BOSONIZATION METHOD IN THE STUDY OF THE FULDE-FERRELL-LARKIN-OVCHINNIKOV PHASE

MÉTODO DE BOSONIZACIÓN EN EL ESTUDIO DE LA FASE FULDE-FERRELL-LARKIN-OVCHINNIKOV

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To my parents, who valued education above all else

Abstract

In the last 40 years to develop a microscopic theory that explains high temperature superconductivity has become one of the most intriguing challenges for theoretical condensed matter physics. One of the questions that accompanies this challenge is to what extent superconductivity can survive in the presence of a magnetic field, or equivalently, whether it is possible for a system to exhibit superconducting properties with unpaired fermions. The Fulde-Ferrell-Larkin-Ovchinnikov (FFLO) phase is one among multiple theoretical proposals of quantum phases where this is possible. Although this proposal dates back to mid 60's, it is only nowadays with the advent of the third quantum revolution that it is possible to attain experimental realizations of this phase in the laboratory using, among other interesting systems, one-dimensional ultracold optical lattices. Thus, we are motivated to investigate this phase using a very special non-perturbative technique restricted to one-dimensional systems, namely, bosonization. In this work we perform a comprehensive presentation of this formalism, and discuss its application to the asymmetric Hubbard model. Then we use these developments to tackle the FFLO model, finding an effective low-energy Hamiltonian for it, along with the canonical transformation that diagonalizes it. Along the way we show how to explicitly handle with the Klein factors, a chief ingredient of the bosonization program. Ultimately, we employ functional integration to compute the singlet superconducting correlation function and analyze its decay behavior, which turns out to be of the form $x^{-\xi}$. By means of this, we find that many of the signatures of the FFLO phase are sustained by the attractive Fermi model, out of half-filling, namely, the dominance of pairing of finite momentum in the ordering of the system for intermediate values of the magnetic field, and a tendency to no longer display superconducting properties for large values of the attractive interaction and the magnetic field. More specifically, for an interaction of $U = -3$ we find that our model is reliable up to a magnetic field of $h = 2.7$, above this value we get non-sensible outputs, namely, complex propagation velocities for the fields. Thus, we interpret this as an indication of a breakdown of any superconducting behavior of the system.

Keywords: Bosonization, superconductivity, FFLO phase, Klein factors, correlation functions.

Resumen

En los últimos 40 años desarrollar una teoría microscópica que explique el fenómeno de superconductividad a altas temperaturas se ha convertido en uno de los retos más intrigantes de la física teórica de la materia condensada. Una de las preguntas que acompaña este desafío es hasta qué punto la superconductividad puede sobrevivir en presencia de un campo magnético, o de manera equivalente, si es posible que un sistema exhiba propiedades superconductoras con algunos de los fermiones desemparejados. La fase de Fulde-Ferrell-Larkin-Ovchinnikov (FFLO) es una de las múltiples propuestas teóricas de fases cuánticas donde esto es posible. A pesar de que esta propuesta se remonta a mediados de los años 60, es solo ahora con la llegada de la tercera revolución cuántica que es posible obtener realizaciones experimentales de esta fase en el laboratorio, usando, entre otros sistemas interesantes, redes ópticas ultrafrías unidimensionales. Es justamente esto último lo que nos motiva a investigar esta fase utilizando una técnica perturbativa muy especial, a saber, bosonización. En el presente trabajo llevamos a cabo una presentación detallada de este formalismo, y discutimos su aplicación al modelo de Hubbard asimétrico. Luego, nos apoyamos en los desarrollos obtenidos para abordar el modelo FFLO, encontrando un hamiltoniano efectivo para bajas energías, junto con la transformación canónica que lo diagonaliza. En el camino mostramos cómo lidiar con los factores de Klein, un ingrediente esencial dentro de la técnica de bosonización. Finalmente, empleamos integración funcional para calcular la función de correlación de un superconductor tipo singlete, y analizamos su decaimiento que resulta ser de la forma $x^{-\xi}$. A partir de esto encontramos que muchas de las características de la fase FFLO están presentes en el modelo de Fermi atractivo, fuera de medio llenado, a saber, la preponderancia de emparejamientos con momento finito en el ordenamiento del sistema para valores intermedios del campo magnético, y una tendencia para no manifestar propiedades superconductoras para valores grandes de la interacción atractiva y el campo magnético. Más específicamente, para una interacción de $U = -3$ encontramos que nuestro modelo es confiable hasta un valor del campo magnético de $h = 2.7$, por encima de este valor obtenemos resultados patológicos, a saber, velocidades de propagación complejas para uno de los campos. Así, interpretamos esto como un indicio de que el sistema ya no presenta ningún comportamiento superconductor.

Palabras clave: Bosonización, superconductividad, fase FFLO, factores de Klein, funciones de correlación.

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Chapter 1

Introduction

The only existing microscopic theory of superconductivity (superfluidity of charged particles) dates back to 1957 with the groundbreaking work of Bardeen, Cooper, and Schrieffer (BCS) [1], who explained this phenomenon as the condensation of paired fermions, called Cooper pairs, due to a phonon-mediated attractive interaction induced by the lattice structure of the material. However, since then technological progress has made possible the discovery of a wide variety of superconducting materials that cannot be described by the BCS theory, e.g. high-temperature superconductors, that is, materials with critical temperature above the boiling point of liquid nitrogen (77 K), first discovered in 1986 by Bednorz and Müller [2]. All those superconducting materials that cannot be explained microscopically using BCS theory are said to display *unconventional superconductivity*.

In the BCS theory Cooper pairs are often made of fermions of different species, namely, two fermions with opposite spins (singlet configuration) and opposite momenta. Thus, the most favorable situation for pairing is when the densities of the two species are the same, so that there are no unpaired fermions in the ground state of the system. Then, some questions arise straightaway: To what extent superconductivity can survive to deviations of such most favorable situation? It is possible for a system to display superconductivity even in the presence of such an *unbalanced pairing*? It turns out that these questions are of great importance even out of the condensed matter community. For instance, in a system of high density quark or nuclear matter as in the core of a neutron star, it is expected that unbalanced pairing between quarks and nucleons takes place, giving rise to a new phase of matter called color superconductivity [3–5].

For a singlet superconductor a relatively simple experimental way to induce such an imbalance in the density of the fermion species is to put the system in the presence of a magnetic field. Magnetism and superconductivity are natural enemies, as a magnetic field destroys superconductivity in two ways, either by coupling to the spin or to the orbital degrees of freedom of the fermions. In the latter case, the magnetic field spoils the known property of superconductors of displaying diamagnetism (Meissner effect) by progressively creating vortices in the material through which the magnetic flux lines penetrate [6]. This situation is depicted in figure 1.1. On the spin degrees of freedom the magnetic field lifts the energy

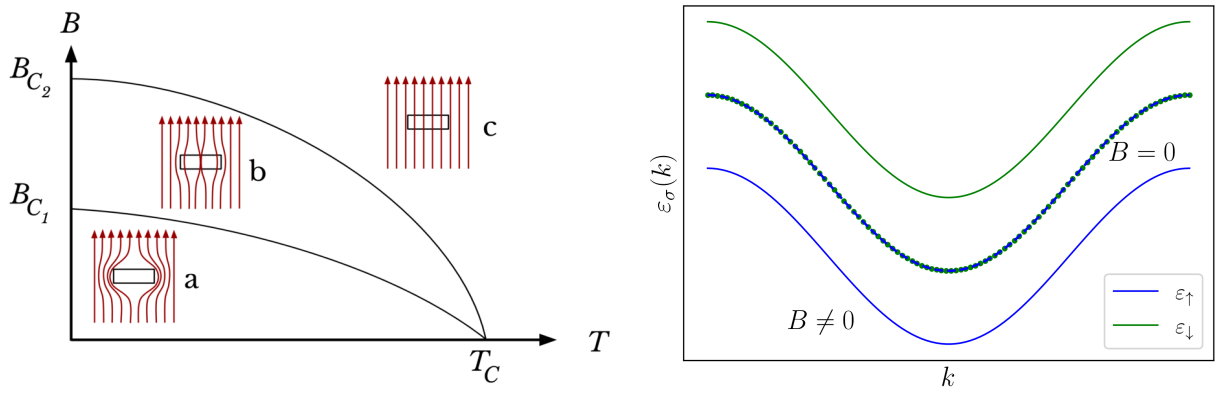

Figure 1.1: Left: Orbital effect of a magnetic field in a type-II superconductor. The density of magnetic flux lines that penetrate the sample increases as the intensity of the magnetic field increases. Above the second critical field B_{C_2} there is a transition to a normal (non-superconducting) state. Right: Zeeman splitting of the energy levels. In this case the magnetic field h is pointing to the $+z$ direction so that the energy of spin-up electrons gets lowered.

degeneracy for the spin-up and -down electrons through the Zeeman effect, as shown in figure 1.1. Consequently, the densities of the species of fermions are going to change, increasing the number of electrons with spin-up, and giving rise to an imbalanced sample.

Thus, to get an experimental realization of such an imbalanced sample in any solid state material we have to manage to overcome the orbital effects and highlight the Zeeman effect. The Maki parameter α_M quantifies the relative relevance of both effects [7, 8],

$$\alpha_M \equiv \sqrt{2}\frac{H_{C_2}^{\mathrm{orb}}}{H_{C_2}^{\mathrm{P}}}, \tag{1.0.1}$$

with $H_{C_2}^{\mathrm{orb}}$ the critical field for a type-II superconductor only including orbital effects, and $H_{C_2}^{\mathrm{P}}$ the critical field in the Pauli paramagnetic limit (when the spins are paired up with the magnetic field). It was proved that one needs $\alpha_M > 1.8$ in order to the spin effects to be dominant [9].

To reach the Pauli paramagnet we need to destroy the singlet configuration and thus break the Cooper pairs, that is, the magnetic field h must overcome the Cooper pair binding energy Δ. This is called the Chandrasekhar-Clogston limit and is reached for $h = \Delta/\sqrt{2}$ [10, 11]. It was believed that above this limit there was a transition to a normal state since Cooper pairs were being broken. Fulde and Ferrel [12] and Larkin and Ovchinnikov [13] independently proposed that it was possible for superconductivity to survive beyond this limit if the Cooper pairs develop finite pairing momentum. That way the order parameter would be spatially modulated, $\Delta(x)$, and both superconductivity and magnetism survive by periodically avoiding each other [14–16], as depicted in figure 1.2. This superconducting phase with such an *unconventional* pairing is called the FFLO phase.

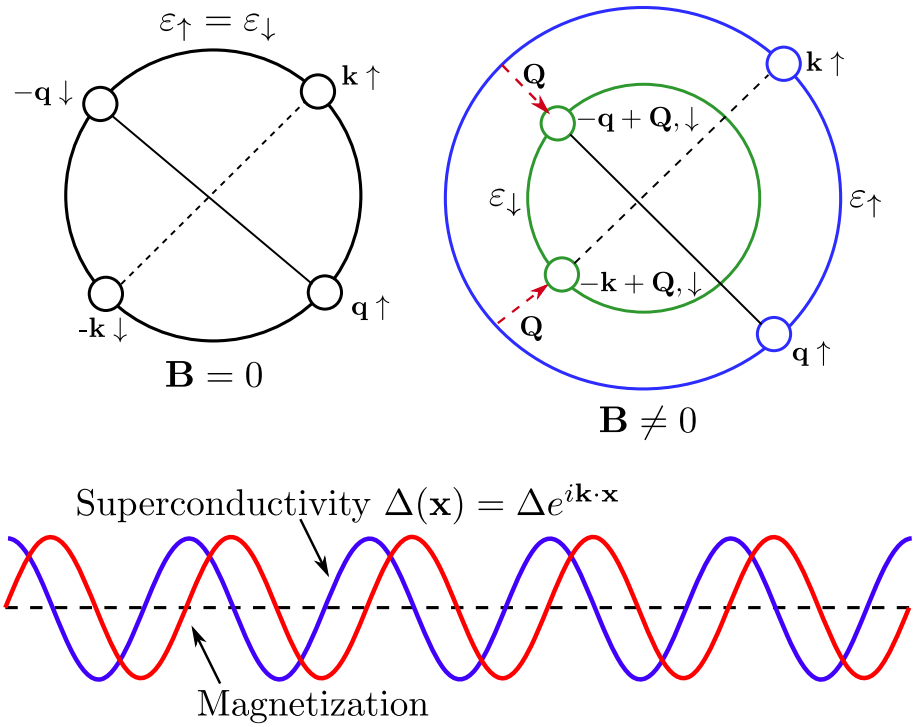

Figure 1.2: Top: The Zeeman field splits the Fermi surface into two, making that any pairing between fermions at these new surfaces must have finite center-of-mass momentum. Bottom: Such a pairing gives rise to a spatially modulated order parameter so that magnetism emerges precisely where superconductivity is absent, and vice versa. This way the system can sustain both phenomena.

A conclusive finding of the FFLO phase has been elusive to experiments for many years. The two main reasons for this are disorder and the dominance of the orbital effects. Indeed the condition $\alpha_M > 1.8$ is such a strong restriction for the type of materials one can consider to get this phase. There are also restrictions on the dimension of the materials one can prove, for eddy currents due to the magnetic field enhance the orbital effects in 3-dimensional materials, making it impossible to reach the FFLO phase [17].

Thus, any experimental evidence for this phase is to be searched in two types of systems: i) Electronic solid state 2-dimensional or 1-dimensional layered materials as heavy fermions $CeCoIn_5$ and $CeCo_2Si_2$ [18–20], organic superconductors [21–23], and Fe-based superconductors [24, 25]. ii) Ultracold atomic gases [26–28]. The latter are of special interest because they are neutral systems, and hence, the orbital effects are completely absent. Moreover, they are highly tunable, for instance, the pseudo-spin of the atoms is fixed in the set up and one can move in the space of parameters from a superfluid state to the boundary where the FFLO is expected to be, by tuning the Feshbach resonances [4, 17, 29]. It is precisely through these highly-tunable systems that one can *simulate* and study more general physical scenarios that eventually allow us to draw conclusions applicable to other realms of physics, e.g. color superconductivity.

In spite of this variety of probe systems, as mentioned before, the experimental evidence

is far from conclusive. For instance, thermodynamic measurements in the heavy fermion compound $CeCoIn_5$ [30, 31] permit to conclude that there exists a new phase inside the superconducting state that the authors claim to be the FFLO phase. However, this phase exhibits some features opposite to what we expect from the FFLO, more specifically, an antiferromagnetic order [32,33]. Even more dramatically, in Ref. [34] the authors found $(dS/dh)_T$ for this system by measuring the magnetic Grüneisen ratio and the specific heat. They detected a broadened downward discontinuity in $(dS/dh)_T$ indicating a negative contribution to the entropy, contrary to what is expected for the FFLO phase due to the emergence of paramagnetic states. Their experimental data also rule out the possibility of the coexistence of this antiferromagnetic phase with the FFLO phase.

Roughly speaking, what we are currently able to do is to find a new state inside the superconducting phase that may have, to a greater or lower extent, some of the features we expect from the FFLO phase. However, more direct evidence is far from attainable with current technologies. For instance, spatial anisotropy in the order parameter, one of the main features of the FFLO model, has never been experimentally observed [14]. That is partially why within the condensed matter community there exist other proposals of exotic phases characterized by a finite pairing momentum, including deformed Fermi surface pairing (DFSP) [35, 36] states and breached pairing (BP) states [37–39].

We have decided to focus on the study of the FFLO phase because currently it is the phase more enthusiastically explored in the light of one-dimensional ultracold atomic gases. They are precisely these one-dimensional realizations what enable us to use a very special non-perturbative analytic technique, namely, bosonization.

The present work is organized as follows: In Chapter 2, we present the bosonization program starting from an heuristic point of view and arriving to the sophisticated field-theoretical formulation. The latter is applied in section 2.5 to the asymmetric Hubbard model, which is introduced as an intermediate step to reach the goal of this work. Chapter 3 is devoted to the study of our system of interest, that is, the FFLO phase in attractive 1D spin-1/2 chains. We shall see how the work on the asymmetric Hubbard model gives us valuable insights to tackle this system. In that fashion, we find a low-energy effective Hamiltonian for the system and the set of canonical fields that lead us to diagonalize it. This in turn allows us to compute the singlet superconducting correlation function with the aim of finding non-zero momentum superconducting pairs. Finally, in Chapter 4 we present the conclusions and perspectives of the work.

Chapter 2

Bosonization

Bosonization is an useful technique to study strongly-correlated electron systems in one dimension, though it was developed simultaneously by condensed matter and particle physicists. The latter were initially interested in a low-dimensional toy model for strong interactions, and later on the applications of bosonization to string theory [40, 41]. In the context of strongly correlated systems, the idea behind this technique is that at low energies the collective excitations of the system, i.e., particle-hole excitations, have bosonic nature [42], so that employing a bosonic language highlights the physics of the system as well as allows us to compute quantities like correlation functions more easily.

Such a prolific technique may represent a major challenge for the beginner researcher, due to the fact that part of the literature requires some background on field theoretical methods applied to condensed matter systems, see e.g. [43–46], which is a demanding prerequisite for senior undergraduates. And, although the other part of the literature follows what Delft and Schoeller call the "constructive approach" to bosonization [47], that is, an operator formalism, there are very different conventions for defining the Hilbert spaces and the field operators, see e.g. [47–49].

In this chapter we adopt a balanced position: although we do not drop the field theoretical insights altogether, we prioritize the constructive approach and try to shed some light on the motivations to choose one convention before another. Our goal is to make the presentation of the technique as self-contained as possible for an undergraduate student familiar with second quantization and lattice systems, by making quantum field theory an appealing credential, not a prerequisite.

2.1 Some words on one dimension

Bosonization is restricted to one-dimensional systems, even though people have tried to generalize it to higher dimensions [50, 51]. In two and three dimensions most of the systems of interacting fermions at low temperatures are described by the Landau-Fermi liquid theory [52]. There, excited states are obtained by exciting a certain number of particles across the Fermi surface, and all these states can be thought of as different configurations of some

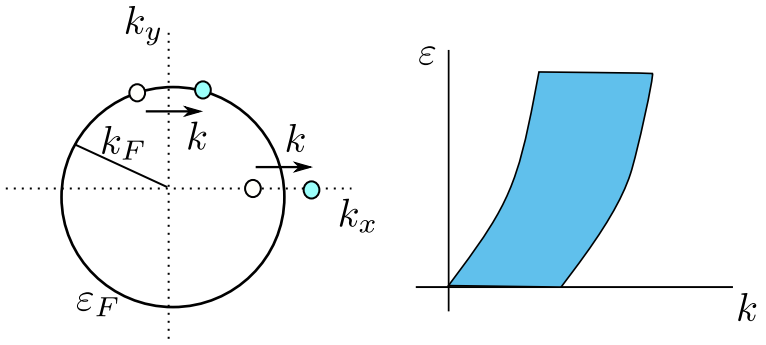

Figure 2.1: Particle-hole excitations in two dimensions. The existence of several allowed energies for the same value of k in the spectrum of the particle-hole excitations (figure in the right) spoils a collective behavior of the low-energy excitations [40].

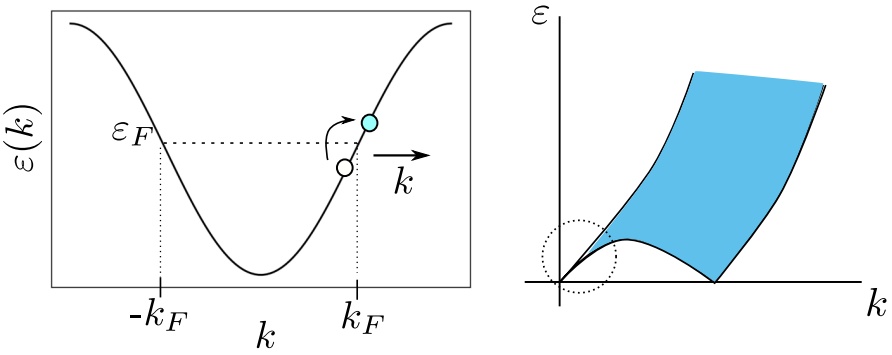

Figure 2.2: Particle-hole excitations in one dimension. The existence of a linear dispersion relation at low energies allows these elementary excitations to propagate coherently.

elementary excitations, namely, the creation of particles or holes just above or below the Fermi surface, respectively. Due to the interactions these elementary excitations are 'dressed' by other particles forming a new entity, a quasi-particle: an excited particle or hole surrounded by a cloud of particles or holes. This is true as long as the interactions are turned on adiabatically but fast enough to not overcome the lifetime of the quasi-particles. Equivalently, it is true as long as we remain close to the Fermi surface. Most of the elementary excitations do not have a linear dispersion relation as shown in the figure 2.1. For a given momentum k there is a continuum range of allowed energies ε, which prevents the quasi-particles to propagate coherently.

In one dimension the picture is quite different due to the unavoidable presence of collective phenomena. Landau-Fermi liquid theory breaks down in one dimension and we have to rely on a new framework, namely, the Tomonaga-Luttinger theory [53,54]. For a Tomonga-Luttinger liquid *all* the low-energy excitations are particle-hole excitations with linear dispersion relation (see figure 2.2), so they can be regarded as sound waves capable of propagating coherently and give rise to new particles: our future bosons.

Most part of the fun lies in one dimension because the effect of the interactions is the strongest and there are high odds to obtain novel physics. There are several methods to study these

low-dimensional systems, either exact methods as the Bethe ansatz [55] or numerical as the Density Matrix Renormalization Group method (DMRG) [56–58], and the correlated Gaussian method [59]. However, for both types of methods the spectrum obtained is limited to very special models, or as in the numerical methods, one has limitations in the number of degrees of freedom, and has to deal with finite-size effects in the system [60]. Bosonization emerges then as a clever alternative since it is a non-perturbative method, and hence we are able to extract the *exact* long-range properties of the ground-state, within a cutoff regime, and one can even find the low-energy spectrum [44].

2.2 Heuristic bosonization

Consider a system of N spinless fermions loaded in a one-dimensional lattice of size L, with some boundary conditions to be specified later. The ground state consists of all the first N energy levels occupied, defining the Fermi level of the system. Even for a finite lattice of size L, we can regard the space variable as continuous by taking the limit $a \to 0$, with a the lattice constant. In such a case the system is described in second quantization through the field operators $\psi(x)$:

$$\psi(x) = \frac{1}{\sqrt{L}} \sum_{k=-\infty}^{\infty} e^{ikx} c_k, \qquad k = \frac{2\pi}{L}\left(n - \frac{\delta}{2}\right), \tag{2.2.1}$$

where $\delta = 0, 1$ specifies the boundary conditions, either periodic or anti-periodic, respectively, such that $\psi(x + L) = e^{-i\pi\delta}\psi(x)$.

As mentioned before, at low energies *all* the elementary excitations of the system are particle-hole excitations that lie close to the Fermi surface which in this case, due to the dimensionality, consists of two disconnected points $\pm k_F$. Hence, if we are only interested in the behavior of the system at low energies, say in a window of momentum of width 2Λ centered at $\pm k_F$, we can get an equivalent description for the system by linearizing the dispersion relation $\varepsilon(k)$ around the Fermi energy ε_F, as shown in the figure 2.3.

The linear spectra are given by

$$\begin{aligned} \varepsilon(k) &= \varepsilon(\pm k_F) + \left.\frac{\partial \varepsilon}{\partial k}\right|_{k=\pm k_F} (k \mp k_F) \\ \varepsilon(k) &= \varepsilon_F \pm v_F (k \mp k_F), \end{aligned} \tag{2.2.2}$$

where $v_F \equiv \partial\varepsilon/\partial k\big|_{k=k_F}$ is the Fermi velocity.

The first thing we can tell about our new description of the system is that we shall deal with massless particles, for the spectra are linear. There is another even more dramatic consequence of the linearization: any finite Fermi energy of the real system gives rise to an infinite number of particles in this equivalent description, since now the spectra are unbounded from below. Therefore, in our auxiliary description we are truly dealing with field theory, because we are facing an infinite number of degrees of freedom. Due to this, the Fock space might not be obvious and we shall discuss it with detail.

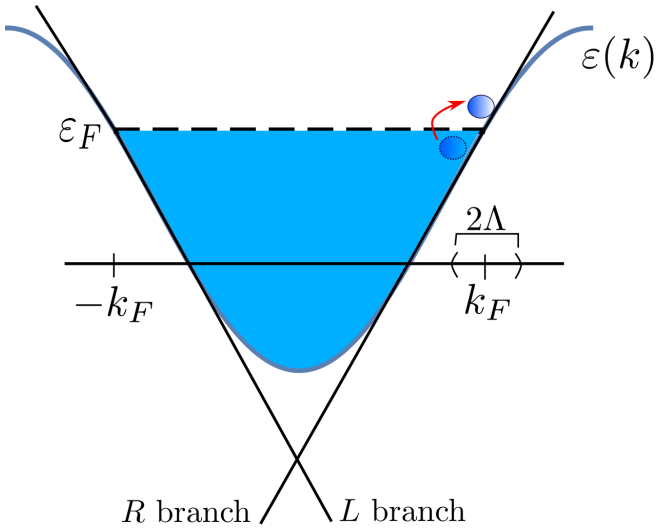

Figure 2.3: Linerization of the energy spectrum around the Fermi surface $\pm k_F$. At low energies the system can be equivalently described by two independent linear dispersion relations called the right branch and left branch.

2.2.1 Single branch system

We begin the study of the emergent features of this equivalent description by focusing on a single branch, say the right one. Without any loss of generality consider that in figure 2.3 we have $k_F = 0$ so that $\varepsilon_F = 0$.

In principle there is no issue upon dealing with an infinite number of degrees of freedom, the actual problem lies in the fact that according to figure 2.3 some of these particles can have negative energies. Fortunately, this is an old-dated problem cleverly solved by Dirac in 1930 [61] in his relativistic theory of the electron. The tenet consists in defining the ground state of the system $|0\rangle_0$ such that it is stable. We can do this as long as we consider the emergent massless particles to be fermions, and postulate that in the ground state of the system all the levels of negative energy are occupied. Thus, the Pauli exclusion principle makes these levels inaccessible for the particles with positive energy. So then, the vacuum state is *defined* as in equation (2.2.3), and depicted in figure 2.4,

$$\begin{aligned} c_k\,|0\rangle_0 &= 0, \ \text{for } k > 0 \ \text{(because everything is empty)} \\ c_k^\dagger\,|0\rangle_0 &= 0, \ \text{for } k \leq 0 \ \text{(because everything is occupied).} \end{aligned} \tag{2.2.3}$$

The Fock space $\mathcal{F}$ is spanned by all the states that can be constructed from $|0\rangle_0$[1] with the operators c_k and $c_k^\dagger$. The usual way to classify these states is according to their total number of fermions. However, for any given state that number is always infinite by the definition of the vacuum. So then, in order to manage sensible physical quantities, we consider the difference between the total number of fermions in a given state and that of the vacuum

[1] The meaning of the subscript 0 will become clear later.

