L.-X. Cen, N.-J. Wu, F.-H. Yang, and J.-H. An, "Local transformation of mixed states of two qubits to bell diagonal states," *Phys. Rev. A*, vol. **65**, p. 052318, May 2002. [Online]. Available: http://link.aps.org/doi/10.1103/PhysRevA.65.052318

[92] E. Isasi, D. Mundarain, and J. Stephany, "On the dynamics of maximum extractable entanglement for open systems," *J. Phys. B: Atomic, Molecular and Optical Physics*, vol. **41**, no. 23, p. 235504, 2008. [Online]. Available: http://stacks.iop.org/0953-4075/41/i=23/a=235504

[93] F. Verstraete and M. M. Wolf, "Entanglement versus bell violations and their behavior under local filtering operations." *Phys. Rev. Lett.*, vol. **89**, p. 170401, Oct 2002. [Online]. Available: http://link.aps.org/doi/10.1103/PhysRevLett.89.170401

[94] D. Collins, N. Gisin, N. Linden, S. Massar, and S. Popescu, "Bell inequalities for arbitrarily high-dimensional systems." *Phys. Rev. Lett.*, vol. **88**, p. 040404, Jan 2002. [Online]. Available: http://link.aps.org/doi/10.1103/PhysRevLett.88.040404

[95] C. E. Susa and J. H. Reina, "Nonlocal fluctuations and control of dimer entanglement dynamics," *Phys. Rev. A*, vol. **82**, p. 042102, Oct 2010. [Online]. Available: http://link.aps.org/doi/10.1103/PhysRevA.82.042102

[96] ——, "Correlations in optically controlled quantum emitters," *Phys. Rev. A*, vol. **85**, p. 022111, Feb 2012. [Online]. Available: http://link.aps.org/doi/10.1103/PhysRevA.85.022111

[97] M. Horodecki, P. Horodecki, and R. Horodecki, "Mixed-state entanglement and distillation: Is there a "bound" entanglement in nature?" *Phys. Rev. Lett.*, vol. **80**, pp. 5239–5242, Jun 1998. [Online]. Available: http://link.aps.org/doi/10.1103/PhysRevLett.80.5239