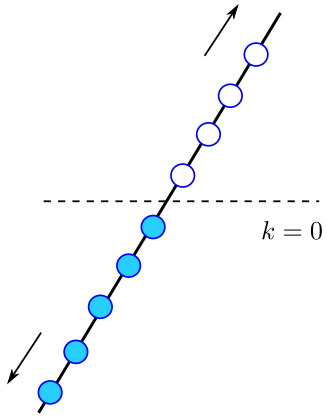

Figure 2.4: The vacuum state of the single branch system is the Dirac sea: Every state with negative k is occupied. Filled circles represent particles while blank circles represent holes.

state, which turns out to be a finite number. Thus, the particle number operator reads

$$N \equiv \sum_k \left(c_k^\dagger c_k - {}_0\langle 0|c_k^\dagger c_k|0\rangle_0 \right). \tag{2.2.4}$$

The subtraction of the infinite quantity ${}_0\langle 0|c_k^\dagger c_k|0\rangle_0$ reminds us of the definition of normal order in quantum field theory. Given an operator O that can be written as products of c's and $c^\dagger$'s, we define its normal ordered version, $:O:$, by moving all the c_k's with $k > 0$ and $c_k^\dagger$'s with $k \leq 0$ to the right of all the c_k's with $k < 0$ and $c_k^\dagger$'s with $k \geq 0$, and taking into account that they are fermionic operators so that each transposition of the operators carries a factor -1. Thus, the particle number operator can also be defined as

$$N \equiv \sum_k :c_k^\dagger c_k:. \tag{2.2.5}$$

The definitions (2.2.4) and (2.2.5) are equivalent because in this case $O = c_k^\dagger c_k$ is quadratic in the creation and annihilation operators, but it is not always true that $:O: = O - \langle O \rangle$ [42], therefore we take the definition of the normal order as more fundamental.

2.2.2 Two-branches system

We want to find a connection between the two descriptions showed in figure 2.3. From equation (2.2.2), we see that one linearization is performed at $k > 0$ while the other at $k < 0$, thus we are motivated to split likewise the mode expansion for the field operator (2.2.1):

$$\psi(x) = \frac{1}{\sqrt{L}} \left(\sum_{k<0} e^{ikx} c_k + \sum_{k>0} e^{ikx}\, c_k \right). \tag{2.2.6}$$

We can perform a shift in momentum around $\pm k_F$ in each sum:

$$\psi(x) = \frac{1}{\sqrt{L}} \left(e^{-ik_F x} \sum_{k=-\infty}^{k_F} e^{ikx} c_{k-k_F} + e^{ik_F x} \sum_{k=-k_F}^{\infty} e^{ikx} c_{k+k_F} \right). \tag{2.2.7}$$

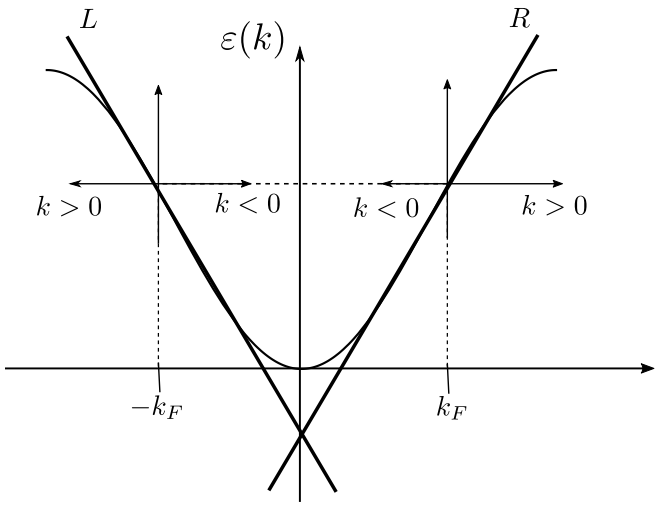

Figure 2.5: The definition of the new operators associated to the massless fermions of each branch is consistent with the name of right-movers and left-movers, since for $k > 0$ we see that particles of the R (L) branch are moving to the right (left).

In order to collect everything in a single summation we set $k \to -k$ in the first sum,

$$\psi(x) = \frac{1}{\sqrt{L}}\left(e^{-ik_F x} \sum_{k=-k_F}^{\infty} e^{-ikx} c_{-k-k_F} + e^{ik_F x} \sum_{k=-k_F}^{\infty} e^{ikx} c_{k+k_F} \right). \tag{2.2.8}$$

As mentioned before, the particles associated to the R and L branches are massless fermions, let us call them *right-movers* and *left-movers*, respectively. The correspondent annihilation operators per branch are defined as,

$$\begin{aligned} c_{k+k_F} &\equiv c_{k,R} \\ c_{-k-k_F} &\equiv c_{k,L}. \end{aligned} \tag{2.2.9}$$

What is the motivation behind these definitions? Consider that $k > 0$, then $k + k_F > k_F$ and $-k - k_F < -k_F$, so for both branches we are taking values of k such that $\varepsilon(k) > \varepsilon_F$. The analogue holds for $k < 0$. Moreover, for $k > 0$ the particles move to the right if they belong to the R branch, and to the left for the L one. In summary, we have the situation depicted in the figure 2.5.

There is another crucial consequence of the definition of the operators $c_{k,L}$ and $c_{k,R}$. It is clear that for the R branch we recover the definition of the Fock space given in section 2.2.1. Now, by setting $c_{k,L} = c_{-k-k_F}$ the same definition holds for the Fock space of the fermions of the L branch, that is, $c_{k,L} |0\rangle_0 = 0$ for $k > 0$ and $c^\dagger_{k,L} |0\rangle_0 = 0$ for $k < 0$. Then, the reason to stick with the definitions (2.2.9) is that we can construct the Fock space for both species of fermions in the same way, and we achieve that the particles in the L (R) branch to be moving to the left (right) for $k > 0$.

There is another striking feature in figure 2.5, notice that the coordinates systems placed at $\pm k_F$ are mirrors of each other. This is a manifestation of an important symmetry of the

auxiliary system: chirality. Hence, we are studying the low-energy behavior of (spinless) free fermions through (spinless) massless chiral fermions.

Now that the definitions (2.2.9) were discussed, we can use them in the expansion (2.2.8) to get

$$\psi(x) = \frac{1}{\sqrt{L}}\left(e^{-ik_F x}\sum_{k=-k_F}^{\infty} e^{-ikx}c_{k,L} + e^{ik_F x}\sum_{k=-k_F}^{\infty} e^{ikx}c_{k,R}\right). \tag{2.2.10}$$

Since we know that for both branches $k < 0$ corresponds to $\varepsilon < \varepsilon_F$, we can extend the summation over all real values of k as long as we are only interested in the low-energy behavior. Therefore,

$$\psi(x) \overset{!}{=} \frac{1}{\sqrt{L}}\left(e^{-ik_F x}\sum_{k=-\infty}^{\infty} e^{-ikx}c_{k,L} + e^{ik_F x}\sum_{k=-\infty}^{\infty} e^{ikx}c_{k,R}\right), \tag{2.2.11}$$

where we have introduced the notation "$\overset{!}{=}$" to indicate that the equality is valid only in the low-energy regime.

The last equation suggests a definition for the field operators associated to each specie of chiral fermions:

$$\psi_\nu(x) \equiv \frac{1}{\sqrt{L}}\sum_{k=-\infty}^{\infty} e^{\alpha ikx}c_{k\nu}, \tag{2.2.12}$$

where $\nu = \{R, L\}$ carries the information of the branch and $\alpha = \pm 1$ for right-movers and left-movers, respectively.

Notice that for the particles in the left branch the mode expansion involves plane waves moving to the left, which agrees with the name of left-movers, and can be regarded as another reason for defining $c_{k,L}$ as c_{-k-k_F}.

Thus, we finally get a relationship between the original system of spinless fermions and the auxiliary system of massless chiral fermions,

$$\psi(x) = e^{-ik_F x}\psi_L(x) + e^{ik_F x}\psi_R(x). \tag{2.2.13}$$

We could regard equation (2.2.13) as a definition of two new fields $\psi_{R/L}(x)$ which are assumed to vary slowly on the scale of the lattice (which is a fancy way to say that they capture the low-energy behavior). This is the approach followed by the introductory material oriented to field theory as Refs. [43, 44]. We have tried to provide a more intuitive and motivated presentation.

Anyway, it is true that there is very little physical content in (2.2.13), so that it could be taken as a definition. Most of the physics is actually contained in the Hamiltonian of the system, $H_0 = \sum_{k\in \mathrm{BZ}} \varepsilon(k)c_k^\dagger c_k$, and how it can be written in terms of these new fields,

$$H_0 = \sum_{k\in \mathrm{BZ}} \varepsilon(k)c_k^\dagger c_k = \sum_{k=-\infty}^{\infty} [\varepsilon(k_F) + v_F k]\left(c_{k,L}^\dagger c_{k,L} + c_{k,R}^\dagger c_{k,R}\right) \quad (a \to 0) \tag{2.2.14}$$

$$= v_F \int dx \Big[\psi_L^\dagger(x)(i\partial_x)\psi_L(x) + \psi_R^\dagger(x)(-i\partial_x)\psi_R(x)\Big]. \tag{2.2.15}$$

Even if we take (2.2.13) as a definition, the Hamiltonian (2.2.15) shows that we end up working with a one-dimensional massless Dirac system, with v_F taking the role of the velocity of light c. Furthermore, one can show that the system verifies a *Lorentz-like invariance* by finding the fields $\psi_{R/L}$ in the Heisenberg picture:

$$\begin{aligned}\psi_{R/L}(x,t) &= e^{iH_0 t/\hbar}\psi_{R/L}(x)e^{-iH_0 t/\hbar} \\ &= \frac{1}{\sqrt{L}}\sum_k e^{\pm ikx}e^{iH_0 t}c_{k,R/L}e^{-iH_0 t} \\ &= \frac{1}{\sqrt{L}}\sum_k e^{\pm ik(x\mp v_F t)}c_{k,R/L}.\end{aligned} \tag{2.2.16}$$

The last expressions shows that one could get a Lorentz-like invariant expression for the fields by setting $x^\mu = (\pm v_F t, x)$, $k^\mu = (\omega_k, k) = (k, k)$, where $\omega_k = k$ since the particles are massless, and taking a metric $g_{\mu\nu} = \mathrm{diag}(-1, 1)$ to define the scalar product $x^\mu k_\mu = k(\mp v_F t + x)$.

2.2.3 Bosons

In 1934 Bloch realized that one-dimensional non-interacting fermions have the same type of low-energy excitations as a harmonic chain [62], that is, that they resemble sound waves. In 1950 Tomonaga employed Bloch's method of sound waves to study a system of interacting fermions [53]. He realized that the Fourier components of the density fluctuations operators have a central role when studying the interaction. His breakthrough was to split such operators into two parts, corresponding to what we have called right-movers and left-movers. In the following we show how in that way we reach bosonic operators.

We begin by defining the density operators for the ν branch as

$$\rho_\nu(x) \equiv :\psi_\nu^\dagger(x)\psi_\nu(x): = \frac{1}{L}\sum_{k,k'} e^{\alpha i(k'-k)x}:c_{k\nu}^\dagger c_{k'\nu}:. \tag{2.2.17}$$

Since the summations range from $k, k' = -\infty$ to $k, k' = \infty$, we can set $k = k' + q$, as long as q respects the boundary conditions. Therefore,

$$\begin{aligned}\rho_\nu(x) &= \frac{1}{L}\sum_{k',q} e^{-\alpha iqx}:c_{k'+q,\nu}^\dagger c_{k'\nu}: \\ &= \frac{1}{L}\sum_k :c_{k\nu}^\dagger c_{k\nu}: + \frac{1}{L}\sum_{q\neq 0} e^{-i\alpha qx}\sum_k :c_{k+q,\nu}^\dagger c_{k\nu}: \\ &= \frac{1}{L}N_\nu + \frac{1}{L}\sum_{q\neq 0} e^{-i\alpha qx}\sum_k :c_{k+q,\nu}^\dagger c_{k\nu}:.\end{aligned} \tag{2.2.18}$$

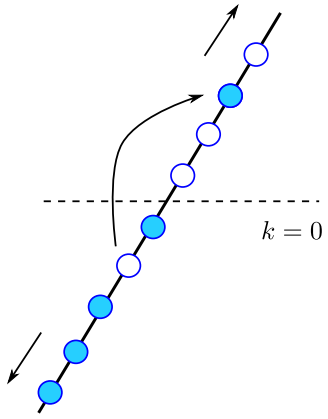

Figure 2.6: Illustration of a particle-hole excitation in a single branch that may be generated by $:c^\dagger_{k+q,R}c_{k,R}:$.

It is clear that the operator in the second term of the last expression quantifies the density fluctuations of the system. Hence, we can define the Fourier components of those fluctuations that were of much interest for Tomonaga:

$$\rho_{q\nu} \equiv \sum_k :c^\dagger_{k+q,\nu}c_{k\nu}: \,, \text{ for } q \neq 0. \tag{2.2.19}$$

There is an immediate drawback with such promising quantities, namely, they are not Hermitian,

$$\rho^\dagger_{q\nu} = \rho_{-q,\nu}, \tag{2.2.20}$$

which can be regarded as an indication that there should be something even more fundamental than $\rho_{q\nu}$. As we will see, these are the bosonic operators.

Let us investigate the physical meaning of the Fourier components of the density fluctuations. Consider $\rho_{q,R} = \sum_k :c^\dagger_{k+q,R}c_{k,R}:$ for $q > 0$. Then, it is clear that the product $:c^\dagger_{k+q,R}c_{k,R}:$ creates a particle-hole excitation with momentum q when acting on the vacuum (as long as $k < 0$), as depicted in the picture 2.6. Thus, the Fourier component $\rho_{q,R}$ is a *superposition of all the possible particle-hole excitations* of right-movers. What about the left-movers? Since our construction of the operators has been such that the definition of the vacuum is the same for both branches, then $\rho_{p,L}$ will also create a particle-hole excitation in the left branch when $p > 0$. This shows how convenient are our definitions. However, as mentioned at the beginning of the chapter, part of the difficulties of following the operator formulation of bosonization are the different conventions that one may find along the literature, for instance, while our definition of the Fock spaces of the branches matches that of Ref. [48], it is different to the one in Ref. [49].

The Fourier components, $\rho_{k\nu}$, are the bridge towards the bosons, and to see it clearly we find their algebra. There are different ways to proceed: (i) To use the inverse Fourier transformation to reduce the problem to that of finding the algebra of the densities in position

representation or (ii) to realize that the normal order is immaterial for $\rho_{q\nu}$, provided that by definition $q \neq 0$. Since the second approach is followed in Refs. [45, 48], we follow (i),

$$\rho_\nu(x) = \frac{1}{L}\sum_q e^{-i\alpha q x} \sum_k :c^\dagger_{k+q,\nu} c_{k\nu}: \implies \rho_{q\nu} = \int dx\, \rho_\nu(x) e^{i\alpha q x}. \tag{2.2.21}$$

Thus,

$$\begin{aligned}
[\rho_{p\nu}, \rho_{p'\nu}] &= \int dx\, dy\, e^{i\alpha p x} e^{i\alpha p' y} [\rho_\nu(x), \rho_\nu(y)] \\
&= \int dx\, dy\, e^{i\alpha p x} e^{i\alpha p' y} \left(\frac{-\alpha i}{2\pi}\right) \partial_x \delta(x-y) \\
&= -\frac{Lp}{2\pi}\delta_{p,-p'},
\end{aligned} \tag{2.2.22}$$

where for the computation of $[\rho_\nu(x), \rho_\nu(y)]$ we refer the reader to the appendix A.1.

Using the equation (2.2.20) into (2.2.22) we can get something really close to the algebra of bosonic operators by turning $\delta_{p,-p'}$ into $\delta_{p,p'}$. To get rid of the factor $-Lp/2\pi$ we scale the operators properly and get

$$\left[i\left(\frac{2\pi}{Lp}\right)^{1/2} \rho_{p\nu}, i\left(\frac{2\pi}{Lp}\right)^{1/2} \rho^\dagger_{p'\nu} \right] = \delta_{p,p'}. \tag{2.2.23}$$

At this point, we might be tempted to define the boson creation operator as the operator proportional to $\rho^\dagger_{p\nu}$. However, we previously figured out that, for $p > 0$, it is actually the operator $\rho_{p\nu}$ the one that creates particle-hole excitations. Thus,

$$b^\dagger_{p\nu} \equiv i\left(\frac{2\pi}{Lp}\right)^{1/2} \rho_{p\nu}, \text{ for } p > 0. \tag{2.2.24}$$

We have showed, heuristically, how the Fourier components of the density fluctuations give rise to bosonic operators that represent the particle-hole excitations in each branch. More physical content of this new picture can be found by writing the Hamiltonian (2.2.15) for spinless free fermiones in terms of the new bosonic operators:

$$:H_0: = \sum_\nu \left(\frac{\pi v_F}{L} N^2_\nu + v_F \sum_{p>0} |p| b^\dagger_{p\nu} b_{p\nu} \right), \tag{2.2.25}$$

where now we have to introduce a normal order for the bosonic operators $b^\dagger_{p\nu}$ defined in the same way as for the $c_{k\nu}$'s, except that now the transposition of operators does not give rise to a negative sign, due to the change in the algebra of the operators.

It is straightforward to interpret the second term in (2.2.25), it represents the shift of the fermions to excited states, that is, the particle-hole excitations. What about the first term in the Hamiltonian? If we were to measure the energy of the system without it, we would

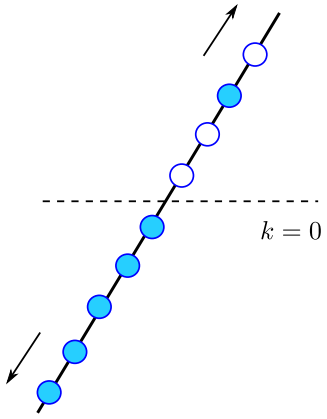

Figure 2.7: Possible configuration of the spectrum of one of the branches. The filled circles represent particles (fermions) while the blank circles represent holes. A particle has been added to the third excited level.

never get contributions from configurations like the one depicted in figure 2.7, for in it none fermion has been shifted, but one fermion is added at the third excited level, and such a process cannot be represented by the $b^\dagger$'s operators. It is precisely the first term of the Hamiltonian the one that accounts for these processes, it is the energy cost of placing N_ν fermions *above* the ground state ($N_\nu = \sum_k :c^\dagger_{k\nu} c_{k\nu}:$) in the ν branch. So then, we can regard this configuration as first placing one electron immediately above the vacuum, $k = 1$, and then shifting it up to $k = 3$[2].

Our conclusion from this heuristic presentation is that the bosonization technique allows us to get an equivalent low-energy description for the system in terms of two processes: adding fermions through N_ν and shifting them through the b's and $b^\dagger$'s.

2.3 Constructive bosonization

The previous discussion about the configuration depicted in figure 2.7 opens a question on the scope of the bosonic description: Is it possible to describe *everything* about a fermionic system in terms of bosonic operators? In the case of a system of free fermions we could get a Hamiltonian written in terms of $b^\dagger_{p\nu}$ and $b_{p\nu}$, but we realized that some features are not captured by them. For instance, how from (2.2.25) would one compute the energy of the system in the state $c^\dagger_{3R}|0\rangle_0$?

This difficulty is related to what we already addressed, namely, that the bosonic operators shift the fermions but *they do not create them.* Mathematically, the underlying map we are constructing is between density fluctuations and bosons, so there is no direct relationship between $c^\dagger_{p\nu}$ and $b^\dagger_{p\nu}$. So then, if we were to compute ${}_0\langle 0|c_{3R} H_0 c^\dagger_{3R}|0\rangle_0$ we really needed to think for a while on how to do it from the bosonic operators.

[2]We are being a bit sloppy at this point because k is not really an integer but it is labeled through one. Recall the boundary conditions we set up in (2.2.1) where $k = 2\pi/L(n - \delta/2)$, so hereinafter whenever we write $k = l$ we actually mean $k = 2\pi/L(l - \delta/2)$.

What we are looking for is the existence of a *bosonic representation*, that is, a set of operator identities in Fock space between bosonic and fermionic operators, what Delft and Schoeller call constructive bosonization [47]. We already pointed out the main issue, we need to write $c_{k\nu}$ and $c^\dagger_{k\nu}$ in terms of bosons. To that end, we shall define new operators F_ν called *Klein factors*, which allow us to change the number of particles (fermions) in the system.

2.3.1 Fock space

We aim to generalize the Fock space described in section 2.2. Consider a system of free fermions such that after linearization we end up with M species of fermions labeled by η (in the study of free fermions we had $M = 2$ and $\eta = \nu$). The vacuum state of the system is defined as

$$\begin{aligned} c_{k\eta} \left|0\right\rangle_0 &\equiv 0, \text{ for } k > 0 \\ c^\dagger_{k\eta} \left|0\right\rangle_0 &\equiv 0, \text{ for } k \leq 0, \end{aligned} \tag{2.3.1}$$

where $c^\dagger_{k\eta}$ ($c_{k\eta}$) creates a particle (hole) of the specie η with momentum k.

The Fock space $\mathcal{F}$ of the system is spanned by the states constructed from $\left|0\right\rangle_0$ through the action of the operators $c_{k\eta}$ and $c^\dagger_{k\eta}$. Once more, a natural way to classify these states is through their number of particles (measured by N_η, that is, with respect to the vacuum $\left|0\right\rangle_0$). If $\mathcal{H}_{\{N_\eta\}}$ is the Hilbert space of *all* the states of $\sum_\eta N_\eta$ particles and holes, then we have the decomposition $\mathcal{F} = \oplus_\eta \mathcal{H}_{\{N_\eta\}}$. The states with lower energy in $\mathcal{H}_{\{N_\eta\}}$ are given by

$$\left|\{N_\eta\}\right\rangle_0 \equiv \bigotimes_\eta \left|N_\eta\right\rangle_0 = \begin{cases} \bigotimes_\eta c^\dagger_{N_\eta,\eta} c^\dagger_{N_\eta-1,\eta} \cdots c^\dagger_{1,\eta} \left|0\right\rangle_0 & \text{if } N_\eta > 0 \\ \bigotimes_\eta c_{N_\eta,\eta} c_{N_\eta+1,\eta} \cdots c_{-1,\eta} \left|0\right\rangle_0 & \text{if } N_\eta < 0, \end{cases} \tag{2.3.2}$$

where $c^\dagger_{N_\eta,\eta}$ ($c_{N_\eta,\eta}$) creates a particle (hole) of the specie η with momentum $k = 2\pi/L(N_\eta - \delta/2)$ for $N_\eta > 0$ ($N_\eta < 0$).

Having discussed the meaning of the bosons as a superposition of particle-hole excitations, we are prepared to understand the meaning of the subscript 0 that we introduced in the notation for the states. It is aimed to remind us that even though $\left|\{N_\eta\}\right\rangle_0$ is an excited state of fermions, it contains *no* particle-hole excitations and therefore it serves as a vacuum state *for the bosonic operators*, provided that

$$b_{p\eta} \left|\{N_\eta\}\right\rangle_0 = 0, \text{ for all } p, \eta. \tag{2.3.3}$$

Therefore, the vacuum of the fermions $\left|0\right\rangle_0$ is not the same vacuum of the bosons, and actually we have infinitely many versions of the latter since any state $\left|\{N_\eta\}\right\rangle_0$ as defined in (2.3.2) has no bosons.

2.3.2 Klein factors

Now we are in position to properly introduce the Klein factors, which turn out to be the way out to the issues pointed out at the end of section 2.2. For M species of fermions we *define* the Klein factors F_η through the following properties [47]:

$$\left[F_\eta, b^\dagger_{k\eta'}\right] = \left[F_\eta, b_{k\eta'}\right] = \left[F^\dagger_\eta, b^\dagger_{k\eta'}\right] = \left[F^\dagger_\eta, b_{k\eta'}\right] = 0, \text{ for all } \eta, \eta', k, \tag{2.3.4a}$$

$$F_\eta^\dagger |N_1, N_2, \ldots, N_M\rangle_0 \equiv (-1)^{\sum_{i=1}^{\eta-1} N_i} |N_1, N_2, \ldots, N_\eta + 1, \ldots, N_M\rangle_0 \tag{2.3.4b}$$

$$F_\eta |N_1, N_2, \ldots, N_M\rangle_0 \equiv (-1)^{\sum_{i=1}^{\eta-1} N_i} |N_1, N_2, \ldots, N_\eta - 1, \ldots, N_M\rangle_0 \tag{2.3.4c}$$

$$F_\eta^\dagger F_\eta = F_\eta F_\eta^\dagger = 1. \tag{2.3.4d}$$

The phase $(-1)^{\sum_{i=1}^{\eta-1} N_i}$ is a consequence of the anti-commutation relations among operators of different species. Thus, this phase is essential to preserve the fermionic nature of the particles. An explicit construction of the Klein factors can be found in Ref. [47].

To put things into context, consider the case of spinless free fermions studied in section 2.2 where $M = 2$ and $\eta = \nu$:

$$\begin{aligned} F_L |\{N_\nu\}\rangle &= (-1)^{N_R} |N_R, N_L - 1\rangle \\ F_L^\dagger |\{N_\nu\}\rangle &= (-1)^{N_R} |N_R, N_L + 1\rangle . \end{aligned} \tag{2.3.5}$$

It is worth writing down the general expression for M species because it makes it straightforward to apply this to our actual system, in which spin will be present so that $M = 4$. With the defining properties (2.3.4) one can investigate the algebra of the Klein factors, which verifies [47, 48]:

$$\left\{F_\eta^\dagger, F_{\eta'}^\dagger\right\} = \left\{F_\eta, F_{\eta'}\right\} = 0, \text{ solely for } \eta \neq \eta', \tag{2.3.6}$$

$$\left\{F_\eta^\dagger, F_{\eta'}\right\} = 2\delta_{\eta,\eta'}, \text{ for all } \eta, \eta', \tag{2.3.7}$$

$$\left[N_\eta, F_{\eta'}^\dagger\right] = \delta_{\eta,\eta'} F_\eta^\dagger \tag{2.3.8}$$

$$[N_\eta, F_{\eta'}] = -\delta_{\eta,\eta'} F_\eta. \tag{2.3.9}$$

Equation (2.3.6) shows a remarkable feature about these operators: Even though we change the number of particles through Klein factors and they maintain the fermionic algebra, *they are not fermionic operators* because $(F_\eta^\dagger)^2 \neq 0$.

2.3.3 Towards the bosonization identity

Now our goal is to write the fermionic fields $\psi_\eta^\dagger(x)$, $\psi_\eta(x)$ (see (2.2.12) for an example when $M = 2$ and $\eta = \nu$) in terms of the bosons $b_{p\eta}^\dagger$, the number of particles N_η, and Klein factors F_η. The full computations of the commutators are shown in Appendix A.

$$[b_{p\eta}, \psi_{\eta'}(x)] = i\left(\frac{2\pi}{Lp}\right)^{1/2} e^{-i\alpha p x} \psi_\eta(x)\delta_{\eta,\eta'} \equiv \beta_p(x)\psi_\eta(x) \text{ for } p > 0. \tag{2.3.10}$$

Similarly, we find that,

$$\left[b_{p\eta}^\dagger, \psi_{\eta'}(x)\right] = \beta_p^* \psi_\eta(x)\delta_{\eta,\eta'}. \tag{2.3.11}$$

Let us consider the action of the commutator $[b_{p\eta}, \psi_{\eta'}(x)]$ on the state $|\{N_\eta\}\rangle_0$. Since $b_{p\eta} |\{N_\eta\}\rangle_0 = 0$, we get

$$[b_{p\eta}, \psi_\eta(x)] |\{N_\eta\}\rangle_0 = b_{p\eta}\psi_\eta(x) |\{N_\eta\}\rangle_0 . \tag{2.3.12}$$

So then, from (2.3.10) we find that

$$b_{p\eta}\psi_\eta(x)\left|\{N_\eta\}\right\rangle_0 = \beta_p(x)\psi_\eta(x)\left|\{N_\eta\}\right\rangle_0 . \tag{2.3.13}$$

The relationship in (2.3.13) is very important, for it shows that the state $\psi_\eta(x)\left|\{N_\eta\}\right\rangle_0$ is an eigenstate of the bosonic annihilation operator $b_{p\eta}$ with eigenvalue $\beta_p(x)$, $p>0$. In other words, $\psi_\eta(x)\left|\{N_\eta\}\right\rangle_0$ is a *coherent state*. From the elementary theory of coherent states we know that [63],

$$\left|z_p\right\rangle = \exp\left[\sum_p\left(-\frac{|z_p|^2}{2}+z_p b_p^\dagger\right)\right]\left|0\right\rangle_{\text{bosons}} , \tag{2.3.14}$$

but we have to adapt the former relation to the states we are handling with. From the discussion in section 2.3.1, we know that the analogue to the vacuum of bosons $\left|0\right\rangle_{\text{bosons}}$ in our case is $\left|\{N_\eta\}\right\rangle_0$. On the other hand, the factor $\exp\left(-\sum_p |z_p|^2/2\right)$ is a normalization factor, which might be different in our case. Therefore, the best we can tell from (2.3.13) is that

$$\psi_\eta(x)\left|\{N_\eta\}\right\rangle_0 \propto \exp\left[\sum_{p>0}\beta_p(x)b_{p\eta}^\dagger\right]\left|\{Q_\eta\}\right\rangle_0 , \text{ with } Q \text{ an integer.} \tag{2.3.15}$$

How do we find Q? If we write $\psi_\eta(x)\left|\{N_\eta\}\right\rangle_0$ using the mode decomposition of $\psi_\eta(x)$ we shall find out that $\psi_\eta(x)\left|\{N_\eta\}\right\rangle_0$ is a superposition of states with $N_\eta-1$ particles. So then, $\psi_\eta(x)\left|\{N_\eta\}\right\rangle_0 \in \mathcal{H}_{\{N_1\}}\oplus\ldots\mathcal{H}_{\{N_\eta-1\}}\oplus\ldots$. In other words, $\left|\{Q_\eta\}\right\rangle_0 = F_\eta\left|\{N_\eta\}\right\rangle_0$,

$$\psi_\eta(x)\left|\{N_\eta\}\right\rangle_0 = A_\eta(x)\exp\left[\sum_{p>0}\beta_p(x)b_{p\eta}^\dagger\right]F_\eta\left|\{N_\eta\}\right\rangle_0 , \tag{2.3.16}$$

where we have considered the general case in which the proportionality constant is x-dependent.

It is only left to find $A_\eta(x)$ which can be done by facing the former expression with ${}_0\langle\{N_\eta\}|\,F_\eta^\dagger$, and using the properties of the Klein factors:

$$\begin{aligned}
{}_0\langle\{N_\eta\}|F_\eta^\dagger\psi_\eta(x)|\{N_\eta\}\rangle_0 &= A_\eta(x)\,{}_0\langle\{N_\eta\}|F_\eta^\dagger\exp\left[\sum_{p>0}\beta_p(x)b_{p\eta}^\dagger\right]F_\eta|\{N_\eta\}\rangle_0 \\
&= A_\eta(x)\,{}_0\langle\{N_\eta\}|\exp\left[\sum_{p>0}\beta_p(x)b_{p\eta}^\dagger\right]F_\eta^\dagger F_\eta|\{N_\eta\}\rangle_0 \\
&= A_\eta(x)\,{}_0\langle\{N_\eta\}|\left(1+\sum_{n=1}^{\infty}\frac{1}{n!}\left(\sum_{p>0}\beta_p(x)b_{p\eta}^\dagger\right)^n\right)|\{N_\eta\}\rangle_0 \\
&= A_\eta(x).
\end{aligned} \tag{2.3.17}$$

Therefore, the mode decomposition tells us that $\frac{1}{\sqrt{L}}\sum_k e^{i\alpha kx}\,{}_0\langle\{N_\eta\}|F_\eta^\dagger\, c_{k\eta}|\{N_\eta\}\rangle_0 = A_\eta(x)$. Recall that $k = 2\pi(n-\delta/2)/L$ due to the boundary conditions, hence, the element $c_{k\eta}\,|\{N_\eta\}\rangle_0$ is non-zero for $n \leq N_\eta$. However, when facing it with ${}_0\langle\{N_\eta\}|\, F_\eta^\dagger$ we get a non-vanishing output only for $n = N_\eta$. Thus,

$$\frac{1}{\sqrt{L}}\exp\left[i\alpha\frac{2\pi}{L}\left(N_\eta - \frac{\delta}{2}\right)x\right] = A_\eta(x), \tag{2.3.18}$$

and equation (2.3.16) reads

$$\psi_\eta\,|\{N_\eta\}\rangle_0 = \frac{1}{\sqrt{L}}\exp\left[i\alpha\frac{2\pi}{L}\left(N_\eta - \frac{\delta}{2}\right)x\right]\exp\left[\sum_{p>0}\beta_p(x)b_{p\eta}^\dagger\right]F_\eta\,|\{N_\eta\}\rangle_0\,. \tag{2.3.19}$$

We pursue an equality between operators. That is something we cannot get from (2.3.19) because the state $|\{N_\eta\}\rangle_0$ is not a general state in $\mathcal{F}$. Recall that we claimed in section 2.3.1 that a general state of $\sum_\eta N_\eta$ particles and holes is spanned by all the particle-hole excitations of $|\{N_\eta\}\rangle_0$. Actually, that statement was rigorously proved by Haldane [64]. Thus, we are interested in finding $\psi_\eta(x)\,|\omega\rangle$ for $|\omega\rangle \equiv f\big(\{b_{p\eta}^\dagger\}\big)\,|\{N_\eta\}\rangle_0$, with f a well-behaved function. Since from (2.3.19) we know $\psi_\eta(x)\,|\{N_\eta\}\rangle_0$, we only need to find the commutator between $\psi_\eta(x)$ and $f\big(\{b_{p\eta}^\dagger\}\big)$. In order to do so, we use the following identities which are proved in the Appendix A.2.1:

$$[A,B] = DA \implies Af(B) = f(B+D)A \tag{2.3.20}$$

$$[A,[A,B]] = [B,[A,B]] = 0 \implies e^{-B}f(A)e^{B} = f(A+[A,B]). \tag{2.3.21}$$

Recall from (2.3.11) that $\left[\psi_\eta(x), b_{p\eta'}^\dagger\right] = -\beta_p^*(x)\psi_\eta(x)\delta_{\eta,\eta'}$. This commutator has the structure $[A,B] = DA$ so that we can employ (2.3.20) to write

$$\psi_\eta(x)f\Big(\{b_{p\eta'}^\dagger\}\Big) = f\Big(\{b_{p\eta'}^\dagger - \beta_p^*(x)\delta_{\eta,\eta'}\}\Big)\psi(x). \tag{2.3.22}$$

Then, the action of $\psi_\eta(x)$ on the state $|\omega\rangle$ is given by

$$\begin{aligned}\psi_\eta(x)\,|\omega\rangle &= f\big(\{b_{p\eta}^\dagger - \beta_p^*(x)\delta_{\eta,\eta'}\}\big)\frac{1}{\sqrt{L}}\exp\left[i\alpha\frac{2\pi}{L}\left(N_\eta - \frac{\delta}{2}\right)x\right]\exp\left[\sum_{p>0}\beta_p(x)b_{p\eta}^\dagger\right]F_\eta\,|\{N_\eta\}\rangle_0 \\ &= \frac{1}{\sqrt{L}}\exp\left[i\alpha\frac{2\pi}{L}\left(N_\eta - \frac{\delta}{2}\right)x\right]F_\eta\exp\left[\sum_{p>0}\beta_p(x)b_{p\eta}^\dagger\right]f\big(\{b_{p\eta}^\dagger - \beta_p^*\delta_{\eta,\eta'}\}\big)\,|\{N_\eta\}\rangle_0\,.\end{aligned} \tag{2.3.23}$$

To get a relationship between operators we have to construct the state $|\omega\rangle$ in the right-hand side, the strategy to do so relies on the use of the second identity (2.3.21). Consider $A = b_{p\eta'}^\dagger$ and $B = \sum_{k>0}\beta_k^*(x)b_{k\eta}$, so that

$$[A,B] = \sum_{k>0}\beta_k^*(x)\left[b_{p\eta'}^\dagger, b_{k\eta}\right] = -\beta_p(x)^*\delta_{\eta,\eta'}. \tag{2.3.24}$$

Since $[A,[A,B]]=[B,[A,B]]=0$, the identity (2.3.21) reads

$$\exp\left[-\sum_{k>0}\beta_k^*(x)b_{k\eta}\right]f\Big(\{b_{p\eta'}^\dagger\}\Big)\exp\left[\sum_{k>0}\beta_k^*(x)b_{k\eta}\right]=f\Big(\{b_{p\eta}^\dagger-\beta_k^*(x)\delta_{\eta,\eta'}\}\Big). \tag{2.3.25}$$

Feeding the former into (2.3.23) yields

$$\begin{aligned}\psi_\eta(x)\,|\omega\rangle &= \frac{1}{\sqrt{L}}\exp\left[i\alpha\frac{2\pi}{L}\Big(N_\eta-\frac{\delta}{2}\Big)x\right]F_\eta\exp\left[\sum_{p>0}\beta_p(x)b_{p\eta}^\dagger\right]\exp\left[-\sum_{k>0}\beta_k^*(x)b_{k\eta}\right]\\ &\quad\times f\Big(\{b_{p\eta'}^\dagger\}\Big)\exp\left[\sum_{k>0}\beta_k^*(x)b_{k\eta}\right]|\{N_\eta\}\rangle_0\\ &= \frac{1}{\sqrt{L}}\exp\left[i\alpha\frac{2\pi}{L}\Big(N_\eta-\frac{\delta}{2}\Big)x\right]F_\eta\exp\left[\sum_{p>0}\beta_p(x)b_{p\eta}^\dagger\right]\exp\left[-\sum_{p>0}\beta_p^*(x)b_{p\eta}\right]|\omega\rangle\,.\end{aligned} \tag{2.3.26}$$

We can finally write down an equality between operators (remember that $\beta_p(x)$ is a function defined in (2.3.10)) given by,

$$\psi_\eta(x)=\frac{1}{\sqrt{L}}\exp\left[i\alpha\frac{2\pi}{L}\Big(N_\eta-\frac{\delta}{2}\Big)x\right]F_\eta\exp\left[\sum_{p>0}\beta_p(x)b_{p\eta}^\dagger\right]\exp\left[-\sum_{p>0}\beta_p^*(x)b_{p\eta}\right]. \tag{2.3.27}$$

Here we have written the chiral fermionic fields in terms of the bosonic operators, the Klein factors, and the particle number operator. This is a major result within the bosonization program, it is called the *Mattis-Mandelstam formula*, after some of its discoverers [65–67].

2.4 Bosonic field operators

It is worthwhile to discuss constructive bosonization because we rigorously draw an equivalence between the fermionic and bosonic descriptions at the level of the Hilbert spaces. However, for the purpose of computing quantities it is often better to rely on the field-theoretical approach [43–46].

Thus, motivated on the description of a massless scalar field, we *define* the *chiral bosonic field* operator as

$$\phi_\eta(x)\equiv-\sum_{p>0}\frac{e^{-\lambda p/2}}{\sqrt{Lp}}\big(e^{i\alpha px}b_{p\eta}+e^{-i\alpha px}b_{p\eta}^\dagger\big),\qquad(\hbar=1) \tag{2.4.1}$$

where $\lambda>0$ is an infinitesimal parameter needed to regularize ultraviolet divergent sums that may arise in certain non normal-ordered expressions and commutators [47]. The factor $e^{-\lambda p/2}$ should be regarded as necessary to ensure convergence at all intermediate steps, but final results should be written taking $\lambda\to0^+$, whenever this yields a sensible answer [48].

In order to employ the full field theory machinery, we need to find a set of canonical dynamical fields. Then, we introduce the *dual bosonic fields*,

$$\theta_\gamma(x) \equiv \frac{1}{\sqrt{2}} \sum_\nu \phi_\eta(x), \tag{2.4.2}$$

$$\phi_\gamma(x) \equiv \frac{1}{\sqrt{2}} [\phi_{L\gamma} - \phi_{R\gamma}], \tag{2.4.3}$$

where we construct linear combinations of the chiralities R/L but the rest of the labels contained in η remain unaltered, and they are now encoded in the collective index γ.

Appendix A.3.1 shows that these quantities define a set of canonical fields for $L \to \infty$ (alongside other important relations involving the field operators), that is to say,

$$[\phi_\gamma(x), \Pi_\gamma(y)] = i\delta(x-y), \tag{2.4.4}$$

with $\Pi_\gamma(x) \equiv \partial_x \theta_\gamma(x)$.

2.4.1 Bosonization identity

The full potential of the field theory approach is unfolded once we rewrite the Mattis-Mandelstam formula (2.3.27) in terms of the chiral bosonic fields, ϕ_η. We begin by recalling the Baker-Campbell-Hausdorff (BCH) formula [63],

$$\exp(X)\exp(Y) = \exp\left(X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] - \frac{1}{12}[Y,[X,Y]] + \dots\right). \tag{2.4.5}$$

In (2.3.27), we identify $X = \sum_{p>0} \beta_p(x) b^\dagger_{p\eta}$ and $Y = -\sum_{p>0} \beta^*_p(x) b_{p\eta}$ so that:

$$[X,Y] = -\sum_{p>0}\sum_{k>0} \beta_p(x)\beta^*_k(x)\left[b^\dagger_{p\eta}, b_{k\eta}\right] = \sum_{p,k>0} \beta_p(x)\beta^*_k\delta_{k,p} = \sum_{p>0} |\beta_p(x)|^2. \tag{2.4.6}$$

Notice that this commutator *diverges*. This follows from the fact that we are destroying the normal ordering in (2.3.27) when trying to combine the exponentials. To keep the commutator finite we need to regularize the sums, then we set $\sum_{p>0} \to \sum_{p>0} e^{-\lambda p}$ [48], and the commutator reads:

$$\begin{aligned}
[X,Y] &= \sum_{p>0} e^{-\lambda p} |\beta_p(x)|^2 \\
&= \frac{2\pi}{L} \sum_{p>0} \frac{e^{-\lambda p}}{p} \\
&= \sum_{n>0} \frac{e^{-\lambda \frac{2\pi}{L} n}}{n} \\
&= -\ln\left(1 - e^{-\lambda \frac{2\pi}{L}}\right).
\end{aligned} \tag{2.4.7}$$

In the third line we used periodic boundary conditions ($\delta = 0$) without loss of generality because the fields shall be employed in the limit $L \to \infty$, at which the choice of boundary conditions is immaterial. This argument shall be widely used in most algebraic calculations. Whereas in equation (2.4.1) we included the convergence factor $e^{-\lambda p/2}$ as part of the definition of the chiral bosonic field, here the regularization is regarded as a re-definition of the summation. These two proceeding ways are equivalent (as we are about to show, we recover the bosonic fields (2.4.1) even though having regularized the sums) but they rely on different areas of physics: To introduce the convergence factor in the definition of the fields is customary in condensed matter, while doing it through the sums is the quantum field theory approach. On the fundamental level, the latter path makes more sense, since the regularization is something that appears half-way to overcome a particular issue upon calculating some physical quantity, and we should not change our definitions prior the emergence of such an issue.

On the other hand, since $\lambda \to 0^+$, the exponential can be expanded up to first order in λ, leading us to $[X, Y] = -\ln(\lambda 2\pi/L)$. It turns out that the commutator is proportional to the identity, so the BCH formula ends up being

$$\begin{aligned} \exp(X)\exp(Y) &= \exp\left(X + Y + \frac{1}{2}[X, Y]\right) \\ &= \left(\frac{L}{2\pi\lambda}\right)^{1/2} \exp(X + Y). \end{aligned} \tag{2.4.8}$$

Feeding the last result into the Mattis-Mandelstan formula, we get

$$\begin{aligned} \psi_\eta(x) &= \frac{1}{\sqrt{2\pi\lambda}} \exp\left[i\alpha\frac{2\pi}{L}\left(N_\eta - \frac{\delta}{2}\right)x\right] F_\eta \exp\left[\sum_{p>0}\left(\beta_p(x) b^\dagger_{p\eta} - \beta^*_p(x) b_{p\eta}\right)\right] \\ &= \frac{1}{\sqrt{2\pi\lambda}} \exp\left[i\alpha\frac{2\pi}{L}\left(N_\eta - \frac{\delta}{2}\right)x\right] F_\eta \exp\left[-i\sqrt{2\pi}\sum_{p>0}\frac{-1}{\sqrt{Lp}}\left(e^{-i\alpha p x} b^\dagger_{p\eta} + e^{i\alpha p x} b_{p\eta}\right)\right]. \end{aligned} \tag{2.4.9}$$

To get the definition of the field $\phi_\eta(x)$ we only need the convergence factor, but it is already contained in the summation $\sum_{p>0}$ so we reach the most important result of the field-theoretical bosonization: a relationship between both chiral fermionic and chiral bosonic fields,

$$\psi_\eta(x) = \frac{1}{\sqrt{2\pi\lambda}} \exp\left[i\alpha\frac{2\pi}{L}\left(N_\eta - \frac{\delta}{2}\right)x\right] F_\eta \exp\left[-i\sqrt{2\pi}\phi_\eta(x)\right]. \tag{2.4.10}$$

We have presented the bosonization framework, first from an heuristic view aimed to the beginner researcher where we discussed the motivations behind some definitions, and we realized that we could introduce bosonic operators into the description of our system through the Fourier components of the density fluctuations. Secondly, we boosted the operator formalism presenting what Delft and Schoeller call constructive bosonization: An equivalence between fermionic and bosonic descriptions at the level of Hilbert spaces. Ultimately, with the aim of computing physical quantities, we introduced the field-theoretical approach that culminates in obtaining the bosonization identity (2.4.10), which is going to be the cornerstone for further developments. Now, we are in a position to study a particular model in the light of this framework.

2.5 Application: Asymmetric Hubbard model

In what follows we discuss the asymmetric Hubbard model (AHM) [68] with a two-fold purpose. On the one hand, it allows us to become familiar with the bosonization technique, while on the other hand we prepare the ground to tackle the Hamiltonian of the actual system to look for the appearance of the FFLO phase. The Hamiltonian of the AHM is given by [68]

$$H_{\text{AHM}} = -\sum_{j=1}^{N_s}\sum_{\delta=\pm1}\sum_{\sigma} t_\sigma c^\dagger_{j,\sigma}\, c_{j+\delta,\sigma} + U\sum_{j=1}^{N_s} n_{j\uparrow}n_{j\downarrow}, \tag{2.5.1}$$

where the asymmetry is manifested in the fact that the hopping integral t_σ is spin dependent, so that fermions with different spin have different probability amplitudes to hop along the lattice. N_s is the total number of sites, δ denotes the two neighboring sites, and U the usual interaction strength.

We should regard this Hamiltonian for a system studied at the thermodynamic limit ($L \to \infty$) otherwise the hopping term would be manifestly non-Hermitian. Then, from the very beginning we are considering a system of infinite size. We take a different path, namely, we first bosonize the finite-size system, which has proved to be a non-trivial task full of subtleties [69]. By taking this path we shall witness such difficulties. To do so, we work out a more familiar Hamiltonian (though equivalent to (2.5.1) in the limit $L \to \infty$) for the AHM:

$$H_{\text{AHM}} = -\sum_{j=1}^{N_s}\sum_{\sigma}\left(t_\sigma\, c^\dagger_{j,\sigma}c_{j+1,\sigma} + \text{H.c}\right) + U\sum_{j=1}^{N_s} n_{j\uparrow}n_{j\downarrow}. \tag{2.5.2}$$

We are interested in bosonizing the Hamiltonian (2.5.2), if we were purists and followed the constructive approach of bosonization, then the first thing we would have to do is to go to the momentum representation and linearize the energy spectrum. This step is merely pedagogical provided that, regardless the shape of the dispersion relation, we can always linearize it as long as we work in one-dimension. However, for this system we have an additional ingredient, namely, we take into account the spin of the fermions. Typically, spin would not alter the spatial degrees of freedom (at least not in the non-relativistic regime where spin-orbit coupling is negligible) but the system displays an asymmetry, such that the hopping integral t_σ is spin dependent. This means that the dispersion relation is also spin dependent

$$\varepsilon_\sigma(k) = -2t_\sigma \cos(ka), \tag{2.5.3}$$

so that we are actually dealing with two dispersion relations, such a situation is depicted in figure 2.8.

We shall see the effect of the different dispersion relations in the interaction term and in the bosonized Hamiltonian. However, since now we realize that new questions arise due to the existence of two different dispersion relations in the same system. For example, in figure 2.8 we see that $k_F(\uparrow) < k_F(\downarrow)$ so that an excitation of order $k_F(\uparrow)$ may (or may not) be a

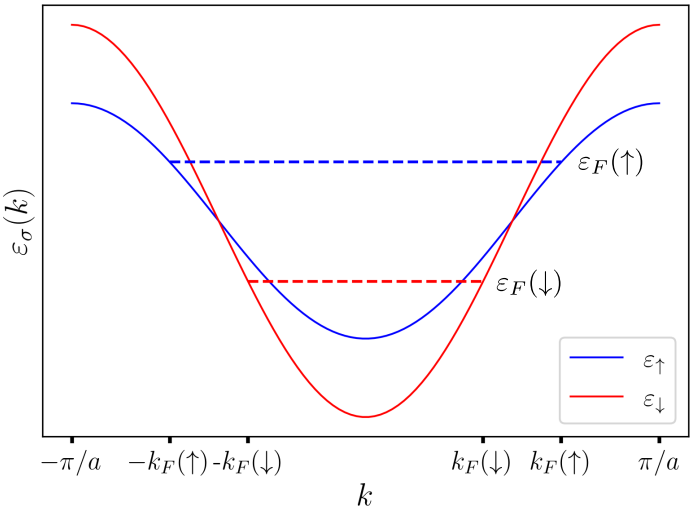

Figure 2.8: The asymmetry in the hopping integral opens the possibility of having two different dispersion relations, which in turn means to have two different Fermi surfaces $k_F(\sigma)$.

low-energy excitation for the spin-down particles, while it would be certainly a high-energy excitation for the spin-up particles. In other words, now we cannot assert without doubt that an excitation with momentum close to the Fermi momentum is a high-energy excitation, since the reply would be: Well, it depends on *which* Fermi momentum. We emphasize that this does not have to do with the presence of the spin but with the asymmetry of the hopping.

Going back to the "heuristic" approach, we know that to have two different dispersion relations implies that we have two different linearizations and hence, two different Hilbert spaces with left and right-movers. Then, after linearization, we get four different species of fermions, opposite to what happens in a customary fermionic chain [70] where the spin just make the information to be two-folded and we can simply work with a collective index for both chirality and spin.

2.5.1 Finite-size system

We shall work our fields in the continuum limit so let us set $a \to 0$, keeping L finite:

$$c_{j\sigma} = \frac{1}{\sqrt{N_s}} \sum_{\text{B.Z}} e^{ikaj} c_{k\sigma} \xrightarrow{a \to 0} \sqrt{a}\psi_\sigma(x) = \frac{1}{\sqrt{L}} \sum_{k=-\infty}^{\infty} e^{ikx} c_{k\sigma}, \tag{2.5.4}$$

where $k = 2\pi(n - \delta/2)/L$ with $n \in \mathbb{Z}$, and $\delta = 0, 1$ depending on the boundary condition we choose (periodic or anti-periodic, respectively).

From our work on bosonization in the previous sections, the relevant expressions for bosonizing the Hamiltonian (2.5.2) are compiled in the following set:

$$\psi_{\sigma\nu}(x) \equiv \frac{1}{\sqrt{L}} \sum_{k=-\infty}^{\infty} e^{i\alpha kx} c_{k,\sigma\nu}, \tag{2.5.5a}$$

$$\psi_\sigma(x) = e^{-ik_F(\sigma)x}\,\psi_{\sigma L}(x) + e^{ik_F(\sigma)x}\psi_{\sigma R}, \tag{2.5.5b}$$

$$\psi_{\sigma\nu}(x) = \frac{1}{\sqrt{2\pi\lambda}}\,F_{\sigma\nu}\exp\left[i\alpha\cdot\frac{2\pi}{L}\left(N_{\sigma\nu}-\frac{\delta}{2}\right)x - i\sqrt{2\pi}\phi_{\sigma\nu}(x)\right], \tag{2.5.5c}$$

$$\phi_\sigma(x) \equiv \frac{1}{\sqrt{2}}[\phi_{\sigma L}(x) - \phi_{\sigma R}(x)], \tag{2.5.5d}$$

$$\theta_\sigma(x) \equiv \frac{1}{\sqrt{2}}\sum_\nu \phi_{\sigma\nu}(x), \tag{2.5.5e}$$

$$\Pi_\sigma(x) \equiv \partial_x\theta_\sigma(x). \tag{2.5.5f}$$

Recall that $\alpha = 1$ stands for right-movers and $\alpha = -1$ for left-movers, ν denotes the chirality, $\nu = \{R, L\}$. $N_{\sigma\nu}$ counts the number of *fermions* with spin σ *above* the vacuum in the ν branch. Compared to (2.4.10), in (2.5.5c) we have combined the exponentials with $N_{\sigma\nu}$ and $\phi_{\sigma\nu}$ which follows from the fact that the operators commute. Finally, λ is the cut-off to eliminate short wave-lengths contributions.

The first step to bosonize (2.5.2) is to write the Hamiltonian in terms of fermionic field operators. To this end, we have to set $a \to 0$ but keep L finite,

$$c_{j\sigma} \xrightarrow{a\to 0} \sqrt{a}\,\psi_\sigma(x) = \sqrt{a}\Big[e^{-ik_F(\sigma)x}\psi_{\sigma L}(x) + e^{ik_F(\sigma)x}\,\psi_{\sigma R}(x)\Big],\ \text{with } x = ja. \tag{2.5.6}$$

We write everything in terms of these chiral fermionic fields. The hopping term would transform in the following way

$$\begin{aligned} c^\dagger_{j\sigma}c_{j+1,\sigma} = a\Big[&e^{-ik_F(\sigma)a}\psi^\dagger_{\sigma L}(x)\psi_{\sigma L}(x+a) + e^{ik_F(\sigma)(2x+a)}\psi^\dagger_{\sigma L}(x)\psi_{\sigma R}(x+a) \\ &+ e^{-ik_F(\sigma)(2x+a)}\psi^\dagger_{\sigma R}(x)\psi_{\sigma L}(x+a) + e^{ik_F(\sigma)a}\psi^\dagger_{\sigma R}(x)\psi_{\sigma R}(x+a)\Big]. \end{aligned} \tag{2.5.7}$$

By going to the momentum representation, we can show that the *oscillatory* terms proportional to $e^{\pm ik_F(\sigma)(2x+a)}$ are not part of the low-energy regime because when adding (integrating) over j (x) we get something proportional to $\delta_{2k_F(\sigma),k-k'}$ ($\delta(2k_F(\sigma)-k+k')$). This makes complete sense because those terms are of the form $\psi^\dagger_{\sigma\nu}\psi_{\sigma\nu'}$ with $\nu \neq \nu'$, so they destroy a particle in one branch and create another in the other branch, being this a forbidden process. Therefore, in the low-energy regime with the allowed contributions we obtain

$$c^\dagger_{j\sigma}c_{j+1,\sigma} \overset{!}{=} a\Big[e^{-ik_F(\sigma)a}\psi^\dagger_{\sigma L}(x)\psi_{\sigma L}(x+a) + e^{ik_F(\sigma)a}\psi^\dagger_{\sigma R}(x)\psi_{\sigma R}(x+a)\Big]. \tag{2.5.8}$$

Equation (2.5.8) is enough to write down the kinetic term of the Hamiltonian in the continuum limit. Now, the building block of the interaction term is

$$\begin{aligned} n_{j\sigma} = c^\dagger_{j\sigma}c_{j\sigma} = a\Big[&\psi^\dagger_{\sigma L}(x)\psi_{\sigma L}(x) + \psi^\dagger_{\sigma R}(x)\psi_{\sigma R}(x) + e^{ik_F(\sigma)2x}\psi^\dagger_{\sigma L}(x)\psi_{\sigma R}(x) \\ &+ e^{-ik_F(\sigma)2x}\psi^\dagger_{\sigma R}(x)\psi_{\sigma L}(x)\Big], \end{aligned} \tag{2.5.9}$$

$$n_{j\sigma}n_{j\sigma'} = a^2\Big[\psi^\dagger_{\sigma L}(x)\psi_{\sigma L}(x) + \psi^\dagger_{\sigma R}(x)\psi_{\sigma R}(x) + e^{ik_F(\sigma)2x}\psi^\dagger_{\sigma L}(x)\psi_{\sigma R}(x)$$

$$+ e^{-ik_F(\sigma)2x}\psi^\dagger_{\sigma R}(x)\psi_{\sigma L}(x)\Big]\Big[\sigma \leftrightarrow \sigma'\Big]. \tag{2.5.10}$$

From our experience with the kinetic term we know that, once we perform the product in (2.5.10), not all the 16 terms contribute since some of them may represent high-energy behaviors. Notice that there are four terms that are non-oscillatory which come from the product of terms with no exponential factors. On the other hand, we neglect products of two terms, one with an exponential factor and the other one without it, because the result becomes oscillatory after summing over j. This discards 8 terms. Upon multiplying terms both with exponential factors, we get two terms with phases $\pm i[k_F(\sigma) + k_F(\sigma')]\,(2x)$, and since $k_F(\sigma) \geq 0$ by definition, and regardless of the spin, this *in general* gives rise to an oscillatory term, so we also discard it. Ultimately, there are two terms with phases $\pm i[k_F(\sigma) - k_F(\sigma')]\,(2x)$ which may give rise to a low-energy contribution, depending on how large is $k_F(\sigma) - k_F(\sigma')$ compared to $\min\{k_F(\sigma), k_F(\sigma')\}$. Another very good reason to retain those two terms for the moment is that in the symmetric case, when k_F does not depend on σ, such a phase would vanish giving rise to an honest low-energy contribution. After all these considerations, in the low-energy regime we should consider, in principle, only 6 out of the 16 terms:

$$\begin{aligned} n_{j,\sigma}n_{j\sigma'} \overset{!!}{=} a^2\Big[&\underbrace{\psi^\dagger_{\sigma L}(x)\psi_{\sigma L}(x)\psi^\dagger_{\sigma' L}(x)\psi_{\sigma' L}(x)}_{1} + \underbrace{\psi^\dagger_{\sigma L}(x)\psi_{\sigma L}(x)\psi^\dagger_{\sigma' R}(x)\psi_{\sigma' R}(x)}_{2} \\ &+ \underbrace{\psi^\dagger_{\sigma R}(x)\psi_{\sigma R}(x)\psi^\dagger_{\sigma' L}(x)\psi_{\sigma' L}(x)}_{3} + \underbrace{\psi^\dagger_{\sigma R}(x)\psi_{\sigma R}(x)\psi^\dagger_{\sigma' R}(x)\psi_{\sigma' R}(x)}_{4} \\ &+ e^{i[k_F(\sigma)-k_F(\sigma')]\cdot 2x}\,\psi^\dagger_{\sigma L}(x)\psi_{\sigma R}(x)\psi^\dagger_{\sigma' R}(x)\psi_{\sigma' L}(x) \\ &+ e^{-i[k_F(\sigma)-k_F(\sigma')]\cdot 2x}\,\psi^\dagger_{\sigma R}(x)\psi_{\sigma L}(x)\psi^\dagger_{\sigma' L}(x)\psi_{\sigma' R}(x)\Big] \end{aligned} \tag{2.5.11}$$

$$\begin{aligned} \overset{!!}{=} a^2\Big[&\psi^\dagger_{\sigma L}(x)\psi_{\sigma L}(x)\psi^\dagger_{\sigma' L}(x)\psi_{\sigma' L}(x) + \psi^\dagger_{\sigma L}(x)\psi_{\sigma L}(x)\psi^\dagger_{\sigma' R}(x)\psi_{\sigma' R}(x) \\ &\psi^\dagger_{\sigma R}(x)\psi_{\sigma R}(x)\psi^\dagger_{\sigma' L}(x)\psi_{\sigma' L}(x) + \psi^\dagger_{\sigma R}(x)\psi_{\sigma R}(x)\psi^\dagger_{\sigma' R}(x)\psi_{\sigma' R}(x)\Big]. \end{aligned} \tag{2.5.12}$$

Terms 1 and 4 are interpreted as intra-branch terms with no flip of the spin (forward scattering) while terms 2 and 3 are interpreted as inter-branch terms with no flip of the spin (also forward scattering). The last two terms in (2.5.11) do not preserve the number of particles per branch, for instance, the term proportional to $\psi^\dagger_{\sigma L}\psi_{\sigma R}\psi^\dagger_{\sigma' R}\psi_{\sigma' L}$ is lowering the number of spin-σ particles in the R branch and raising it in the L branch. Such processes that take fermions from one branch to another are also high-energy processes such that we shall discard them as well. These situations are depicted in figure 2.9.

Upon finding the terms that survive from (2.5.10) we discarded two terms because their phases are $\pm i[k_F(\sigma) + k_F(\sigma')]\,(2x)$, those terms represent Umklapp processes which *in the most general case* are high-energy contributions. However, in the very special case of half-filling (taken as $k_{F\uparrow} + k_{F\downarrow} = \pi/a$ [68]) those terms are not oscillatory and then we must consider them. This is why in (2.5.11) we used the notation "$\overset{!!}{=}$", since in this opportunity we are not restricting the analysis to the low-energy regime but also considering the system *far from the half-filling condition*, for it being out of our future interests.

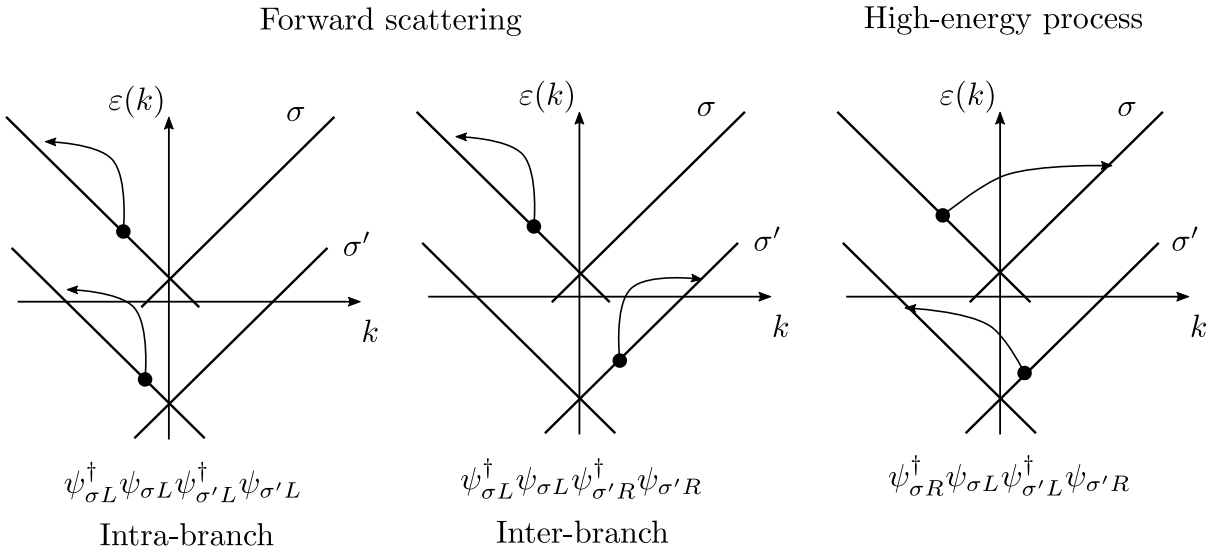

Figure 2.9: Graphic representation of the scattering processes.

Equations (2.5.8) and (2.5.12) are the building blocks of the Hamiltonian (2.5.2) in the continuum limit $a \to 0$. Now, we can use the bosonization identity (2.5.5c) to bosonize them. We begin with the kinetic part:

$$c^\dagger_{j\sigma}c_{j+1,\sigma} \overset{!}{=} a\Big[e^{-ik_F(\sigma)a}\psi^\dagger_{\sigma L}(x)\psi_{\sigma L}(x+a) + e^{ik_F(\sigma)a}\psi^\dagger_{\sigma R}(x)\psi_{\sigma R}(x+a)\Big]. \tag{2.5.13}$$

We only have to bosonize terms of the form $\psi^\dagger_{\sigma\nu}(x)\psi_{\sigma\nu}(x+a)$ (i.e. same chirality):

$$\begin{aligned}\psi^\dagger_{\sigma\nu}(x)\psi_{\sigma\nu}(x+a) =& \frac{1}{2\pi\lambda}F^\dagger_{\sigma\nu}\exp\left[-i\alpha\frac{2\pi}{L}\left(N_{\sigma\nu}-\frac{\delta}{2}\right)x\right]F_{\sigma\nu}\exp\left[i\sqrt{2\pi}\phi_{\sigma\nu}(x)\right]\\ &\times\exp\left[i\alpha\frac{2\pi}{L}\left(N_{\sigma\nu}-\frac{\delta}{2}\right)(x+a)\right]\exp\left[-i\sqrt{2\pi}\phi_{\sigma\nu}(x+a)\right].\end{aligned} \tag{2.5.14}$$

In the last equation we used the facts that: (i) $\phi^\dagger_{\sigma\nu} = \phi_{\sigma\nu}$. (ii) The Klein factors $F_{\sigma\nu}$ commute with the chiral bosonic field $\phi_{\sigma\nu}$, which can be seen using the mode decomposition of the latter. The tricky part is that the Klein factors do not commute with $N_{\sigma\nu}$, recall that they verify

$$[N_\eta, F^\dagger_{\eta'}] = \delta_{\eta,\eta'}F^\dagger_\eta \qquad [N_\eta, F_{\eta'}] = -\delta_{\eta,\eta'}F_\eta, \tag{2.5.15}$$

with η a collective index for the different species of fermions, in this case $\eta = \{\uparrow R, \uparrow L, \downarrow R, \downarrow L\}$.

Using these commutation relations we can see that

$$F^\dagger_{\sigma\nu}\exp\left[-i\alpha\frac{2\pi}{L}\left(N_{\sigma\nu}-\frac{\delta}{2}\right)x\right] = \exp\left[-i\alpha\frac{2\pi}{L}\left(N_{\sigma\nu}-\frac{\delta}{2}-1\right)x\right]F^\dagger_{\sigma\nu}. \tag{2.5.16}$$

Therefore, equation (2.5.14) yields

$$\psi^\dagger_{\sigma\nu}(x)\psi_{\sigma\nu}(x+a) = \frac{1}{2\pi\lambda}\exp\left[i\alpha\frac{2\pi}{L}x + i\alpha\frac{2\pi}{L}a\left(N_{\sigma\nu}-\frac{\delta}{2}\right)\right]\exp\left[i\sqrt{2\pi}\phi_{\sigma\nu}(x)\right]$$

$$\times \exp\Big[-i\sqrt{2\pi}\phi_{\sigma\nu}(x+a)\Big]. \tag{2.5.17}$$

We could get rid of the Klein factors because we are dealing with a product of fields with the same chirality and spin, so that their action eventually balance through unitarity. On the other hand, we want to combine the exponentials of the chiral bosonic fields, so that we invoke the BCH formula and use the fact that (see (A.3.37) in Appendix A.3.1)

$$[\phi_\eta(x), \phi_{\eta'}(y)] = \frac{i\alpha}{2}\mathrm{sgn}(x-y)\delta_{\eta,\eta'}, \tag{2.5.18}$$

which leads us to

$$\begin{aligned}\psi^\dagger_{\sigma\nu}(x)\psi_{\sigma\nu}(x+a) &= \frac{1}{2\pi\lambda}\exp\left[i\alpha\frac{2\pi}{L}x + i\alpha\frac{2\pi a}{L}\left(N_{\sigma\nu}-\frac{\delta}{2}\right)\right]\exp\left[\alpha\frac{i\pi}{2}\mathrm{sgn}(-a)\right]\\ &\quad\times\exp\Big[i\sqrt{2\pi}(\phi_{\sigma\nu}(x)-\phi_{\sigma\nu}(x+a))\Big]\\ &= \frac{1}{2\pi\lambda}\exp\left[i\alpha\frac{2\pi}{L}x + i\alpha\frac{2\pi a}{L}\left(N_{\sigma\nu}-\frac{\delta}{2}\right) - \alpha\frac{i\pi}{2}\right]\\ &\quad\times\exp\Big[i\sqrt{2\pi}(-a\partial_x\phi_{\sigma\nu}(x))\Big],\ \text{at first order in } a\ (a\to 0).\end{aligned}$$

With the result (2.5.19) we can bosonize (2.5.8),

$$\begin{aligned}c^\dagger_{j,\sigma}c_{j+1,\sigma} =& \frac{a}{2\pi\lambda}\Bigg\{\exp\left[-i\frac{2\pi}{L}x - i\frac{2\pi a}{L}\left(N_{\sigma L}-\frac{\delta}{2}\right) + i\frac{\pi}{2} - ik_F(\sigma)a - ia\sqrt{2\pi}\partial_x\phi_{\sigma\nu}(x)\right]\\ &+\exp\left[i\frac{2\pi}{L}x + i\frac{2\pi a}{L}\left(N_{\sigma R}-\frac{\delta}{2}\right) - i\frac{\pi}{2} + ik_F(\sigma)a - ia\sqrt{2\pi}\partial_x\phi_{\sigma R}(x)\right]\Bigg\}.\end{aligned} \tag{2.5.19}$$

Considering that the kinetic part in the Hamiltonian (2.5.2) has the form $(t_\sigma c^\dagger_{j\sigma}c_{j+1,\sigma}+\text{H.c})$, if we *assume* that the hopping integral is real then we can write it as $t_\sigma(c^\dagger_{j\sigma}c_{j+1,\sigma}+\text{H.c})$. In such a case, to find $(c^\dagger_{j\sigma}c_{j+1,\sigma}+\text{H.c})$ is straightforward once we have $c^\dagger_{j\sigma}c_{j+1,\sigma}$ since all the operators inside the exponentials are Hermitian, so then the $+$H.c makes the exponentials to become into cosines,

$$\begin{aligned}c^\dagger_{j,\sigma}c_{j+1,\sigma}+\text{H.c} =& \frac{a}{\pi\lambda}\Bigg\{\cos\left[-\frac{2\pi}{L}x - \frac{2\pi a}{L}\left(N_{\sigma L}-\frac{\delta}{2}\right) + \frac{\pi}{2} - k_F(\sigma)a - a\sqrt{2\pi}\partial_x\phi_{\sigma L}(x)\right]\\ &+\cos\left[\frac{2\pi}{L}x + \frac{2\pi a}{L}\left(N_{\sigma R}-\frac{\delta}{2}\right) - \frac{\pi}{2} + k_F(\sigma)a - a\sqrt{2\pi}\partial_x\phi_{\sigma R}(x)\right]\Bigg\}.\end{aligned} \tag{2.5.20}$$

Herein we have some alternatives. We might be tempted to proceed similarly as in (2.5.19) where we expanded up to first order in a because $a\to 0$. However, in this case we have some quantities inside the cosines that are not proportional to a, so that we cannot regard the whole argument of the cosines as being small. The resolution for this could be to use sum of angles identities to separate the terms that are proportional to a from those that are not,

and then make the correspondent approximation. A shorter path is to use sum-to-product trigonometric identities, which yields,

$$c^\dagger_{j\sigma}c_{j+1,\sigma} + \text{H.c} = \frac{2a}{\pi\lambda}\cos\left[-\frac{\pi a}{L}(N_{\sigma R} - N_{\sigma L}) - a\sqrt{\frac{\pi}{2}}\partial_x(\phi_{\sigma L} + \phi_{\sigma R})\right] \times \cos\Big[-k_F(\sigma)a - \frac{2\pi}{L}x - \frac{\pi a}{L}(N_{\sigma R} + N_{\sigma L}) + \frac{\pi a}{L}\delta + \frac{\pi}{2} - a\sqrt{\frac{\pi}{2}}\partial_x(\phi_{\sigma L} - \phi_{\sigma R})\Big]. \tag{2.5.21}$$

Notice that by adding and subtracting the chiral bosonic field we get the dual bosonic fields ϕ_σ and θ_σ, defined in equations (2.5.5d) and (2.5.5e). It is clear now why sum-to-product identities is a better path,

$$c^\dagger_{j,\sigma}c_{j+1,\sigma} + \text{H.c} = \frac{2a}{\pi\lambda}\cos\left[\frac{\pi a}{L}(N_{\sigma L} - N_{\sigma R}) + a\sqrt{\pi}\partial_x\theta_\sigma\right] \times \sin\left[k_F(\sigma)a + \frac{2\pi}{L}x + \frac{\pi a}{L}(N_{\sigma L} + N_{\sigma R} - \delta) + a\sqrt{\pi}\partial_x\phi_\sigma\right]. \tag{2.5.22}$$

This is all for the kinetic energy, let us focus now on the interaction term (2.5.12). The intra-branch terms and inter-branch terms can be identified automatically with products of densities

$$\psi^\dagger_{\sigma\nu}(x)\psi_{\sigma\nu}(x)\psi^\dagger_{\sigma'\nu}(x)\psi_{\sigma'\nu}(x) = \rho_{\sigma\nu}(x)\rho_{\sigma'\nu}(x) \tag{2.5.23}$$

$$\psi^\dagger_{\sigma\nu}(x)\psi_{\sigma\nu}(x)\psi^\dagger_{\sigma'\nu'}(x)\psi_{\sigma'\nu'}(x) = \rho_{\sigma\nu}(x)\rho_{\sigma'\nu'}(x). \tag{2.5.24}$$

Since these terms are constructed as products of pairs of operators with the same chirality and spin, the action of the Klein factors is balanced as we showed explicitly when bosonizing the kinetic part.

So that the interaction term turns out to be very simple,

$$n_{j\sigma}n_{j\sigma'} \overset{!!}{=} a^2\sum_{\nu,\nu'}\rho_{\sigma\nu}(x)\rho_{\sigma'\nu'}(x). \tag{2.5.25}$$

Equations (2.5.22) and (2.5.25) are all we need to write down the Hamiltian for the AHM in terms of the bosonic fields, in the continuum limit $a \to 0$, in the low-energy regime and far from the half-filling condition:

$$H_{\text{AHM}} = H_{\text{kin}} + H_{\text{int}} \tag{2.5.26}$$

$$H_{\text{kin}} = -\int dx \sum_\sigma \frac{2t_\sigma}{\pi\lambda}\Bigg\{\cos\left[\frac{\pi a}{L}(N_{\sigma R} - N_{\sigma L}) - a\sqrt{\pi}\partial_x\theta_\sigma\right]\sin\left[k_F(\sigma)a + \frac{2\pi}{L}\left(x - \frac{\delta}{2}\right) - \frac{\pi a}{L}(N_{\sigma R} + N_{\sigma L}) - a\sqrt{\pi}\partial_x\phi_\sigma\right]\Bigg\} \tag{2.5.27}$$

$$H_{\text{int}} = U \int dx\; a \sum_{\nu,\nu'} \rho_{\uparrow\nu}(x)\rho_{\downarrow'\nu'}(x). \tag{2.5.28}$$

This is enough to be convinced of the difficulties that exist when trying to analyze the finite-size system. Even though we have a closed expression for the bosonized Hamiltonian, it does not depend only on the bosonic operators but also on many N's. Remarkably, the action of the Klein factors gets balanced both in the kinetic and interaction terms. Also, we have many terms that do not get simplified nor can be combined into simpler ones. On the other hand, it is necessary to take $L \to \infty$ provided that all the observables should be computed in the thermodynamic limit.

The reader might be wondering why we bothered setting up the Klein factors at the level of the Fock space and in the bosonization identity (2.4.10), if they turn out to be immaterial in the bosonization of our Hamiltonian. This is actually one of the reasons why a large part of the literature either omit them or slightly discuss them. However, as we shall see in Chapter 3 they are relevant upon computing the correlation functions [42].

2.5.2 Taking the thermodynamic limit

Taking $L \to \infty$ in the Hamiltonian (2.5.28) yields

$$\begin{aligned} H_{\text{AHM}} = &- \int dx \sum_{\sigma} \frac{2t_\sigma}{\pi\lambda} \left\{ \cos\left[a\sqrt{\phi}\partial_x\theta_\sigma\right] \sin\left[k_F(\sigma)a + a\sqrt{\pi}\partial_x\phi_\sigma\right] \right\} \\ &+ U \int dx\; a \sum_{\nu,\nu'} \rho_{\uparrow\nu}(x)\rho_{\downarrow\nu}(x) \end{aligned} \tag{2.5.29}$$

This is a considerable simplification compared to (2.5.26). We shall simplify it further noting that in the kinetic part we have trigonometric functions with arguments proportional to a, so that we can expand them in power series.

$$\begin{aligned} &\cos\left[a\sqrt{\pi}\partial_x\theta_\sigma\right] \sin\left[k_F(\sigma)a + a\sqrt{\pi}\partial_x\phi_\sigma\right] \\ &= \cos\left[\sqrt{\pi}a\partial_x\theta_\sigma\right] \left\{ \sin[k_F(\sigma)a] \cos\left[a\sqrt{\pi}\partial_x\phi_\sigma\right] + \cos\left[k_F(\sigma)a\right] \sin\left[a\sqrt{\pi}\partial_x\phi_\sigma\right] \right\}. \end{aligned} \tag{2.5.30}$$

We expand

$$\begin{aligned} \cos\left[a\sqrt{\pi}\partial_x\theta_\sigma\right] \cos\left[a\sqrt{\pi}\partial_x\phi_\sigma\right] &= 1 - \frac{\pi}{2}a^2(\partial_x\phi_\sigma)^2 - \frac{\pi}{2}a^2(\partial_x\theta_\sigma)^2 + \mathcal{O}(a^4) \\ \cos\left[a\sqrt{\pi}\partial_x\theta_\sigma\right] \sin\left[\sqrt{\pi}a\partial_x\phi_\sigma\right] &= a\sqrt{\pi}\partial_x\phi_\sigma + \mathcal{O}(a^3). \end{aligned} \tag{2.5.31}$$

Despite in equation (2.5.19) we approximated up to first order in a, in the previous equations it would be wise to approximate up to second order because the kinetic part of the Hamiltonian has a factor $1/\lambda$ that eventually cancels out one order in a, so that our final expression is, consistently, of first order in a.

$$H_{\text{kin}} = - \int dx \sum_{\sigma} \frac{2t_\sigma}{\pi\lambda} \left\{ \sin[k_F(\sigma)a] \left[1 - \frac{\pi}{2}a^2(\partial_x\phi_\sigma)^2 - \frac{\pi}{2}a^2(\partial_x\theta_\sigma)^2\right] + \cos[k_F(\sigma)a] a\sqrt{\pi}\partial_x\phi_\sigma \right\}. \tag{2.5.32}$$

Notice that the zeroth order contribution in a leads to a divergent term after integrating. To fix this, it is necessary to employ the *normal order for bosons*. Using it the divergent factor goes away and we are left with:

$$\begin{aligned} :H:_{\text{kin}} &= \int dx \sum_\sigma \frac{2t_\sigma}{2\lambda} a^2 \sin[k_F(\sigma)a]\left[(\partial_x\phi_\sigma)^2 + (\partial_x\theta_\sigma)^2\right] - \int dx \sum_\sigma \frac{2t_\sigma a}{\sqrt{\pi}\lambda} \cos[k_F(\sigma)a]\partial_x\phi_\sigma \\ &= \int dx \sum_\sigma \frac{v_F(\sigma)a}{2\lambda}\left[(\partial_x\phi_\sigma)^2 + (\partial_x\theta_\sigma)^2\right] + \int dx \sum_\sigma \frac{\varepsilon_\sigma(k_F)a}{\sqrt{\pi}\lambda}\partial_x\phi_\sigma, \end{aligned} \tag{2.5.33}$$

where in the last line we have introduced the Fermi velocity $v_F(\sigma) \equiv d\varepsilon_\sigma/dk\big|_{k=k_F}$ and the dispersion relation $\varepsilon_\sigma(k) = -2t_\sigma\cos(ka)$.

Since a and λ are of the same order ($a \to 0$ in the continuum limit and $\lambda \to 0^+$ by definition) we are left with an expression of zeroth order in a for the kinetic part:

$$:H:_{\text{kin}} = \int dx \sum_\sigma \frac{v_F(\sigma)}{2}\left[(\partial_x\phi_\sigma)^2 + (\partial_x\theta_\sigma)^2\right] + \int dx \sum_\sigma \frac{\varepsilon_\sigma(k_F)}{\sqrt{\pi}}\partial_x\phi_\sigma. \tag{2.5.34}$$

Therefore, the normal-ordered Hamiltonian is given by

$$:H_{\text{AHM}}: = \int dx \sum_\sigma \frac{v_F(\sigma)}{2}\left[(\partial_x\phi_\sigma)^2 + \Pi_\sigma^2\right] + \int dx \sum_\sigma \frac{\varepsilon_\sigma(k_F)}{\sqrt{\pi}}\partial_x\phi_\sigma + U\int dx\, a\sum_{\nu\nu'}\rho_{\uparrow\nu}(x)\rho_{\downarrow\nu'}(x). \tag{2.5.35}$$

This expression looks great and we are almost done, it is still left to express the densities in terms of the bosonic fields. Inverting (2.5.5d) and (2.5.5e), by using equations (A.3.44) and (2.5.5f), and taking the thermodynamic limit, we end up with

$$\rho_{\sigma\nu}(x) = \frac{1}{2\sqrt{\pi}}(\partial_x\phi_\sigma - \alpha\Pi_\sigma) \implies \sum_{\nu\nu'}\rho_{\uparrow\nu}(x)\rho_{\downarrow\nu'}(x) = \frac{1}{\pi}(\partial_x\phi_\uparrow)(\partial_x\phi_\downarrow). \tag{2.5.36}$$

Thus, we can write down the fully-bosonized Hamiltonian for the AHM:

$$:H_{\text{AHM}}: = \sum_\sigma \int dx \frac{v_F(\sigma)}{2}\left[(\partial_x\phi_\sigma)^2 + \Pi_\sigma^2\right] + \sum_\sigma \int dx \frac{\varepsilon_\sigma(k_F)}{\sqrt{\pi}} + Ua\int dx \frac{1}{\pi}(\partial_x\phi_\uparrow)(\partial_x\phi_\downarrow) \tag{2.5.37}$$

$$\equiv H_{\text{kin}} + H_{\text{surf}} + H_{\text{int}}. \tag{2.5.38}$$

Which are the considerations we made along the way of finding this Hamiltonian? We are working in the thermodynamic limit, in the low-energy regime and with a filling other than half-filling. Besides, we *assume* that the hopping integral t_σ is real, and use the normal order for bosons.

2.5.3 Testing spin-charge separation

So far we have written the Hamiltonian of the system in a field-theoretical bosonic language. It does not provide much information about the physics of the system, aside from the implications of being able to employ the bosonization technique. On the other hand, one of the benefits of studying the AHM is that we can reach the Hubbard model by setting $t_\downarrow = t_\uparrow = t$. The latter has been widely studied and its bosonized version is well-known to lead to two major findings: i) A separation of the system into spin and charge degrees of freedom and ii) the existence of various phases depending on the filling of the system and the nature of the interaction, regardless how strong it is (that is, it only depends on the sign of U, and there is no finite critical value) [42, 44, 71]. We conclude this section by investigating to what extent the hopping asymmetry for the AHM deviates us from the physics of the Hubbard model.

In the Hamiltonian (2.5.38) the interaction term gives rise to a cross term $(\partial_x\phi_\uparrow)(\partial_x\phi_\downarrow)$, we shall see that this term is decoupled by the so-called spin and charge fields that appear in the study of the Hubbard model, though it does not diagonalize the whole Hamiltonian. The cross term can be written as the quadratic form

$$(\partial_x\phi_\uparrow)(\partial_x\phi_\downarrow) = \frac{1}{2}\begin{pmatrix}\partial_x\phi_\uparrow & \partial_x\phi_\downarrow\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\partial_x\phi_\uparrow\\ \partial_x\phi_\downarrow\end{pmatrix}. \tag{2.5.39}$$

We find that the eigenvectors of the matrix in (2.5.39) are $\begin{pmatrix}-1 & 1\end{pmatrix}^T$ and $\begin{pmatrix}1 & 1\end{pmatrix}^T$, up to normalization factors, and the eigenvalues are -1 and 1, respectively. Hence, the diagonalization of the matrix is accomplished through

$$\begin{aligned} A &\equiv \frac{1}{\sqrt{2}}\begin{pmatrix}-1 & 1\\ 1 & 1\end{pmatrix}\\ (\partial_x\phi_\uparrow)(\partial_x\phi_\downarrow) &= \frac{1}{2}\begin{pmatrix}\partial_x\phi_\uparrow & \partial_x\phi_\downarrow\end{pmatrix}A\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}A^{-1}\begin{pmatrix}\partial_x\phi_\uparrow\\ \partial_x\phi_\downarrow\end{pmatrix}\\ (\partial_x\phi_\uparrow)(\partial_x\phi_\downarrow) &= \frac{1}{2}(\partial_x\phi_c)^2 - \frac{1}{2}(\partial_x\phi_s)^2, \end{aligned} \tag{2.5.40}$$

where we have defined the charge and spin fields as,

$$\begin{aligned} \phi_c &\equiv \frac{1}{\sqrt{2}}(\phi_\uparrow + \phi_\downarrow)\\ \phi_s &\equiv \frac{1}{\sqrt{2}}(\phi_\uparrow - \phi_\downarrow). \end{aligned} \tag{2.5.41}$$

In terms of these new fields the interaction part of the Hamiltonian (2.5.38) turns out to be

$$H_{\text{int}} = Ua\int dx\left[\frac{1}{2\pi}(\partial_x\phi_c)^2 - \frac{1}{2\pi}(\partial_x\phi_s)^2\right]. \tag{2.5.42}$$

Now, we should also write the kinetic part of the Hamiltonian in terms of the new fields. To

do so we stick with the matrix approach,

$$\begin{aligned} H_{\text{kin}} &= \sum_{\sigma} \int dx \frac{v_F(\sigma)}{2} \left[(\partial_x \phi_\sigma)^2 + \Pi_\sigma^2 \right] \\ &= \frac{1}{2} \int dx \left[\begin{pmatrix} \partial_x \phi_\uparrow & \partial_x \phi_\downarrow \end{pmatrix} \begin{pmatrix} v_{F\uparrow} & 0 \\ 0 & v_{F\downarrow} \end{pmatrix} \begin{pmatrix} \partial_x \phi_\uparrow \\ \partial_x \phi_\downarrow \end{pmatrix} + \begin{pmatrix} \Pi_\uparrow & \Pi_\downarrow \end{pmatrix} \begin{pmatrix} v_{F\uparrow} & 0 \\ 0 & v_{F\downarrow} \end{pmatrix} \begin{pmatrix} \Pi_\uparrow \\ \Pi_\downarrow \end{pmatrix} \right]. \end{aligned} \tag{2.5.43}$$

Under the transformation $\{\phi_\uparrow, \phi_\downarrow\} \to \{\phi_c, \phi_s\}$ the relevant quantities transform as

$$\begin{pmatrix} \partial_x \phi_\uparrow \\ \partial_x \phi_\downarrow \end{pmatrix} \to A^{-1} \begin{pmatrix} \partial_x \phi_\uparrow \\ \partial_x \phi_\downarrow \end{pmatrix} = \begin{pmatrix} -\partial_x \phi_s \\ \partial_x \phi_c \end{pmatrix} \tag{2.5.44}$$

$$\begin{pmatrix} v_{F\uparrow} & 0 \\ 0 & v_{F\downarrow} \end{pmatrix} \to A^{-1} \begin{pmatrix} v_{F\uparrow} & 0 \\ 0 & v_{F\downarrow} \end{pmatrix} A = \frac{1}{2} \begin{pmatrix} v_{F\uparrow} + v_{F\downarrow} & v_{F\downarrow} - v_{F\uparrow} \\ v_{F\downarrow} - v_{F\uparrow} & v_{F\downarrow} + v_{F\uparrow} \end{pmatrix} \equiv \frac{1}{2} \begin{pmatrix} v_+ & v_- \\ v_- & v_+ \end{pmatrix} \tag{2.5.45}$$

$$\begin{pmatrix} \Pi_\uparrow \\ \Pi_\downarrow \end{pmatrix} \to A^{-1} \begin{pmatrix} \Pi_\uparrow \\ \Pi_\downarrow \end{pmatrix} = \begin{pmatrix} -\Pi_s \\ \Pi_c \end{pmatrix}. \tag{2.5.46}$$

So then, the kinetic part reads

$$\begin{aligned} H_{\text{kin}} &= \int dx \, \frac{1}{2} \left[(-\partial_x \phi_s \quad \partial_x \phi_c) \cdot \frac{1}{2} \begin{pmatrix} v_+ & v_- \\ v_- & v_+ \end{pmatrix} \begin{pmatrix} -\partial_x \phi_s \\ \partial_x \phi_c \end{pmatrix} + (-\Pi_s \quad \Pi_c) \cdot \frac{1}{2} \begin{pmatrix} v_+ & v_- \\ v_- & v_+ \end{pmatrix} \begin{pmatrix} -\Pi_s \\ \Pi_c \end{pmatrix} \right] \\ &= \int dx \, \frac{1}{2} \left\{ \frac{1}{2} \left[v_+ (\partial_x \phi_s)^2 + v_+ (\partial_x \phi_c)^2 \right] - v_- (\partial_x \phi_c)(\partial_x \phi_s) \right\} \\ &\quad + \int dx \, \frac{1}{2} \left\{ \frac{1}{2} \left[v_+ \Pi_s^2 + v_+ \Pi_c^2 \right] - v_- \Pi_c \Pi_s \right\}. \end{aligned} \tag{2.5.47}$$

Ultimately, in terms of the new fields the Hamiltonian turns out to be

$$\begin{aligned} :H_{\text{kin}} + H_{\text{int}}: =& \frac{v_+}{4} \int dx \left[\left(1 + \frac{2Ua}{\pi v_+} \right) (\partial_x \phi_c)^2 + \Pi_c^2 \right] + \frac{v_+}{4} \int dx \left[\left(1 - \frac{2Ua}{\pi v_+} \right) (\partial_x \phi_s)^2 + \Pi_s^2 \right] \\ &+ \frac{v_-}{2} \int dx [(\partial_x \phi_c)(\partial_x \phi_s) + \Pi_c \Pi_s] \end{aligned} \tag{2.5.48}$$

$$:H_{\text{surf}}: = \sum_{\sigma} \int dx \frac{\varepsilon_\sigma(k_F)}{\sqrt{\pi}} \partial_x \phi_\sigma(x). \tag{2.5.49}$$

We define the Luttinger parameter and the propagation velocity for each degree of freedom as

$$\frac{1}{K_c} \equiv \sqrt{1 + \frac{2Ua}{\pi v_+}}, \qquad v_c \equiv \frac{v_+}{2K_c}, \tag{2.5.50}$$

$$\frac{1}{K_s} \equiv \sqrt{1 - \frac{2Ua}{\pi v_+}}, \qquad v_s \equiv \frac{v_+}{2K_s}. \tag{2.5.51}$$

Therefore, the Hamiltonian of the AHM in terms of the charge and spin fields is given by

$$:H_{\text{AHM}}: = \frac{v_c}{2} \int dx \left[\frac{1}{K_c} (\partial_x \phi_c)^2 + K_c \Pi_c^2 \right] + \frac{v_s}{2} \int dx \left[\frac{1}{K_s} (\partial_x \phi_s)^2 + K_s \Pi_s^2 \right]$$

$$+\frac{v_-}{2}\int dx[(\partial_x\phi_s)(\partial_x\phi_c)+\Pi_c\Pi_s]+\sum_\sigma\int dx\frac{\varepsilon_\sigma(k_F)}{\sqrt{\pi}}\partial_x\phi_\sigma(x). \tag{2.5.52}$$

From this Hamiltonian it is clear that the asymmetry prevents the system to display a spin-charge separation. The third term in the Hamiltonian is a coupling term between the spin and charge degrees of freedom, that arises when we rewrite the kinetic energy with respect to these fields. This term is proportional to v_{F-} which is zero in the limiting case when $t_\uparrow = t_\downarrow$, that is, when we recover the Hubbard model. Since the spin and charge fields do not diagonalize the Hamiltonian, we should not attempt to understand the phase transitions of the system in terms of whether the spin or charge sectors acquire a mass gap (for more reference see e.g. [42]).

Having become familiar with the bosonization framework by writing the Hamiltonian of the AHM in the bosonic language, we are ready to tackle our actual system looking for unconventional superconductivity. We shall see that there exists a parallel between the Hamiltonian of the AHM and our effective Hamiltonian, the one that is able to display the FFLO phase, which is actually the main objective of the present Thesis and is broadly discussed in Chapter 3, together with our more relevant results on the subject.

Chapter 3

The FFLO phase in the attractive Fermi model

As explained in chapter 1 the FFLO phase is expected to arise from the interplay of superconductivity and magnetism, therefore we consider a one-dimensional lattice of fermions described by the following Hamiltonian

$$H = \sum_{j=1}^{N_s} \left[-t \sum_{\sigma} (c_{j\sigma}^{\dagger} c_{j+1,\sigma} + \text{H.c}) + h c_j^{\dagger} \sigma_z c_j - \mu \sum_{\sigma} c_{j\sigma}^{\dagger} c_{j\sigma} + U n_{j\uparrow} n_{j\downarrow} \right], \tag{3.0.1}$$

where t is the probability amplitude for the hopping between adjacent sites of the lattice, h is the linear Zeeman field magnitude pointing to the $-z$ direction, μ is the chemical potential, $U < 0$ is the attractive on-site interaction strength, and where we have introduced a Pauli matrix notation for the magnetic field:

$$c_j^{\dagger} \sigma_z c_j \equiv \begin{pmatrix} c_{j\uparrow}^{\dagger} & c_{j\downarrow}^{\dagger} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} c_{j\uparrow} \\ c_{j\downarrow} \end{pmatrix}. \tag{3.0.2}$$

The Hamiltonian (3.0.1) can be regarded as the well-known Hubbard model with the addition of two external fields, h and μ. The superconducting behavior is introduced through the on-site interaction, and the magnetic features are tuned through the Zeeman field. The chemical potential enforces particle conservation.

One can gather the effect of the external fields into a single term,

$$h c_j^{\dagger} \sigma_z c_j - \mu \sum_{\sigma} c_{j\sigma}^{\dagger} c_{j\sigma} = -(\mu - h) c_{j\uparrow}^{\dagger} c_{j\uparrow} - (\mu + h) c_{j\downarrow}^{\dagger} c_{j\downarrow} \equiv -\mu_{\uparrow} c_{j\uparrow}^{\dagger} c_{j\uparrow} - \mu_{\downarrow} c_{j\downarrow}^{\dagger} c_{j\downarrow}. \tag{3.0.3}$$

Therefore (3.0.1) can be written as

$$H = \sum_{j=1}^{N_s} \left[-t \sum_{\sigma} \left(c_{j\sigma}^{\dagger} c_{j+1,\sigma} + \text{H.c} \right) - \sum_{\sigma} \mu_{\sigma} c_{j\sigma}^{\dagger} c_{j\sigma} + U\, n_{j\uparrow} n_{j\downarrow} \right]. \tag{3.0.4}$$

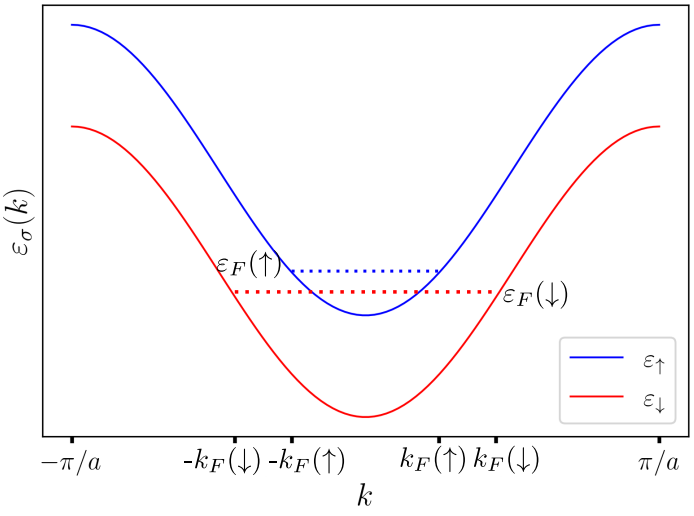

Figure 3.1: The polarization of the sample lowers the energy of the spin-down particles giving rise to two different dispersion relations with two different Fermi surfaces $k_F(\sigma)$. This is the well-known Zeeman effect.

The Hamiltonian for the non-interacting system ($U = 0$) in momentum representation reads

$$H_0 = \sum_k \sum_\sigma (-2t\cos(ka) - \mu_\sigma) c^\dagger_{k\sigma} c_{k\sigma}. \tag{3.0.5}$$

This is a striking result because the dispersion relation of the system is σ-dependent: $\varepsilon(k) = \varepsilon_\sigma(k) = -2t\cos(ka) - \mu_\sigma$. This is the reason why we explored the bosonization technique through the AHM instead of any other system, it also displayed a σ-dependent dispersion relation (see (2.5.3)). Hence, we are able to bosonize the Hamiltonian (3.0.4) by using some of the results already obtained in section 2.5. However, it is important to point out some differences between both systems: For the AHM the σ dependence comes from the asymmetry in the hopping integral, t_σ, while for the attractive Fermi model it comes from the presence of the magnetic field that polarizes the sample and, due to the chosen quantization axis direction, it lowers the energy of the spin-down particles.

Then, a question naturally arises, namely, *if the source of the σ-dependence is different in both systems, is the bosonization of the Hamiltonians the same?* From the "heuristic" point of view what matters is how the linearization of $\varepsilon_\sigma(k)$ would look like. A sketch of the dispersion relation for the system is depicted in figure 3.1. Comparing this figure to figure 2.8 it is clear that the split of the dispersion relations has a different nature for each system: In this case the well-known Zeeman effect shifts the energy of one of the species of fermions, while for the AHM we had a modulation of the amplitude of the curves due to t_σ.

Therefore, for the sake of the bosonization program what matters is that now we also end up with different dispersion relations depending on the spin of the fermions, meaning that we shall have two different linearizations, as in the AHM. Hence, the bosonization should

resemble what we already found,

$$H=\sum_{j=1}^{N_s}\Big[\underbrace{-t\sum_{\sigma}\left(c^{\dagger}_{j\sigma}c_{j+1,\sigma}+\text{H.c}\right)+U\,n_{j\uparrow}n_{j\downarrow}}_{\text{(I): Same as the AHM, eq. (2.5.2) with } t_\uparrow=t_\downarrow}\quad-\quad\underbrace{\sum_{\sigma}\mu_\sigma c^{\dagger}_{j\sigma}c_{j\sigma}}_{\text{(II): New term to be bosonized.}}\Big]. \tag{3.0.6}$$

This claim should be examined closely, for to set $t_\uparrow = t_\downarrow$ might seem to destroy the σ-dependence in the AHM, which is precisely the link to our actual system. The key is that the kinetic and surface terms in the AHM are bosonized into

$$\sum_{\sigma}\int dx\,\frac{v_F(\sigma)}{2}\left[(\partial_x\phi_\sigma)^2+\Pi_\sigma^2\right]+\sum_{\sigma}\int dx\,\frac{\varepsilon_\sigma(k_F)}{\sqrt{\pi}}\partial_x\phi_\sigma, \tag{3.0.7}$$

so if we set $t_\uparrow = t_\downarrow$ in the former expression we get $v_F(\sigma) = 2ta\sin\left[k_F(\sigma)a\right]$ which is actually the Fermi velocity for the FFLO model as well. However, if we do the same in $\varepsilon_\sigma(k)$ we would get $-2t\cos(k_F(\sigma)a)$ which is not its dispersion relation, this is the major change to take into account. Therefore the term (I) in (3.0.6) reads

$$\sum_{j=1}^{N_s}\left[-t\sum_{\sigma}\left(c^{\dagger}_{j\sigma}c_{j+1,\sigma}+\text{H.c}\right)+U\,n_{j\uparrow}n_{j\downarrow}\right] \tag{3.0.8}$$

$$=\sum_{\sigma}\int dx\,\frac{v_F(\sigma)}{2}\left[(\partial_x\phi_\sigma)^2+\Pi_\sigma^2\right]+\sum_{\sigma}\int dx\,\frac{-2t\cos(k_F(\sigma)a)}{\sqrt{\pi}}\partial_x\phi_\sigma+Ua\int dx\,\frac{1}{\pi}(\partial_x\phi_\uparrow)(\partial_x\phi_\downarrow). \tag{3.0.9}$$

Thus, the extra job is to bosonize the density term $c^{\dagger}_{j\sigma}c_{j\sigma}$. We begin with equation (2.5.9):

$$c^{\dagger}_{j\sigma}c_{j\sigma}=a\Big[\rho_{\sigma L}(x)+\rho_{\sigma R}(x)+e^{ik_F(\sigma)(2x)}\psi^{\dagger}_{\sigma L}(x)\psi_{\sigma R}(x)+e^{-ik_F(\sigma)(2x)}\psi^{\dagger}_{\sigma R}(x)\psi_{\sigma L}(x)\Big]. \tag{3.0.10}$$

Working in the limit $L\to\infty$ and using the bosonization identity (2.4.10), the equation (A.3.44), and the commutator (A.3.37) to combine the exponentials, we get the following

$$c^{\dagger}_{j\sigma}c_{j\sigma}=a\Bigg\{\frac{1}{\sqrt{\pi}}\partial_x\phi_\sigma+\frac{1}{2\pi\lambda}F^{\dagger}_{\sigma L}F_{\sigma R}\exp\left[ik_F(\sigma)(2x)+i2\sqrt{\pi}\phi_\sigma(x)\right]$$
$$+\frac{1}{2\pi\lambda}F^{\dagger}_{\sigma R}F_{\sigma L}\exp\left[-ik_F(\sigma)(2x)-i2\sqrt{\pi}\phi_\sigma(x)\right]\Bigg\}. \tag{3.0.11}$$

Recalling our discussion in section 2.5.1 about high-energy contributions, notice that in the general case, that is, *far from the half-filling condition*, the second and third terms do not contribute to the low-energy description of the system, for they are oscillatory terms. Such a conclusion does not change even if we consider the whole term (II) in (3.0.6), $\sum_\sigma \mu_\sigma c^{\dagger}_{j\sigma}c_{j\sigma}$, because the Klein factors spoil any chance to combine the exponentials. On the other hand,

the first term can be combined with the second term in (3.0.9) to get the dispersion relation of the system $\varepsilon_\sigma(k_F) = -2t\cos(k_F(\sigma)a) - \mu_\sigma$.

Therefore, far from the half-filling condition, the Hamiltonian for the FFLO model is given by

$$:H: = \sum_\sigma \int dx \frac{v_F(\sigma)}{2}\left[(\partial_x\phi_\sigma)^2 + \Pi_\sigma^2\right] + \sum_\sigma \int dx\, \frac{\varepsilon_\sigma(k_F)}{\sqrt{\pi}}\,\partial_x\phi_\sigma + Ua\int dx\,\frac{1}{\pi}(\partial_x\phi_\uparrow)(\partial_x\phi_\downarrow). \tag{3.0.12}$$

This Hamiltonian turns out to be exactly the same we found for the AHM (see (2.5.38)). This is a remarkable result because, as explained before, the dispersion relations of both models are different. Automatically from what we did for the AHM we can assert that the FFLO model does not separate into spin and charge degrees of freedom, though this time it is the presence of a magnetic field what spoils such a separation. This early departure from the behavior of the Hubbard model is a promising signal, since we are looking precisely for unconventional superconducting features.

The consideration of being far from half-filling is not only crucial to get this Hamiltonian, but turns to be also crucial for further calculations. Basically, we ruled out the situation when there is exactly one fermion per site in the lattice, for it is very particular and there are relevant mechanisms in our initial Hamiltonian (3.0.1) that can change such a situation, namely, the hopping and the chemical potential. From what we did for the AHM, we are implicitly ruling out in (3.0.12) a cosine term with $2x(k_{F\uparrow} - k_{F\downarrow})$ as part of its argument, and other one with $2x(k_{F\uparrow} + k_{F\downarrow})$ (see section 2.5.1). The latter is ruled out because it only contributes to low-energy behaviors at half-filling, while the former is ruled out because the cosine would not be oscillatory only at the special case $k_{F\uparrow} = k_{F\downarrow}$, i.e., in the absence of magnetic field, which would also means absence of any unconventional superconductivity.

Thus, we have found an *effective* Hamiltonian for the low-energy behavior of the system, and it turns to be *astonishingly simple*,

$$H_{\text{eff}} = \sum_\sigma \int dx\, \frac{v_F(\sigma)}{2}\left[(\partial_x\phi_\sigma)^2 + \Pi_\sigma^2\right] + \sum_\sigma \int dx \frac{\varepsilon_\sigma(k_F)}{\sqrt{\pi}}\partial_x\phi_\sigma + Ua\int dx\frac{1}{\pi}(\partial_x\phi_\uparrow)(\partial_x\phi_\downarrow). \tag{3.0.13}$$

3.1 Diagonalizing the Hamiltonian

The main goal of this work is to investigate whether or not the Hamiltonian (3.0.1) leads to unconventional superconducting properties. For that purpose, we find the superconducting pairing correlation function and check whether it differs from a typical pairing. We will calculate the correlation function through functional integration, for which the Hamiltonian need not be diagonal (see Appendix B.1). Nevertheless, diagonalizing it will prove to be useful in simplifying the calculations.

Based on the works [72–74], we find a canonical transformation that diagonalizes the Hamil-

tonian (3.0.13) in terms of new fields ϕ_1 and ϕ_2[1]:

$$\phi_\uparrow = A_1\phi_1 + A_2\phi_2, \quad \Pi_\uparrow = C_1\Pi_1 + C_2\Pi_2 \tag{3.1.1}$$
$$\phi_\downarrow = B_1\phi_1 + B_2\phi_2, \quad \Pi_\downarrow = D_1\Pi_1 + D_2\Pi_2 \tag{3.1.2}$$
$$\begin{aligned} A_1 &= -\sqrt{\frac{v_\uparrow}{v_1}}\sin\varphi, \quad A_2 = \sqrt{\frac{v_\uparrow}{v_2}}\cos\varphi \\ B_1 &= \sqrt{\frac{v_\downarrow}{v_1}}\cos\varphi, \quad B_2 = \sqrt{\frac{v_\downarrow}{v_2}}\sin\varphi \\ C_1 &= -\sqrt{\frac{v_1}{v_\uparrow}}\sin\varphi, \quad C_2 = \sqrt{\frac{v_2}{v_\uparrow}}\cos\varphi \\ D_1 &= \sqrt{\frac{v_1}{v_\downarrow}}\cos\varphi, \quad D_2 = \sqrt{\frac{v_2}{v_\downarrow}}\sin\varphi, \end{aligned} \tag{3.1.3}$$

with the angle φ defined through

$$\tan(2\varphi) = \frac{2Ua\sqrt{v_\uparrow v_\downarrow}}{\pi(v_\uparrow^2 - v_\downarrow^2)}, \tag{3.1.4}$$

and where the propagation velocities for the new degrees of freedom are given by

$$v_{1/2}^2 = \frac{1}{2}(v_\uparrow^2 + v_\downarrow^2) \mp \frac{1}{2}\sqrt{(v_\uparrow^2 - v_\downarrow^2)^2 + \frac{4U^2a^2}{\pi^2}v_\uparrow v_\downarrow}. \tag{3.1.5}$$

Recall that in section 2.4 the dual bosonic fields were defined as θ_σ and ϕ_σ. So far the θ field has not had much prominence but it is about to become very relevant. Since $\partial_x\theta_\sigma(x) = \Pi_\sigma(x)$ and the derivative is a linear operator, then θ_σ transforms in the same fashion as Π_σ in equations (3.1.1) and (3.1.2).

The transformation is canonical so that the new fields verify $[\phi_i(x), \Pi_j(y)] = i\delta(x-y)\delta_{i,j}$, for $i, j = 1, 2$. Under this transformation the Hamiltonian (3.0.13) becomes

$$\begin{aligned} H_{\text{eff}} &= \sum_{j=1,2}\frac{v_j}{2}\int dx\big[(\partial_x\phi_j)^2 + \Pi_j^2\big] + \sum_\sigma\int dx\frac{\varepsilon_\sigma(k_F)}{\sqrt{\pi}}\partial_x\phi_\sigma \\ &\equiv \sum_{j=1,2}\frac{v_j}{2}\int dx\big[(\partial_x\phi_j)^2 + \Pi_j^2\big] + H_{\text{surf}}. \end{aligned} \tag{3.1.6}$$

Since the transformation is linear, we have not written H_{surf} in terms of the ϕ_j fields, for it shall vanish anyway. The fact that we can diagonalize the Hamiltonian is remarkable: The physics of the system can be understood in terms of two new degrees of freedom ϕ_1 and ϕ_2, that propagate independently ($v_1 \neq v_2$). This separation is possible after discarding the high-energy contributions of the Hamiltonian, more specifically, it does not work for the particular case of half-filling where according to our discussion in section 2.5.1 we would have a cosine with the sum $\phi_\uparrow + \phi_\downarrow$ as part of its argument, and it is clear that such a term does not separate under the previous transformation. It is also worthwhile to stress that the

[1]Henceforth for the sake of simplicity $v_F(\sigma)$ in (3.0.13) is denoted as v_σ.

transformation brings the Hamiltonian into its standard form [42] with no need of introducing Luttinger parameters for the fields, contrary to what happens for the AHM with the charge and spin fields. Finally, notice that if we set $v_\uparrow = v_\downarrow$, i.e. $h = 0$, this transformation becomes the $\{\phi_\uparrow, \phi_\downarrow\} \to \{\phi_c, \phi_s\}$ transformation described in section 2.5.3, as expected.

Having diagonalizaed the Hamiltonian we are ready to investigate the existence of the FFLO phase in the superconductive system described by the Hamiltonian (3.0.1), with its bosonized version in (3.1.6).

3.2 Warming up

As explained in Chapter 1, the novel superconducting phase we are pursuing is characterized by the presence of Cooper pairs with finite pairing momentum. Therefore, we are deeply interested in studying the correlation function of creating Cooper pairs: $\left\langle c_{j\uparrow} c_{j\downarrow} c^\dagger_{j+n,\uparrow} c^\dagger_{j+n,\downarrow} \right\rangle$. This correlation can be understood as the probability amplitude of a transition from the state $c^\dagger_{j\uparrow} c^\dagger_{j\downarrow} |0\rangle$, where two fermions of opposite spin (singlet configuration) are created at the jth site of the lattice, to the state $c^\dagger_{j+n,\uparrow} c^\dagger_{j+n,\downarrow} |0\rangle$ where the pair is located n sites to the right. There is a major caveat here: The vacuum state $|0\rangle$ used to compute the correlation is not the same vacuum defined in equation (2.3.1), which corresponds to a system of free fermions. The correlation function must be calculated over the vacuum associated to the diagonal form of the Hamiltonian (3.1.6), denoted $|\Omega\rangle$ hereinafter.

We begin by recalling (2.5.4) and (2.5.5b), so that we can write

$$c_{j\uparrow} c_{j\downarrow} c^\dagger_{j+n,\uparrow} c^\dagger_{j+n,\downarrow} = -a^2 O(x) O^\dagger(x+n), \text{ with } O(x) \equiv \psi_\uparrow(x)\psi_\downarrow(x). \tag{3.2.1}$$

Thus, we have to bosonize the operator $O(x)$ and then write it in terms of the fields ϕ_1, ϕ_2 that diagonalize the Hamiltonian. Proceeding likewise as in section 2.5, we get

$$\begin{aligned} O(x) =& \frac{e^{-i(k_{F\uparrow}+k_{F\downarrow})x}}{2\pi\lambda} F_{\uparrow L} F_{\downarrow L} \exp\left[-i\sqrt{\pi}\sum_j (\alpha_j \phi_j + \gamma_j \theta_j)\right] \\ &+ \frac{e^{-i(k_{F\uparrow}-k_{F\downarrow})x}}{2\pi\lambda} F_{\uparrow L} F_{\downarrow R} \exp\left[-i\sqrt{\pi}\sum_j (\beta_j \phi_j + \gamma_j \theta_j)\right] \\ &+ \frac{e^{i(k_{F\uparrow}-k_{F\downarrow})x}}{2\pi\lambda} F_{\uparrow R} F_{\downarrow L} \exp\left[-i\sqrt{\pi}\sum_j (\gamma_j \theta_j - \beta_j \phi_j)\right] \\ &+ \frac{e^{i(k_{F\uparrow}+k_{F\downarrow})x}}{2\pi\lambda} F_{\uparrow R} F_{\downarrow R} \exp\left[-i\sqrt{\pi}\sum_j (\gamma_j \theta_j - \alpha_j \phi_j)\right] \\ \equiv& O_1(x) + O_2(x) + O_3(x) + O_4(x), \end{aligned} \tag{3.2.2}$$

where $j = 1, 2$, $\theta_j(x)$ is such that $\partial_x \theta_j(x) = \Pi_j(x)$, furthermore we have introduced parameters α_j, β_j, γ_j, defined in terms of the coefficients of the linear transformation presented in section 3.1:

$$\alpha_j \equiv A_j + B_j \tag{3.2.3}$$

$$\beta_j \equiv A_j - B_j \tag{3.2.4}$$
$$\gamma_j \equiv C_j + D_j. \tag{3.2.5}$$

Upon performing the product $O(x)O^\dagger(y)$ we are left with 16 terms that can be gathered into a 4×4 matrix Q_{mn} constructed by regarding the operator $O(x)$ as the column vector $\big(O_1(x) \quad O_2(x) \quad O_3(x) \quad O_4(x)\big)^T$ and $O^\dagger(y)$ its Hermitian conjugate. We shall explicitly derive some elements of the Q_{mn} matrix to discuss some insights that emerge along the way, but in the end we shall just state the expressions for the rest of the entries. Before doing so, we concentrate on a special issue, namely, how to deal with the Klein factors.

3.2.1 More on Klein factors

The most nontrivial step when working within the bosonization framework is the proper treatment of Klein factors [75]. Nevertheless, they are simply ignored in very many papers, see e.g. [68, 76–80], textbooks, see e.g. [81, 82], and other introductory material, see e.g. [43, 46]. In more gentle scenarios they are slightly mentioned but not properly discussed, see e.g. [42, 44, 49, 83, 84]. Even though it is true that in many cases they can be safely ignored like in a single-chain Luttinger model with spin, or any system of fermions with an internal SU(N) symmetry [85, 86], it is enough to consider a two-chain problem to require a deeper treatment for them [85]. Therefore, in the most general case to neglect the Klein factors is very dangerous and it is known that it may lead to mistakes, see e.g. [47,87]. Hence, their light presentation in most the literature is a major drawback for the beginner researcher. In this respect, our careful and detailed treatment of the Klein factors is a remarkable contribution of this work.

Initially, we introduced the Klein factors to enforce a formal equality between fermionic and bosonic operators, accomplished through the bosonization identity. Nevertheless, there is a remarkable consequence of taking the thermodynamic limit regarding the Klein factors, namely, they become Hermitian $F_\eta^\dagger = F_\eta$ [88]. This can be seen from the fact that $F_\eta^\dagger$ (F_η) creates (annihilates) a partcile in the η-branch with momentum $k \sim 1/L$, so that in the thermodynamic limit the increase (decrease) in momentum is so small that it is like as the energetic properties of the system essentially remain the same. In other words, we cannot distinguish between creating or destroying a particle, for the system has infinite size (that also means $N \to \infty$) so we cannot keep track of these additions or removals. Under these conditions the algebra of the Klein factors turns out to be simplified into

$$\{F_\eta^\dagger, F_\eta'\} = 2\delta_{\eta,\eta'} \quad \xrightarrow{L\to\infty} \quad \{F_\eta, F_\eta'\} = 2\delta_{\eta,\eta'}, \tag{3.2.6}$$

which is the well-known Clifford algebra, another heritage from Dirac's relativistic theory of the electron.

We might had thought that in this limit $F_\eta^\dagger = F_\eta = 1$ but that would not enforce the anti-commutation relations of the fermionic fields (left-hand side of (2.5.5c)). Even though we did not need this to bosonize the Hamiltonian (3.0.1), it shall be crucial to compute the correlation functions.

The fact that in the thermodynamic limit $F_\eta^\dagger = F_\eta$ can be regarded as a "gauge freedom" [40]. It is clear that by introducing the Klein factors into the framework we intentionally enlarged

the Hilbert space generated by the boson operators b and $b^\dagger$ (they do not create fermions but shift them). At the thermodynamic limit, however, since $F_\eta^\dagger = F_\eta$ we must compensate this expansion by some sort of "gauge fixing": The tenet is to choose a representation of the Clifford algebra that turns the Hamiltonian and physical observables into diagonal [40, 89].

From elementary quantum field theory we know that representations of the Clifford algebra have even dimension [90]. In our case of 4 species (chirality and spin) we have a 4-dimensional representation and we can choose [40]

$$\begin{aligned} F_{\uparrow R} &= \sigma_1 \otimes \sigma_1 = \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix} \\ F_{\downarrow R} &= \sigma_3 \otimes \sigma_1 = \begin{pmatrix} \sigma_1 & 0 \\ 0 & -\sigma_1 \end{pmatrix} \\ F_{\uparrow L} &= \sigma_2 \otimes \sigma_1 = \begin{pmatrix} 0 & -i\sigma_1 \\ i\sigma_1 & 0 \end{pmatrix} \\ F_{\downarrow L} &= 1 \otimes \sigma_2 = \begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix}, \end{aligned} \tag{3.2.7}$$

with σ_i the Pauli matrices. We easily check that this representation indeed satisfies the Clifford algebra $\{F_\eta, F_{\eta'}\} = 2\delta_{\eta,\eta'}$ and unitarity.

Thus we can get rid of the enlargement of the Hilbert space in the following way. Consider for instance the product of Klein factors $F_{\uparrow L}F_{\uparrow R}F_{\downarrow R}F_{\downarrow L}$ in this representation

$$F_{\uparrow L}F_{\uparrow R}F_{\downarrow R}F_{\downarrow L} = \begin{pmatrix} 0 & -i\sigma_1 \\ i\sigma_1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix}\begin{pmatrix} \sigma_1 & 0 \\ 0 & -\sigma_1 \end{pmatrix}\begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix} \tag{3.2.8}$$

$$= 1 \otimes \sigma_3. \tag{3.2.9}$$

We observe that the operator $F_{\uparrow L}F_{\uparrow R}F_{\downarrow R}F_{\downarrow L}$ has eigenvalues ± 1. Then, we choose the following gauge: From the vast number of states in the Hilbert space hitherto described, pick the eigenstate of $F_{\uparrow L}F_{\uparrow R}F_{\downarrow R}F_{\downarrow L}$ with eigenvalue $+1$ and stick with it. As we shall see, through this the relevant operators undergo diagonal forms in the Klein factors, simplifying to great extent further calculations. It is important to emphasize that *this is a choice*, we could equally consider the eigenstates with eigenvalue -1. Therefore, the results we are about to obtain should be analyzed in the light of this gauge.

This can also be done from the defining properties of the Klein factors with no reference to any particular representation for them [85]. It is straightforward to check that $(F_{\uparrow L}F_{\uparrow R}F_{\downarrow R}F_{\downarrow L})^2 = 1$, regardless of the representation, so that $F_{\uparrow L}F_{\uparrow R}F_{\downarrow R}F_{\downarrow L}$ has eigenvalues ± 1. On the other hand, one can also check that $F_{\uparrow L}F_{\uparrow R}F_{\downarrow R}F_{\downarrow L}$ commutes with any product of the 4 different Klein factors, regardless of the order of the factors, so that we can simultaneously diagonalize all such products and stick with the portion of the Hilbert space that is diagonal.

3.2.2 Building the Q matrix

We are in a position to discuss the Q matrix. Consider first the $Q_{11} = O_1(x)O_1(y)^*$ element,

$$Q_{11} = \frac{e^{-i(k_{F\uparrow}+k_{F\downarrow})(x-y)}}{(2\pi\lambda)^2} F_{\uparrow L}F_{\downarrow L}F^\dagger_{\downarrow L}F^\dagger_{\uparrow L} \exp\left[-i\sqrt{\pi}\sum_j (\alpha_j\phi_j(x)+\gamma_j\theta_j(x))\right] \times \exp\left[i\sqrt{\pi}\sum_j (\alpha_j\phi_j(y)+\gamma_j\theta_j(y))\right]. \tag{3.2.10}$$

The first thing to notice is that, due to unitarity, the product of Klein factors simplifies to unity. This shall happen with all the diagonal elements Q_{mm}. To collect the exponentials through the BCH formula we need to find the commutator $\sum_{j,j'}\pi[\alpha_j\phi_j+\gamma_j\theta_j\,,\alpha_{j'}\phi_{j'}+\gamma_{j'}\theta_{j'}]$. To that end, we exploit the fact that the transformation $\{\phi_\uparrow,\phi_\downarrow\}\to\{\phi_1,\phi_2\}$ is canonical, and that the commutator between ϕ_σ and $\theta_{\sigma'}$ is given in (A.3.40), so that

$$\sum_{j,j'}\pi[\alpha_j\phi_j(x)+\gamma_j\theta_j(x)\,,\,\alpha_{j'}\phi_{j'}(y)+\gamma_{j'}\theta_{j'}(y)] = -i\pi\,\mathrm{sgn}(x-y)\sum_j\alpha_j\gamma_j. \tag{3.2.11}$$

Thus, the BCH formula leads us to

$$Q_{11} = \frac{e^{-i(k_{F\uparrow}+k_{F\downarrow})(x-y)}}{(2\pi\lambda)^2} \times \exp\left[-i\frac{\pi}{2}\,\mathrm{sgn}(x-y)\sum_j\alpha_j\gamma_j + i\sqrt{\pi}\sum_j[\alpha_j(\phi_j(y)-\phi_j(x))+\gamma_j(\theta_j(y)-\theta_j(x))]\right]. \tag{3.2.12}$$

Secondly, let us consider the Q_{14} term,

$$Q_{14} = \frac{e^{-i(k_{F\uparrow}+k_{F\downarrow})(x+y)}}{(2\pi\lambda)^2} F_{\uparrow L}F_{\downarrow L}F^\dagger_{\downarrow R}F^\dagger_{\uparrow R} \exp\left[-i\sqrt{\pi}\sum_j (\alpha_j\phi_j(x)+\gamma_j\theta_j(x))\right] \times \exp\left[i\sqrt{\pi}\sum_j (\gamma_j\theta_j(y)-\alpha_j\phi_j(y))\right]. \tag{3.2.13}$$

In the thermodynamic limit $F_{\uparrow L}F_{\downarrow L}F^\dagger_{\downarrow R}F^\dagger_{\uparrow R} = -F_{\uparrow L}F_{\uparrow R}F_{\downarrow R}F_{\downarrow L}$, and in the gauge fixed in section 3.2.1 we can safely replace this operator by -1. Notice the importance of a careful treatment of Klein factors beyond the bosonization of the Hamiltonian, the term Q_{14} actually contributes to the correlation function with a negative sign that would be impossible to get if we do not dig into the underlying physics of these operators.

We combine the exponentials likewise for the term Q_{11}. Since

$$\sum_{j,j'}\pi[\alpha_j\phi_j(x)+\gamma_j\theta_j(x)\,,\,\gamma_{j'}\theta_{j'}(y)-\alpha_{j'}\phi_{j'}(y)] = 0, \tag{3.2.14}$$

the BCH formula leads us to

$$Q_{14} = -\frac{e^{-i(k_{F\uparrow}+k_{F\downarrow})(x+y)}}{(2\pi\lambda)^2} \exp\left[i\sqrt{\pi}\sum_j \left[\gamma_j(\theta_j(y)-\theta_j(x)) - \alpha_j(\phi_j(y)+\phi_j(x))\right]\right]. \quad (3.2.15)$$

Finally, the Q_{21} term reads

$$\begin{aligned} Q_{21} =& \frac{e^{-i(k_{F\uparrow}-k_{F\downarrow})x}e^{i(k_{F\uparrow}+k_{F\downarrow})y}}{(2\pi\lambda)^2} F_{\uparrow L}F_{\downarrow R}F^\dagger_{\downarrow L}F^\dagger_{\uparrow L} \exp\left[-i\sqrt{\pi}\sum_j (\beta_j\phi_j(x)+\gamma_j\theta_j(x))\right] \\ &\times \exp\left[i\sqrt{\pi}\sum_j (\alpha_j\phi_j(y)+\gamma_j\theta_j(y))\right]. \end{aligned} \quad (3.2.16)$$

This time the product of Klein factor does not simplify to unity: $F_{\uparrow L}F_{\downarrow R}F^\dagger_{\downarrow L}F^\dagger_{\uparrow L} = F_{\downarrow R}F^\dagger_{\downarrow L}$. The correspondent commutator reads

$$\sum_{j,j'} \pi[\beta_j\phi_j(x)+\gamma_j\theta_j(x)\,,\, \alpha_{j'}\phi_{j'}(y)+\gamma_{j'}\theta_{j'}(y)] = -i\pi\,\mathrm{sgn}(x-y)\sum_j \gamma_j A_j. \quad (3.2.17)$$

Therefore, the matrix element can be rewritten as

$$\begin{aligned} Q_{21} =& \frac{e^{-i(k_{F\uparrow}-k_{F\downarrow})x}e^{i(k_{F\uparrow}+k_{F\downarrow})y}}{(2\pi\lambda)^2} F_{\downarrow R}F^\dagger_{\downarrow L} \\ &\times \exp\left[-i\frac{\pi}{2}\mathrm{sgn}(x-y)\sum_j \gamma_j A_j + i\sqrt{\pi}\sum_j \left[\gamma_j(\theta_j(y)-\theta_j(x)) + \alpha_j\phi_j(y) - \beta_j\phi_j(x)\right]\right]. \end{aligned} \quad (3.2.18)$$

Let us consider the contribution of this term to the correlation function, $\langle Q_{21}\rangle$, for which we would have to face the states $\langle\Omega| F_{\downarrow L}F^\dagger_{\downarrow R}$ and $\exp[\ldots]|\Omega\rangle$. On the one hand, the state $F_{\downarrow R}F^\dagger_{\downarrow L}|\Omega\rangle$ has a different number of fermions in each branch compared to $|\Omega\rangle$. On the other hand, the state $\exp[\ldots]|\Omega\rangle$ does not change the number of fermions in any of the branches compared to $|\Omega\rangle$, for the exponential consists of bosonic operators that do not change the number of fermions, just shift them. Thus, the product $\langle\Omega| F_{\downarrow L}F^\dagger_{\downarrow R}\exp[\ldots]|\Omega\rangle$ vanishes. It is clear then that any other term with an unequal number of F_η's and $F^\dagger_\eta$'s shall not contribute to the correlation function.

We can understand the last finding on intuitive physical grounds. From (3.2.1) and the definition of the Q matrix we can see that Q_{21} is proportional to $\psi_{\uparrow L}(x)\psi_{\downarrow R}(x)\psi^\dagger_{\downarrow L}(y)\psi^\dagger_{\uparrow L}(y)$, then when we ask for $\langle Q_{21}\rangle$, we are looking for the probability amplitude related to the transition from the state $\psi^\dagger_{\downarrow L}(y)\psi^\dagger_{\uparrow L}(y)|\Omega\rangle$, a pair with non-zero momentum, to the state $\psi^\dagger_{\downarrow R}(x)\psi^\dagger_{\uparrow L}(x)|\Omega\rangle$, a pair with zero momentum. Since there is no physical process in our system through which *a pair* could gain or loss momentum, these states are orthogonal.

The rest of the 16 terms are analogue to one of the three cases previously discussed, that is, their correspondent product of Klein factors is either ±1 due to unitarity or the gauge fixing, or it is simplified into a product of only two factors. The latter is the case of terms

Q_{12}, Q_{13}, Q_{21}, Q_{24}, Q_{31}, Q_{34}, Q_{42}, and Q_{43}. As showed before, all these terms do not contribute to the correlation function, since they correspond to correlations of one pair with zero momentum and other one with non-zero momentum (for instance Q_{42} is proportional to $\psi_{\uparrow R}(x)\psi_{\downarrow R}(x)\psi^\dagger_{\downarrow R}(y)\psi^\dagger_{\uparrow L}(y)$). Thus we are left with half of the original terms.

We emphasize once more the importance of a thorough discussion of the Klein factors. If we had ignored them when presenting the bosonization identity, as many authors do, we would have to go through the computation of these 8 nonphysical terms. Paradoxically, it is the explicit account of these operators the only reliable, rigorous way of getting rid of them: It is only through fixing a gauge, that all the relevant terms for our computation are set free of any product of Klein factors. Unitarity may let us to get rid of some of them but, as already discussed, fixing a gauge turns out to be of essential need.

Nevertheless, it does not suffice with that. There are high-energy processes that are hidden in the bosonic language and that are not ruled out by the Klein factors. Consider for instance the term Q_{14}, from equation (3.2.1) and the definition of the Q matrix we see that it is proportional to $\psi_{\uparrow L}(x)\psi_{\downarrow L}(x)\psi^\dagger_{\downarrow R}(y)\psi^\dagger_{\uparrow R}(y)$, so that in terms of *chiral fermions* it becomes clear that we should discard this term, for it takes a spin-up fermion from the L to the R branch, and likewise with spin-down fermions. As explained in section 2.5.1 and shown in figure 2.9, these kind of processes are of high-energy and do not preserve the number of fermions per branch. Therefore, they are out of the scope of our effective theory and we neglect them. Thus, we discard terms Q_{14}, Q_{41}, Q_{23}, Q_{32}, and ultimately, only 4 out of 16 terms contribute to the correlation function.

We stick with the classification in pairs of zero momentum and non-zero momentum, for it proves to be useful for the interpretation of the results. Setting $k^\pm \equiv k_{F\uparrow} \pm k_{F\downarrow}$ we start by writing down the contributions of pairs with **zero momentum**:

$$\begin{aligned} Q_{22} =& \frac{e^{-ik^-(x-y)}}{(2\pi\lambda)^2} \exp\left[-i\frac{\pi}{2}\mathrm{sgn}(x-y)\sum_j \gamma_j\beta_j + i\sqrt{\pi}\sum_j\left[\gamma_j(\theta_j(y)-\theta_j(x)) + \beta_j(\phi_j(y)-\phi_j(x))\right]\right] \\ Q_{33} =& \frac{e^{ik^-(x-y)}}{(2\pi\lambda)^2} \exp\left[i\frac{\pi}{2}\mathrm{sgn}(x-y)\sum_j \gamma_j\beta_j + i\sqrt{\pi}\sum_j\left[\gamma_j(\theta_j(y)-\theta_j(x)) - \beta_j(\phi_j(y)-\phi_j(x))\right]\right]. \end{aligned} \tag{3.2.19}$$

While the terms associated to pairs with **non-zero momentum** are:

$$\begin{aligned} Q_{11} =& \frac{e^{-ik^+(x-y)}}{(2\pi\lambda)^2} \exp\left[-i\frac{\pi}{2}\mathrm{sgn}(x-y)\sum_j \alpha_j\gamma_j + i\sqrt{\pi}\sum_j\left[\gamma_j(\theta_j(y)-\theta_j(x)) + \alpha_j(\phi_j(y)-\phi_j(x))\right]\right] \\ Q_{44} =& \frac{e^{ik^+(x-y)}}{(2\pi\lambda)^2} \exp\left[i\frac{\pi}{2}\mathrm{sgn}(x-y)\sum_j \gamma_j\alpha_j + i\sqrt{\pi}\sum_j\left[\gamma_j(\theta_j(y)-\theta_j(x)) - \alpha_j(\phi_j(y)-\phi_j(x))\right]\right] \end{aligned} \tag{3.2.20}$$

We have separated the relevant elements of the Q matrix into contributions of pairs with zero and non-zero momentum. Such a division is of particular relevance in our work, for

these non-zero momentum contributions are the ones we anticipate to be responsible of the unconventional pairing of the FFLO phase. Although such contributions are present even for a BCS-like pairing, the zero momentum contributions are the ones that usually dominate [83]. Then, we expect that in our case these contributions become at least comparable to those of zero momentum due to the physics of the system.

3.3 Correlation functions

To calculate the vacuum expectation value of the 4 matrix elements listed before we can ignore constant factors and focus solely on the operators, which all have the general structure

$$\exp(X) \equiv \exp\left[i\sqrt{\pi}\sum_j \left[\gamma_j(\theta_j(y)-\theta_j(x))+\delta\mu_j(\phi_j(y)-\phi_j(x))\right]\right], \tag{3.3.1}$$

where $\delta = \pm 1$ and $\mu_j = \{\alpha_j,\, \beta_j\}$.

Therefore, the calculation of all the $\langle Q_{mn}\rangle$ is reduced to finding the quantity $\langle\exp(X)\rangle$, for which we use the identity (3.3.2). This identity is valid for any operator X linear in the bosonic operators b's and $b^\dagger$'s, and it is proved in the appendix C of Ref. [47]. It is remarkable how physics has driven us from 16 terms to the computation of a single term,

$$\langle\exp(X)\rangle = \exp\left(\frac{1}{2}\langle X^2\rangle\right). \tag{3.3.2}$$

To compute $\langle X^2\rangle$ we shall employ functional integration, as explained in Appendix B, more specifically equation (B.2.8). Let us begin with recalling the effective Hamiltonian for our system (3.1.6)

$$H_{\text{eff}} = \sum_{j=1,2}\frac{v_j}{2}\int dx\left[(\partial_x\phi_j)^2+\Pi_j^2\right] + H_{\text{surf}} = \sum_j\int dx\,\mathcal{H}_j(\phi_j,\Pi_j) + \int dx\,\mathcal{H}_{\text{surf}}, \tag{3.3.3}$$

which is quadratic, Weyl-ordered, and separable in the fields ϕ_j. At this point we can appreciate the role of these fields: Although (B.2.8) is valid for interacting theories, to actually compute the integrals we should rely on Gaussian integrals as detailed in Appendix C, and they only hold for quadratic Hamiltonians. It is by introducing the fields ϕ_j and rewriting the Hamiltonian as that of a non-interacting theory that we can calculate the *exact* value of the integrals.

In the context of our system, (B.2.8) establishes that ($\hbar = 1$)

$$\langle A(x,\tau)B(y,\tau')\rangle = \frac{1}{Z}\int\prod_{i=1,2}\mathcal{D}\Pi_i(x,\tau)\mathcal{D}\phi_i(x,\tau)\; A(\phi_j,\Pi_j)B(\phi_j,\Pi_j)\,\exp\Big\{-S_E[\phi_j,\Pi_j]\Big\}, \tag{3.3.4}$$

where hereinafter the functional integrals are understood to be over periodic fields $\phi_j(x,\tau+\beta) = \phi_j(x,\tau)$, and the action in Euclidean time is given by

$$-S_E = \int_0^\beta d\tau\int dx\sum_j\left[i\Pi_j(x,\tau)\partial_\tau\phi_j(x,\tau) - \mathcal{H}_j(\phi_j,\Pi_j)\right] = -\sum_j S_E^{(j)}, \tag{3.3.5}$$

which remarkably separates into two independent actions as a consequence of the separability of the Hamiltonian. This property, in turn, is manifested in the partition function

$$\begin{aligned} Z &= \int \prod_i \mathcal{D}\Pi_i(x,\tau)\mathcal{D}\phi_i(x,\tau)\, \exp\Big\{ - S_E[\phi_j,\Pi_j]\Big\} \\ &= \int \prod_i \mathcal{D}\Pi_i(x,\tau)\mathcal{D}\phi_i(x,\tau)\, \exp\Big\{ - S_E^{(1)}\Big\} \exp\Big\{ - S_E^{(2)}\Big\} \\ &= \prod_i \Big(\int \mathcal{D}\Pi_i(x,\tau)\mathcal{D}\phi_i(x,\tau)\, \exp\Big\{ - S_E^{(i)}\Big\}\Big) \\ &\equiv Z_1 Z_2. \end{aligned} \tag{3.3.6}$$

Since the fields are periodic in τ we can expand them in the (k,ω_n)-space, as explained in section B.2.2 of Appendix B, to get an algebraic expression for $-S_E$. The surface term $\mathcal{H}_{\text{surf}}$ vanishes as expected, for it is a linear combination of terms of the form $\partial_x\phi_j$, each of which is identically zero after integrating over space,

$$\begin{aligned} \int dx\, \partial_x\phi_j &= \int dx \frac{1}{\sqrt{L\beta}} \sum_{k,n} (ik) e^{i(kx-\omega_n\tau)} \phi_j(k,\omega_n) \\ &= \sqrt{\frac{L}{\beta}} \sum_{k,n} (ik)\delta_{k,0} e^{-i\omega_n\tau} \phi_j(k,\omega_n) \\ &= 0. \end{aligned} \tag{3.3.7}$$

Thus, the Euclidean action in the (k,ω_n)-space is given by

$$-S_E = \sum_j \sum_{k,n} \left\{ - \frac{v_j}{2} k^2 \big[\phi_j(k,\omega_n)\phi_j^*(k,\omega_n) + \theta_j(k,\omega_n)\theta_j^*(k,\omega_n)\big] - ik\omega_n \phi_j(k,\omega_n)\theta_j^*(k,\omega_n) \right\}, \tag{3.3.8}$$

which can be cast into a quadratic form

$$-S_E = -\frac{1}{2} \sum_j \sum_{k,n} \begin{pmatrix} \phi_j^*(k,\omega_n) & \theta_j^*(k,\omega_n) \end{pmatrix} \begin{pmatrix} v_j k^2 & ik\omega_n \\ ik\omega_n & v_j k^2 \end{pmatrix} \begin{pmatrix} \phi_j(k,\omega_n) \\ \theta_j(k,\omega_n) \end{pmatrix} \equiv -\frac{1}{2} \sum_j \mathbf{u}_j^\dagger \cdot \mathbf{M}\mathbf{u}_j, \tag{3.3.9}$$

where we have introduced the contraction defined in Appendix C.

On the other hand, the operator X^2 is given by

$$\begin{aligned} X^2 = -\pi \sum_{i,j} \Big[&\gamma_j\gamma_i(\theta_j(y)-\theta_j(x))(\theta_i(y)-\theta_i(x)) + \delta\mu_i\gamma_j(\theta_j(y)-\theta_j(x))(\phi_i(y)-\phi_i(x)) \\ &+ \delta\mu_j\gamma_i(\phi_j(y)-\phi_j(x))(\theta_i(y)-\theta_i(x)) + \mu_j\mu_i(\phi_j(y)-\phi_j(x))(\phi_i(y)-\phi_i(x))\Big]. \end{aligned} \tag{3.3.10}$$

Let us being by computing $\langle(\theta_j(y)-\theta_j(x))(\theta_i(y)-\theta_i(x))\rangle$ for $i\neq j$, equation (3.3.4) leads to

$$\begin{aligned}
&\langle(\theta_j(y,\tau)-\theta_j(x,\tau'))(\theta_i(y,\tau)-\theta_i(x,\tau'))\rangle \qquad (i\neq j)\\
&=\frac{1}{Z}\int\prod_l \mathcal{D}\theta_l(x,\tau)\mathcal{D}\phi_l(x,\tau)\,(\theta_j(y,\tau)-\theta_j(x,\tau'))(\theta_i(y,\tau)-\theta_i(x,\tau'))e^{-S_E}\\
&=\frac{1}{Z}\int\prod_l \mathcal{D}\theta_l(x,\tau)\mathcal{D}\phi_l(x,\tau)\,(\theta_j(y,\tau)-\theta_j(x,\tau'))e^{-S_E^{(j)}}(\theta_i(y,\tau)-\theta_i(x,\tau'))e^{-S_E^{(i)}}\\
&=\prod_i\left(\frac{1}{Z_i}\int\mathcal{D}\theta_i(x,\tau)\mathcal{D}\phi_i(x,\tau)\,(\theta_i(y,\tau)-\theta_i(x,\tau'))e^{-S_E^{(i)}}\right),
\end{aligned}$$

essentially, we have found that for $i\neq j$ the correlation is also separable

$$\langle(\theta_j(y,\tau)-\theta_j(x,\tau'))(\theta_i(y,\tau)-\theta_i(x,\tau'))\rangle=\langle(\theta_j(y,\tau)-\theta_j(x,\tau'))\rangle\,\langle(\theta_i(y,\tau)-\theta_i(x,\tau'))\rangle\,. \tag{3.3.11}$$

It turns out that each of the factors in the right-hand side vanishes

$$\langle\theta_j(y,\tau)-\theta_j(x,\tau')\rangle=\frac{1}{Z_j}\int\mathcal{D}\theta_j(x,\tau)\mathcal{D}\phi_j(x,\tau)\,(\theta_j(y,\tau)-\theta_j(x,\tau'))e^{-S_E^{(j)}}=0, \tag{3.3.12}$$

for the function $\theta_j(y,\tau)-\theta_j(x,\tau')$ is linear in the fields. This is a generalization of the elementary Gaussian integral $\int dx\,xe^{-ax^2}=0$, which can be proved for its functional counterpart in a similar fashion to what we developed in Appendix C.

Thus we conclude that

$$\langle(\theta_j(y,\tau)-\theta_j(x,\tau'))(\theta_i(y,\tau)-\theta_i(x,\tau'))\rangle=0 \qquad (i\neq j). \tag{3.3.13}$$

This computation relies on the separability of the action, which in turn implies the separability of the partition function, and since in the functional integral formalism we deal with functions not operators, then the correlations separate accordingly. This has nothing to do with the particular fields considered, so that the same happens with products of ϕ's and products of ϕ's and θ's:

$$\langle(\phi_j(y,\tau)-\phi_j(x,\tau'))(\phi_i(y,\tau)-\phi_i(x,\tau'))\rangle=0 \qquad (i\neq j) \tag{3.3.14}$$

$$\langle(\phi_j(y,\tau)-\phi_j(x,\tau'))(\theta_i(y,\tau)-\theta_i(x,\tau')\rangle=0 \qquad (i\neq j) \tag{3.3.15}$$

$$\langle(\theta_j(y,\tau)-\theta_j(x,\tau'))(\phi_i(y,\tau)-\phi_i(x,\tau'))\rangle=0 \qquad (i\neq j). \tag{3.3.16}$$

Thus, the non-zero contributions to the expectation value $\langle X^2\rangle$ are found when $i=j$,

$$\begin{aligned}
\langle X^2\rangle=-\pi\sum_j\Big\{&\gamma_j^2\left\langle(\theta_j(y)-\theta_j(x))^2\right\rangle+\delta\mu_j\gamma_j\left\langle(\theta_j(y)-\theta_j(x))(\phi_j(y)-\phi_j(x))\right\rangle\\
&+\delta\mu_j\gamma_j\left\langle(\phi_j(y)-\phi_j(x))(\theta_j(y)-\theta_j(x))\right\rangle+\mu_j^2\left\langle(\phi_j(y)-\phi_j(x))^2\right\rangle\Big\}.
\end{aligned} \tag{3.3.17}$$

Consider the $\theta-\phi$ correlation, following (3.3.4) we have

$$\begin{aligned}
&\langle(\theta_j(y,\tau)-\theta_j(x,\tau'))(\phi_j(y,\tau)-\phi_j(x,\tau'))\rangle = \langle(\phi_j(y,\tau)-\phi_j(x,\tau'))(\theta_j(y,\tau)-\theta_j(x,\tau'))\rangle\\
&= \frac{1}{Z}\int\prod_i \mathcal{D}\theta_i(x,\tau)\mathcal{D}\phi_i(x,\tau)\,(\theta_j(y,\tau)-\theta_j(x,\tau'))(\phi_j(y,\tau)-\phi_j(x,\tau'))e^{-S_E}\\
&= \frac{1}{Z_j}\int\mathcal{D}\theta_j(x,\tau)\mathcal{D}\phi_j(x,\tau)\,(\theta_j(y,\tau)-\theta_j(x,\tau'))(\phi_j(y,\tau)-\phi_j(x,\tau'))e^{-S_E^{(j)}},
\end{aligned}$$

next we transform the product $(\theta_j(y,\tau)-\theta_j(x,\tau'))(\phi_j(y,\tau)-\phi_j(x,\tau'))$ to the (k,ω_n)-space following section B.2.2,

$$\begin{aligned}
&\langle(\theta_j(y,\tau)-\theta_j(x,\tau'))(\phi_j(y,\tau)-\phi_j(x,\tau'))\rangle\\
&= \frac{1}{Z_j}\frac{1}{L\beta}\sum_{k,n}\sum_{q,m}\left(e^{i(ky-\omega_n\tau)}-e^{i(kx-\omega_n\tau')}\right)\left(e^{i(qy-\omega_m\tau)}-e^{i(qx-\omega_m\tau')}\right)\\
&\qquad\times\int\mathcal{D}\theta_j(k,\omega_n)\mathcal{D}\phi_j(k,\omega_n)\;\theta_j(k,\omega_n)\phi_j(q,\omega_m)e^{-S_E^{(j)}}\\
&= \frac{1}{Z_j}\frac{1}{L\beta}\sum_{k,n}\sum_{q,m}\left(e^{i(ky-\omega_n\tau)}-e^{i(kx-\omega_n\tau')}\right)\left(e^{i(qy-\omega_m\tau)}-e^{i(qx-\omega_m\tau')}\right)\\
&\qquad\times\int\mathcal{D}\theta_j(k,\omega_n)\mathcal{D}\phi_j(k,\omega_n)\;\theta_j(k,\omega_n)\phi_j^*(-q,-\omega_m)\exp\left(-\frac{1}{2}\mathbf{u}_j^\dagger\cdot\mathbf{M}\mathbf{u}_j\right).
\end{aligned}$$

The integral can be performed using (C.2.15), for which we have to invert $\mathbf{M}$,

$$\mathbf{M}=\begin{pmatrix}v_jk^2 & ik\omega_n\\ ik\omega_n & v_jk^2\end{pmatrix}\implies\mathbf{M}^{-1}=\frac{1}{v_j^2k^4+k^2\omega_n^2}\begin{pmatrix}v_jk^2 & -ik\omega_n\\ -ik\omega_n & v_jk^2\end{pmatrix},\tag{3.3.18}$$

then equation (C.2.15) leads us to

$$\begin{aligned}
&\langle(\theta_j(y,\tau)-\theta_j(x,\tau'))(\phi_j(y,\tau)-\phi_j(x,\tau'))\rangle\\
&= \frac{1}{Z_j}\frac{1}{L\beta}\sum_{k,n}\sum_{q,m}\left(e^{i(ky-\omega_n\tau)}-e^{i(kx-\omega_n\tau')}\right)\left(e^{i(qy-\omega_m\tau)}-e^{i(qx-\omega_m\tau')}\right)\\
&\qquad\times\frac{\mathcal{N}}{2}\delta_{k,-q}\delta_{n,-m}\left[M_{21}^{-1}(q,\omega_m)+M_{21}^{-1}(-q,-\omega_m)\right].
\end{aligned}\tag{3.3.19}$$

On the other hand, equation (C.2.16) shows that

$$Z_j=\int\mathcal{D}\theta_j(k,\omega_n)\mathcal{D}\phi_j(k,\omega_n)\;e^{-S_E^{(j)}}=\mathcal{N},\tag{3.3.20}$$

so that in (3.3.19) the factors $\mathcal{N}$ cancel out, as claimed several times in Appendix B, and this formally divergent constant has no physical meaning for the correlations.

$$\begin{aligned}
&\langle(\theta_j(y,\tau)-\theta_j(x,\tau'))(\phi_j(y,\tau)-\phi_j(x,\tau'))\rangle\\
&= \frac{1}{L\beta}\sum_{k,n}\sum_{q,m}\left(e^{i(ky-\omega_n\tau)}-e^{i(kx-\omega_n\tau')}\right)\left(e^{i(qy-\omega_m\tau)}-e^{i(qx-\omega_m\tau')}\right)\delta_{k,-q}\delta_{n,-m}\frac{-i\omega_m}{q(v_j^2q^2+\omega_m^2)}
\end{aligned}$$

$$= \frac{1}{L\beta}\sum_{k,n}\left[2 - e^{ik(y-x)}e^{-i\omega_n(\tau-\tau')} - e^{-ik(y-x)}e^{i\omega_n(\tau-\tau')}\right]\frac{-i\omega_n}{k(v_j^2k^2+\omega_n^2)}$$

$$\equiv \sum_k\left(\sum_n h_1(k,n) - \sum_n h_2(k,n)\right). \tag{3.3.21}$$

To perform the summations over n we employ the Poisson's summation formula, which establishes that

$$\sum_{n=-\infty}^{\infty} s(n) = \sum_{m=-\infty}^{\infty}\tilde{s}(m), \qquad \tilde{s}(m) = \int_{-\infty}^{\infty} dn\, s(n)e^{-i2\pi mn}. \tag{3.3.22}$$

Therefore, we begin by finding the Fourier transform of $h_1(k,n)$ (recall that $\omega_n = 2\pi n/\beta$):

$$\begin{aligned}\tilde{h}_1(k,m) &= \int_{-\infty}^{\infty} dn\,\frac{1}{L\beta}\left[2 - e^{ik(y-x)}e^{-i2\pi(\tau-\tau')n/\beta}\right]\frac{-i2\pi n}{k\beta(v_j^2k^2+4\pi^2n^2/\beta^2)}\times e^{-i2\pi mn}\\ &= \frac{1}{Lk}\int_{-\infty}^{\infty} dn\left[2 - e^{ik(y-x)}e^{-i2\pi(\tau-\tau')n/\beta}\right]\frac{-i2\pi n}{v_j^2k^2\beta^2+4\pi^2n^2}\times e^{-i2\pi mn}.\end{aligned} \tag{3.3.23}$$

Even though this Fourier transform can be found using tables, we employ the software Mathematica to calculate the integral,

$$\begin{aligned}\tilde{h}_1(k,m) = &-\frac{1}{2kL}\Big[2\mathrm{sgn}(m)\exp\left(-|kv_j\beta||m|\right)\\ &- e^{ik(y-x)}\mathrm{sgn}(m\beta+\tau-\tau')\exp\left(-|kv_j||m\beta+(\tau-\tau')|\right)\Big],\end{aligned} \tag{3.3.24}$$

therefore Poisson's summation reads

$$\begin{aligned}\sum_n h_1(k,n) = &-\frac{1}{2kL}\sum_{m=-\infty}^{\infty}\Big[2\mathrm{sgn}(m)\exp\left(-|kv_j\beta||m|\right) \qquad (3.3.25)\\ &- e^{ik(y-x)}\mathrm{sgn}(m\beta+\tau-\tau')\exp\left(-|kv_j||m\beta+(\tau-\tau')|\right)\Big]\\ = &\frac{e^{ik(y-x)}}{2kL}\sum_{m=-\infty}^{\infty}\mathrm{sgn}(m\beta+\tau-\tau')\exp\left(-|kv_j||m\beta+(\tau-\tau')|\right), \qquad (3.3.26)\end{aligned}$$

since the first term in equation (3.3.25) is an odd function in the variable m so its sum is going to be zero.

We have to get rid of the absolute value in $|m\beta+(\tau-\tau')|$ in order to sum over m. Since $\tau,\tau'\in[0,\beta]$ then

$$(m-1)\beta < \tau-\tau'+m\beta < (m+1)\beta \implies \tau-\tau'+m\beta\begin{cases}<0, & \text{if } m\le -1\\ >0, & \text{if } m\ge 1,\end{cases} \tag{3.3.27}$$

thus

$$\sum_n h_1(k,n) = -\frac{e^{ik(y-x)}}{2kL}e^{|kv_j|(\tau-\tau')}\sum_{m=-\infty}^{-1}\exp\left(|kv_j|m\beta\right) + \frac{e^{ik(y-x)}}{2kL}\mathrm{sgn}(\tau-\tau')\exp\left(-|kv_j||\tau-\tau'|\right)$$

$$+\frac{e^{ik(y-x)}}{2kL}e^{-|kv_j|(\tau-\tau')}\sum_{m=1}^{\infty}\exp\left(-|kv_j|m\beta\right) \tag{3.3.28}$$

$$\sum_n h_1(k,n) = -\frac{e^{ik(y-x)}}{kL}\sinh\left(|kv_j|(\tau-\tau')\right)\sum_{m=1}^{\infty}\exp\left(-|kv_j|m\beta\right)$$
$$+\frac{e^{ik(y-x)}}{2kL}\mathrm{sgn}(\tau-\tau')\exp\left(-|kv_j||\tau-\tau'|\right) \tag{3.3.29}$$

$$\sum_n h_1(k,n) = -\frac{e^{ik(y-x)}}{kL}\sinh\left(|kv_j|(\tau-\tau')\right)\left(\frac{1}{1-e^{-|kv_j|\beta}}-1\right)$$
$$+\frac{e^{ik(y-x)}}{2kL}\mathrm{sgn}(\tau-\tau')\exp\left(-|kv_j||\tau-\tau'|\right). \tag{3.3.30}$$

Now, upon using (3.3.4) we have drew a parallelism with statistical mechanics, more specifically, with the canonical ensemble of inverse temperature β. However, our original system is not thermal, therefore we should take the zero-temperature limit $\beta\to\infty$ after summing over the Matsubara frequencies ω_n. Notice that, by taking this limit in the last equation we simply get

$$\sum_n h_1(k,n) = \frac{e^{ik(y-x)}}{2kL}\mathrm{sgn}(\tau-\tau')\exp\left(-|kv_j||\tau-\tau'|\right). \tag{3.3.31}$$

It was possible to expand the fields in the (k,ω_n)-space because we set up artificial periodic boundary conditions, now we undo them by taking $L\to\infty$, and translating the sum into an integral accordingly:

$$\begin{aligned}\sum_{k,n} h_1(k,n) &= \frac{1}{2}\mathrm{sgn}(\tau-\tau')\int\frac{dk}{2\pi}\frac{e^{ik(y-x)}}{k}\exp\left(-|kv_j||\tau-\tau'|\right)\\ &= \frac{i}{2\pi}\mathrm{sgn}(\tau-\tau')\int_0^{\infty}\frac{e^{-v_jk|\tau-\tau'|}}{k}\sin(k(y-x)).\end{aligned} \tag{3.3.32}$$

This integral diverges due to its behavior for large k. We can regularize it by introducing some cutoff by re-defining the integral measure $\int dk\to\int dk\, e^{-\lambda|k|}$. The final integral is computed using Mathematica,

$$\begin{aligned}\sum_{k,n} h_1(k,n) &= \frac{i}{2\pi}\mathrm{sgn}(\tau-\tau')\int_0^{\infty}dk\frac{e^{-(v_j|\tau-\tau'|+\lambda)k}}{k}\sin(k(y-x))\\ &= -\frac{i}{2\pi}\mathrm{sgn}(\tau-\tau')\arctan\left(\frac{x-y}{\lambda+v_j|\tau-\tau'|}\right).\end{aligned} \tag{3.3.33}$$

In a similar fashion, we compute $\sum_{k,n} h_2(k,n)$, as defined in (3.3.21), getting

$$\sum_{k,n} h_2(k,n) = -\frac{i}{2\pi}\mathrm{sgn}(\tau-\tau')\arctan\left(\frac{x-y}{\lambda+v_j|\tau-\tau'|}\right), \tag{3.3.34}$$

and finally the $\theta-\phi$ correlation, $\sum_{k,n}(h_1(k,n)-h_2(k,n))$, yields

$$\langle(\theta_j(y,\tau)-\theta_j(x,\tau'))(\phi_j(y,\tau)+\epsilon\phi_j(x,\tau'))\rangle=0, \tag{3.3.35}$$

and we shall get the same for $\langle(\phi_j(y,\tau)+\epsilon\phi_j(x,\tau'))(\theta_j(y,\tau)-\theta_j(x,\tau'))\rangle$.

Likewise we compute the $\theta-\theta$ and $\phi-\phi$ correlations:

$$\left\langle(\theta_j(y,\tau)-\theta_j(x,\tau'))^2\right\rangle=\left\langle(\phi_j(y,\tau)-\phi_j(x,\tau'))^2\right\rangle=\frac{1}{2\pi}\ln\left[\frac{(x-y)^2+(\lambda+v_j|\tau-\tau'|)^2}{\lambda^2}\right]. \tag{3.3.36}$$

Finally we can write down the building block $\langle X^2\rangle$ to find the elements $\langle Q_{mn}\rangle$ (see equation (3.3.17))

$$\left\langle X^2\right\rangle=-\sum_j\frac{1}{2}\left(\gamma_j^2+\mu_j^2\right)\ln\left[\frac{(x-y)^2+(\lambda+v_j|\tau-\tau'|)^2}{\lambda^2}\right]\equiv F(\mu_j), \tag{3.3.37}$$

recalling that $\mu_j=\{\alpha_j,\beta_j\}$ is defined through equations (3.2.3) and (3.2.4).

Notice that $\langle X^2\rangle$ does not depend on δ while X^2 does, then as defined in section 3.2.2, $\langle Q_{22}\rangle$ and $\langle Q_{33}\rangle$ differ only through an oscillating factor, likewise it happens with $\langle Q_{11}\rangle$ and $\langle Q_{44}\rangle$.

3.4 Singlet superconducting correlation function

The correlations $\langle Q_{mn}\rangle$ can be written in terms of the function $F(\mu_j)$ using the identity (3.3.2). We begin by noting that from the definitions in (3.2.3)-(3.2.5), one gets

$$\sum_j\gamma_j\beta_j=0,\qquad\sum_j\gamma_j\alpha_j=2, \tag{3.4.1}$$

therefore the elements $\langle Q_{mn}\rangle$, as defined in section 3.2.2, read,

$$\langle Q_{22}\rangle=\frac{e^{-ik^-(x-y)}}{(2\pi\lambda)^2}\exp\left[\frac{1}{2}F(\beta_j)\right] \tag{3.4.2}$$

$$\langle Q_{33}\rangle=\frac{e^{ik^-(x-y)}}{(2\pi\lambda)^2}\exp\left[\frac{1}{2}F(\beta_j)\right] \tag{3.4.3}$$

$$\langle Q_{11}\rangle=\frac{e^{-ik^+(x-y)}}{(2\pi\lambda)^2}\exp\left[-i\pi\mathrm{sgn}(x-y)\right]\exp\left[\frac{1}{2}F(\alpha_j)\right] \tag{3.4.4}$$

$$\langle Q_{44}\rangle=\frac{e^{ik^+(x-y)}}{(2\pi\lambda)^2}\exp\left[-i\pi\mathrm{sgn}(x-y)\right]\exp\left[\frac{1}{2}F(\alpha_j)\right]. \tag{3.4.5}$$

Recalling equation (3.2.1), the singlet superconducting correlation function is given by

$$\left\langle c_{j\uparrow}c_{j\downarrow}c^\dagger_{j+n,\uparrow}c^\dagger_{j+n,\downarrow}\right\rangle=-a^2(\langle Q_{22}\rangle+\langle Q_{33}\rangle+\langle Q_{11}\rangle+\langle Q_{44}\rangle). \tag{3.4.6}$$

Notice that we can get a nice close expression for $\exp\left[F(\mu_j)/2\right]$ using equation (3.3.37),

$$\exp\left[\frac{1}{2}F(\mu_j)\right] = \prod_j \left(\frac{\lambda}{\sqrt{(x-y)^2+(\lambda+v_j|\tau-\tau'|)^2}}\right)^{(\gamma_j^2+\mu_j^2)/2}. \tag{3.4.7}$$

To compare the correlation function with experimental results, e.g. coincidence detection measurements, we shall focus on their spatial aspects, that is, we set $\tau-\tau' \to 0^-$. On the other hand, in terms of the lattice operators we are investigating the correlation of pairs at the jth and $(j+n)$th site of the lattice, with $j=1,\ldots,L$ and $n=1,\ldots,L-1$, so that in the continuum limit $y-x\in[a,L-a]$, and $\exp\left[-i\pi\mathrm{sgn}(x-y)\right]=-1$. Thus, the correlation function reads

$$\begin{aligned}\left\langle c_{j\uparrow}c_{j\downarrow}c^\dagger_{j+n,\uparrow}c^\dagger_{j+n,\downarrow}\right\rangle =& \frac{\cos(k^+(y-x))}{2\pi^2}\left(\frac{\lambda}{\sqrt{(x-y)^2+\lambda^2}}\right)^{\sum_j(\gamma_j^2+\alpha_j^2)/2} \\ &- \frac{\cos(k^-(y-x))}{2\pi^2}\left(\frac{\lambda}{\sqrt{(x-y)^2+\lambda^2}}\right)^{\sum_j(\gamma_j^2+\beta_j^2)/2},\end{aligned} \tag{3.4.8}$$

where the first term is the non-zero momentum contribution while the second one is the contribution of pairs with zero momentum.

From equation (3.4.8) we observe that the singlet pairing correlation function is a competence between the contributions of the pairs with zero momentum, responsible of coventional BCS superconductivity, and those with non-zero momentum, responsible of novel features as the FFLO phase. These contributions differ through their decaying power law and an oscillating factor. As explained in Chapter 1, the former is a qualitative signature of the FFLO phase, where the system develops an oscillating order parameter in order to sustain both magnetism and superconductivity simultaneously. Therefore, it is remarkable the fact that in (3.4.8) the oscillating factor of the term of pairs with zero momentum depends on the difference of the Fermi momenta, k^-, so that in the absence of magnetic field, $h=0$, $k_{F\uparrow}=k_{F\downarrow}$, the system displays BCS superconductivity and that term is no longer oscillating, as expected. On the other hand, since the oscillating factor of the term of pairs with non-zero momentum depends on the sum of the Fermi momenta, k^+, even with $h=0$ it remains oscillating. Thus, we can characterize the contribution of pairs with non-zero momentum as truly oscillating, as expected for the FFLO phase [14–16].

In what follows we study the interplay between the two contributions for different parameter sets. The general settings are the following: the hopping parameter is the energy unit $t=1$, the size of the lattice is $L=100$ with lattice constant $a=0.1$, the chemical potential is $\mu=0.1$, and we choose a filling of $\nu=0.7\pi/a$. Let us begin with the case of null magnetic field, $h=0$, $\mu_\uparrow=\mu_\downarrow$, so that the system should display BCS superconductivity. This situation is depicted in figure 3.2. Notice that frame (a) shows the expected overall decay behavior for both contributions, with an envelope of the form $(y-x)^{-\xi}$, which starts to oscillate around zero for a separation of around of a. In frame (b) we zoom in those oscillations, and observe that the red curve, correspondent to the zero momentum contribution, decays faster ($\xi=2.01794$) compared to the blue one ($\xi=2.00008$). Therefore, contrary to what we first

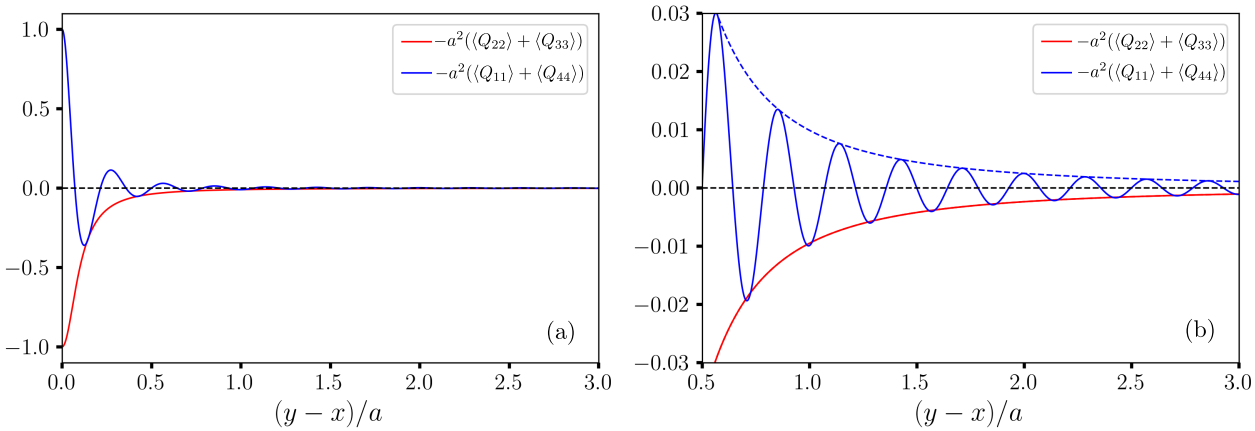

Figure 3.2: Comparison between the two types of contributions to the correlation function for $t = 1$, $U = -0.1$, $h = 0$. Both contributions are normalized to their value when $y - x = 0$. In (b) we zoom in around 0, the envelopes of the curves decay as $(y-x)^{-\xi}$ with $\xi = 2.01794$ for the zero momentum contribution (red curve) and $\xi = 2.00008$ for the non-zero momentum contribution (blue curve). BCS superconductivity turns out to be out of the scope of our effective theory.

expected, figure 3.2 shows that the non-zero momentum pairing dominates even in the absence of magnetic field. It is like if BCS superconductivity were ascribed to Cooper pairs of finite center of mass momentum. However, it turns out that this finding is not as pathological as it seems at first sight: In section 2.5.1 when we bosonized the AHM Hamiltonian, we repeatedly ruled out terms from the interaction part of the Hamiltonian, arguing that they would contribute only at the *particular cases* of half-filling or in the absence of magnetic field (see also the discussion at the end of section 3). Thus, we should not feel surprised by the fact that in the figure 3.2 the BCS features do not dominate, for we have oriented our effective theory to the study of the FFLO phase, and then we should not rely on it to get an accurate description of the system in the absence of magnetic field.

We continue by checking the case of moderate magnetic field, $h = 2.0$, $\mu_\uparrow < \mu_\downarrow$, so that the sample is not completely polarized and we expect the presence of the FFLO phase. This situation is depicted in figure 3.3. In this case, frame (b) shows that the zero momentum contribution decays faster ($\xi = 2.02275$) than the non-zero momentum contribution ($\xi = 2.00012$). Thus, we corroborate that at a moderate value of h the singlet pairing correlation function is dominated by the pairing of non-zero momentum, that is, the kind of pairing we expect from the FFLO phase.

To investigate a third regime is mandatory in order to strengthen the validity of (3.4.8). Although $h = 0$ is out of the scope of our calculations, we are able to check the case of large h. In that case the sample is almost completely polarized, most fermions have their spin pointing down, and therefore the system should have transited to the normal state (or to be about to). Figure 3.4 depicts this scenario where we have taken $h = 3.5$, $\mu_\uparrow << \mu_\downarrow$. Although this figure is not conclusive about whether a transition to the normal state takes place, we

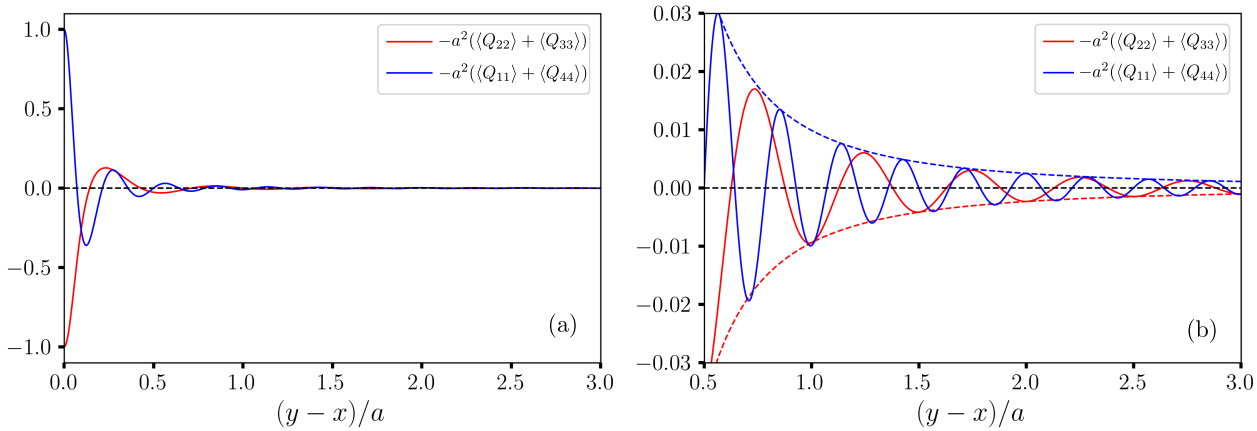

Figure 3.3: Comparison between the two types of contributions to the correlation function for $t = 1$, $U = -0.1$, and $h = 1.75$. Both contributions are normalized to their value when $y - x = 0$. In (b) we zoom in around 0, the envelopes of the curves decay as $(y-x)^{-\xi}$ with $\xi = 2.02274$ for the zero-momentum contribution (red curve) and $\xi = 2.00012$ for the non-zero momentum contribution (blue curve).

remark the following facts: (i) Both contributions decay faster than in the case of figure 3.3, more specifically, both start to oscillate around 0 for a separation of about of $0.5a$ instead of a. (ii) As shown by the green curve, the contributions interfere destructively making the correlation function to have less amplitude than in figure 3.3. (iii) Both contributions decay way faster than in the previous cases, with $\xi = 2.03283$ for the zero momentum contribution and $\xi = 2.00024$ for the non-zero momentum contribution. These observations somewhat show that the superconducting properties of the system are diminished as we increase the magnetic field, enforcing the reliability of (3.4.8).

Now, for the same three regimes let us study the correlation function itself, not its separate contributions, and for three limiting cases of the attractive interaction, $U = -3$, $U = -1$, and $U = -0.1$, as shown in figure 3.5. In all three cases, for each value of h we identify two extreme values in the correlation function, and from our analysis of the previous figures we ascribe the value of minimum correlation (we refer to it as value A hereinafter) to the dominance of the non-zero momentum contribution, while the value of maximum correlation (we refer to it as value B hereinafter) corresponds to the zero momentum contribution. Thus, it makes sense that on the one hand, A decreases as we lower the value of h because then the BCS-like features dominates (notice that we are not taking $h = 0$ as before). On the other hand, B is the largest for the intermediate value of magnetic field, where the FFLO phase is expected to emerge and the non-zero momentum pairing is dominant; while it is the lowest for the smallest value of h, where what dominates is the zero momentum contribution. Remarkably, the largest value A is not attained when h is the largest, for in that case the magnetic field is too strong and starts to spoil any superconducting behavior.

Figure 3.5 shows another interesting feature, namely , notice that for frame (a) the maximum value of h plotted is $h = 2.7$ while from frames (b) and (c) it is $h = 3.5$. This is not arbitrary,

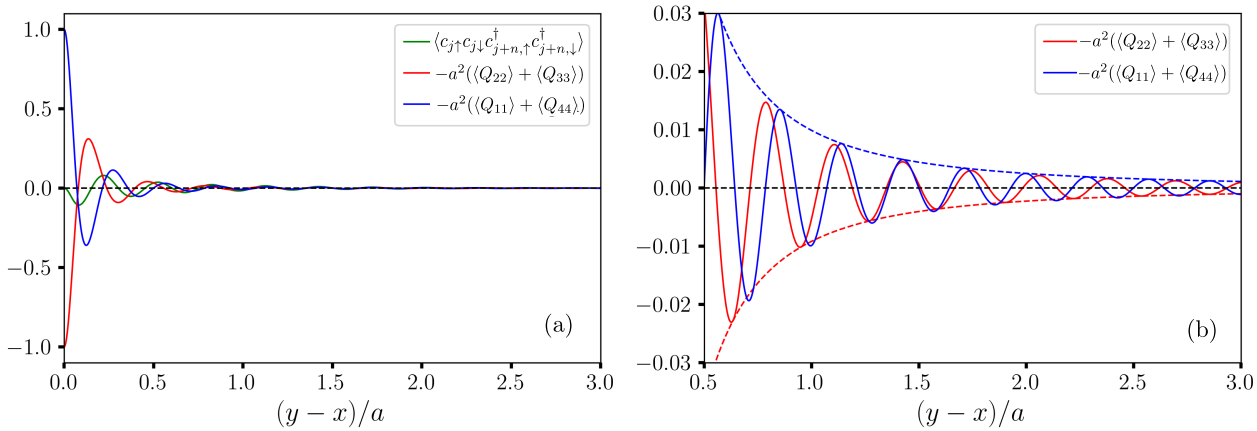

Figure 3.4: Comparison between the two types of contributions to the correlation function for $t = 1$, $U = -0.1$, $h = 3.5$. Both contributions are normalized to their value when $y - x = 0$. In (a) we included the correlation function, that is, the sum of the contributions. In (b) we zoom in around 0, the envelopes of the curves decay as $(y-x)^{-\xi}$ with $\xi = 2.03283$ for the zero momentum contribution (red curve) and $\xi = 2.00024$ for the non-zero momentum contribution (blue label).

it turns out that for $U = -3$ a magnetic field of $h = 3.5$ leads to non-sensible quantities, more specifically, it turns the propagation velocity v_1, defined in equation (3.1.5), into a complex number. This may be interpreted as a breakdown of superconductivity due to the polarization of the sample, and what we have found is that such a critical value for the magnetic field decreases as we increase the strength of the interaction. Ultimately, the chief difference between (a) and (b)-(c) is the relative position of the extreme values A and B for each value of h. Whereas in frame (a) the extreme values take place at relatively close positions, that does not happen for (b) nor (c). This is explained by the fact that the interaction is on-site and attractive, thus as we increase its strength the particles have less freedom to move along the lattice, then any dominant pairing shall occur at approximately the same place. Finally, to increase the attractive interaction makes the absolute value of the extreme values of the correlation to decrease, for any value of h. For instance, for $h = 2.0$ the value A is the largest for $U = -0.1$ (greater than 0.2), is less for $U = -1$ (approximately 0.2), while it is the smallest for $U = -3$ (less than 0.2). The fact that, in broad terms, the correlation function decreases with the interaction strength is explained again as the particles lack freedom to move, that is, to display superfluidity.

After this analysis we have identified two features that support the existence of the FFLO phase in our system: On the one hand, the fact that the contribution of pairs with non-zero momentum displays an oscillating behavior even when $h = 0$, which does not occur with the BCS-like contribution. On the other hand, the fact that for a moderate value of the magnetic field, more specifically $h = 2.0$, it is the non-zero momentum contribution the one that dominates over the BCS-like. The study of the three regimes, small h, moderate h, and large h, for given values of U, strengthens the validity of the main analytic finding of this

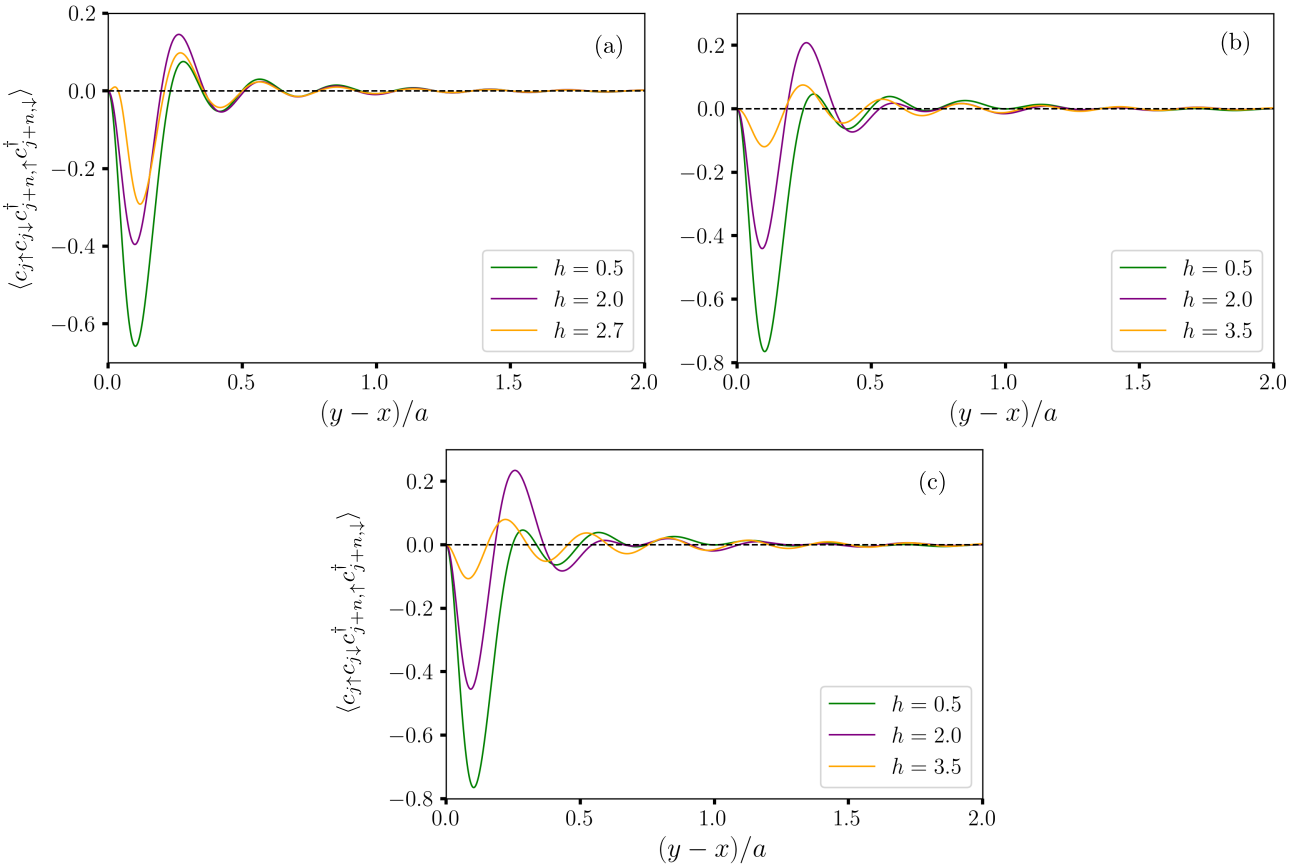

Figure 3.5: Superconducting correlation function for different values of the magnetic field and different strengths of the interaction, (a) $U = -3$, (b) $U = -1$, and (c) $U = -0.1$. All the other parameters remain unaltered as in the previous plots.

work, equation (3.4.8), in particular because we observe the tendency of the system to no longer sustain superconductivity for large h and U.

Chapter 4

Conclusions and perspectives

The contributions of this work rely in the first place on understanding formal aspects like bosonization and functional integration, both of great importance for modern topics in condensed matter and particle physics. In the second place, they rely on the application of these methods and their underlying ideas to treat a particular model that sustains the FFLO phase, which is of relevance for understanding the theory of superfluidity.

4.1 Conclusions

As mentioned at the beginning of Chapter 2, in spite of the vast literature on bosonization it turns out to be a difficult topic to the beginner researcher, not because of its intellectual depth but because of the lack of consistency in the definitions and conventions followed by different authors. Thus, Chapter 2 is intended to bridge that gulf between the literature and the background knowledge of a senior undergraduate student: We aimed to begin with a comprehensive approach, motivating the given definitions, and providing a logical chain of reasons to move forward in a particular direction rather than another, until reaching a more sophisticated formulation within field theory.

The relevant results of Chapter 2 begin with the derivation of the bosons operators from the Fourier components of the density fluctuations in (2.2.24), then we highlight the rigorous construction of the Fock space of the theory in (2.3.2), which in turn permits a careful treatment of the Klein factors. Finally, we remark the detailed derivation of the bosonization identity (2.4.10) and put on safe grounds the subtleties that may appear on the application of the program, by discussing in detail the bosonization of the AHM in section 2.5.

In Chapter 3 we focused on the FFLO model, whose physics could be easily approached after having discussed the AHM. The early chief result in this chapter is to obtain a remarkably simple effective Hamiltonian for the system in the bosonic language, equation (3.0.13), and its posterior diagonalization through the canonical transformation described in section 3.1. This is actually a wonderful finding, since the most documented work about bosonization of 1-dimensional Fermi systems in the presence of a Zeeman field is developed under the assumption that charge and spin degrees of freedom are decoupled (hence, they diagonalize

the Hamiltonian) at low energies for small polarization [91]. As showed for the AHM, this is true only at the particular case of half-filling. Thus, our study of the FFLO Hamiltonian in terms of the fields ϕ_i is more general and, to the best of our knowledge, novel.

Later in this chapter we showed in a clear and detailed way how to deal with the Klein factors. As mentioned in section 3.2.1, this is a major contribution of our work, for most the literature either completely omit such a discussion or barely mention it, and it became evident when finding the non-zero contributions of the Q matrix to the correlation functions. This completes our thorough presentation of the bosonization program.

Then we employed a rather sophisticated formalism, namely, functional integration, to compute the singlet superconducting correlation function for the system in equation (3.4.8). Once more, it turns out that this is a remarkable achievement, for similar studies of this system employ mostly numerical techniques, see e.g. [92–94], and a similar analytic approach is restricted solely to half-filling [91].

Lastly, by analyzing the decay of this correlation we are able to identify to what extent the system develops some long-range order. We study three regimes according to the value of the magnetic field, null h, moderate h, and large h. Since the effective theory developed is oriented to highlight the effects of the FFLO phase, the description for the case $h = 0$ (figure 3.2) turns out not to be reliable, for many terms relevant *only in this particular case* were discarded along the way. To include these terms in the theory requires the use of another sophisticated technique, namely, renormalization, for they would contribute to equation (2.5.38) as additional terms to the interaction part of the Hamiltonian that carry trigonometric functions of the bosonic fields ϕ_σ. The presence of such terms would, among other things, prevent us to diagonalize the Hamiltonian of the system by means of the canonical transformation described in section 3.1.

In spite of this, the findings at moderate h and large h follow what we expected. For the latter, figure 3.4 shows that the superconducting correlation function is small, suggesting an eventual transition to the normal due to the complete polarization of the sample. This finding is also supported by the fact that for $U = -3$ there exist large values of h that make the propagation velocity v_1 in (3.1.5) to be complex, an indication of a possible breakdown of any superconducting behavior. On the other hand, figure 3.3 shows that for an intermediate value of the magnetic field the ordering of the system is dominated by the non-zero momentum contribution, that is, the physics of any phase associated to that choice of the parameters is dictated by the presence of pairs of opposite spins but with non-zero center of mass momentum. This is also observed in figure 3.5 where the association of extreme values with each contribution suggests the existence of two orderings of different nature, each one driven by one of the contributions. Also, the study of the interplay between U and h for the correlation function strengthens the validity of equation (3.4.8) because we observe the tendency of the system to not to sustain superconductivity for large h and U. These findings permit us to conclude that, by using a non-perturbative approach, many of the signatures of the FFLO phase are sustained by the attractive Fermi model, out of half-filling.

4.2 Perspectives

An immediate perspective is to maintain in equation (2.5.11) the terms that contribute at $h = 0$ and use renormalization method to be able to reach BCS superconductivity within the effective theory. We identify another three avenues to develop further the ideas presented in this work. One experimental way to look for the FFLO phase is to track topological states within the superconducting regime, for instance, the presence of Andreev bound states (particular topological states) in organic superconductors is presented as a hallmark of the FFLO phase [95, 96]. Any analytic search for such states requires to treat the system as finite, and investigate the physics that takes place at the edges of the lattice. Thus, our early results in section 2.5.1 where we bosonized the Hamiltonian for the AHM are the starting point to conduct research on this direction. Topological phases of matter are of great relevance for technological applications, for instance, they provide a promising way to protect the information in a quantum computer [97]. Another way to contrast the results with experimental measurements, namely, spectroscopic data, is to compute the Fourier transform of the correlation function (3.4.8) to analyze the physics of the system in the light of linear response theory [98].

In the second place, the question of to what extent superconductivity survives in the presence of imbalanced pairing can be followed by the question of how easily this new phase is spoiled by thermal fluctuactions. Remarkably, the method employed to compute the singlet superconducting correlation function allows us to study the system at finite temperature. Actually, we already got the results in such a case, recall that equation (3.3.30) holds for our system being in contact with a heat bath of temperature $1/\beta$. Thus, it is straightforward to study the interplay between imbalanced pairing and temperature with the methods we described in this work.

Finally, since most studies of similar systems that can be found in the literature use numerical methods like density-matrix renormalization group, while we do not know, up to date, of analogue analytic works, as part of the process of validating our results it is necessary to complement them with numerical results, which in turn can be contrasted with the literature.

Appendix A

Supplementary calculations

In this appendix we derive some early chief results and identities used throughout the work. Such a variety of results are presented in a single appendix because they are derived using the operator formalism, that is, the computations are mainly algebraic. This is in contrast to the calculation of the correlations functions which uses a more sophisticated technique like functional integration.

A.1 Commutator of the chiral densities

In subsection 2.2.3 we followed a particular path to find the algebra of the Fourier components of the density fluctuations, $[\rho_{p\nu}, \rho_{p'\nu'}]$, namely, we did not rely on the known algebra of the c's and $c^\dagger$'s operators directly, but we used the inverse Fourier transform to work with $\rho_\nu(x)$. In this manner we will be able to discuss some technicalities that appear in the context of field theory.

To exploit the normal order in the computation of $[\rho_\nu(x), \rho_{\nu'}(y)] = [:\psi^\dagger_\nu(x)\psi_\nu(x):, :\psi^\dagger_{\nu'}(y)\psi_{\nu'}(y):]$ we compute it through its expectation value in the vacuum state. From Wick's theorem [99] we know that only the full pairings of the fields contribute to the expectation value, and hence

$$\langle \rho_\nu(x)\rho_{\nu'}(y)\rangle = \left\langle \psi^\dagger_\nu(x)\psi^\dagger_{\nu'}(y)\right\rangle \langle \psi_\nu(x)\psi_{\nu'}(y)\rangle + \left\langle \psi^\dagger_\nu(x)\psi_{\nu'}(y)\right\rangle \left\langle \psi_\nu(x)\psi^\dagger_{\nu'}(y)\right\rangle , \tag{A.1.1}$$

and from the mode expansion of the chiral fermionic fields in (2.2.12) it follows that

$$\langle \rho_\nu(x)\rho_{\nu'}(y)\rangle = \left\langle \psi^\dagger_\nu(x)\psi_\nu(y)\right\rangle \left\langle \psi_\nu(x)\psi^\dagger_\nu(y)\right\rangle \delta_{\nu,\nu'}. \tag{A.1.2}$$

Therefore, we concentrate on the calculation of $\left\langle \psi^\dagger_\nu(x)\psi_\nu(y)\right\rangle$. We foresee that such expectation value is going to be found in terms of the c's and $c^\dagger$'s. Thus, from the definition of the vacuum state $|0\rangle_0$ it follows that[1]

$$\left\langle c^\dagger_{k\nu}\, c_{k'\nu'}\right\rangle = \theta(-k)\delta_{k,k'}\delta_{\nu,\nu'} \tag{A.1.3}$$

[1] In this section all the expectations value make reference to the vacuum state $|0\rangle_0$ although do not include the subscript 0.

$$\left\langle c_{k\nu}\, c^{\dagger}_{k'\nu'} \right\rangle = \theta(k)\delta_{k,k'}\delta_{\nu,\nu'}, \tag{A.1.4}$$

where $\theta(x)$ is the Heaviside step function defined as

$$\theta(x) \equiv \begin{cases} 1 & x > 0 \\ \frac{1}{2} & x = 0 \\ 0 & x < 0. \end{cases} \tag{A.1.5}$$

On the other hand, the mode expansion (2.2.12) leads us to

$$\left\langle \psi^{\dagger}_{\nu}(x)\psi_{\nu}(y) \right\rangle = \frac{1}{L}\sum_{k,q} e^{-i\alpha k x}\, e^{i\alpha q y} \left\langle c^{\dagger}_{k\nu} c_{q\nu} \right\rangle \tag{A.1.6}$$

$$= \frac{1}{L}\sum_{k} e^{-i\alpha k(x-y)}\, \theta(-k) \tag{A.1.7}$$

$$= \int_0^{\infty} \frac{dk}{2\pi}\, e^{i\alpha k(x-y)} \tag{A.1.8}$$

$$= \frac{1}{2\pi}\frac{i}{i\lambda + \alpha(x-y)}. \tag{A.1.9}$$

In the third line we have considered the thermodynamic limit $L \to \infty$ which makes the summation over k an integral, under the usual rule $1/L\sum_k \to \int dk/2\pi$. To get the result in the last line we used one of the technicalities we mentioned at the beginning of this appendix, namely, we implicitly re-defined the integral over k, $\int dk \to \int dk e^{-\lambda|k|}$, with $\lambda > 0$ infinitesimal. This is an essential physical consideration, for we expect the expectation value to be finite while the integral in (A.1.8) is not well-behaved for large values of k. Thus, the λ parameter controls the, otherwise unavoidable, ultraviolet divergence. Another reason to introduce this is that with $\lambda = 0$ the expectation value is ill-defined when $x = y$. For that reason, even though formally we should take $\lambda \to 0^+$ at the end of the calculations, in this case it is crucial to keep the λ. This was present in different contexts throughout the work, see for instance the discussion in section 2.4.

In a similar fashion we can compute $\left\langle \psi_{\nu}(x)\psi_{\nu}(y)^{\dagger} \right\rangle$ finding that

$$\left\langle \psi^{\dagger}_{\nu}(x)\psi_{\nu}(y) \right\rangle = \left\langle \psi_{\nu}(x)\psi^{\dagger}_{\nu}(y) \right\rangle = \frac{1}{2\pi}\frac{i}{i\lambda + \alpha(x-y)}. \tag{A.1.10}$$

Using this result and (A.1.2) the expectation value of $[\rho_\nu(x), \rho_{\nu'}(y)]$ reads:

$$\langle [\rho_\nu(x), \rho_{\nu'}(y)] \rangle = \left(\left\langle \psi^{\dagger}_{\nu}(x)\psi_{\nu}(y) \right\rangle^2 - \left\langle \psi^{\dagger}_{\nu}(y)\psi_{\nu}(x) \right\rangle^2 \right)\delta_{\nu,\nu'} \tag{A.1.11}$$

$$= \left[\left(\frac{1}{2\pi}\frac{i}{i\lambda + \alpha(x-y)} \right)^2 - \left(\frac{1}{2\pi}\frac{i}{i\lambda + \alpha(y-x)} \right)^2 \right]\delta_{\nu,\nu'} \tag{A.1.12}$$

$$= \left(\frac{i}{2\pi}\right)^2 (-\alpha)\, \partial_x \left(\frac{1}{i\lambda + \alpha(x-y)} + \frac{1}{i\lambda - \alpha(x-y)} \right)\delta_{\nu,\nu'} \tag{A.1.13}$$

$$= -\alpha \frac{1}{2\pi}^2\, i2\, \partial_x \left(\frac{\lambda}{\lambda^2 + (x-y)^2} \right)\delta_{\nu,\nu'}. \tag{A.1.14}$$

Taking $\lambda \to 0^+$ and using the following limit representation for the Dirac delta function [100],

$$\delta(x-a) = \lim_{\eta\to 0^+} \frac{1}{\pi}\frac{\eta}{\eta^2+(x-a)^2}, \tag{A.1.15}$$

we are left with

$$\langle[\rho_\nu(x), \rho_{\nu'}(y)]\rangle = \frac{\alpha}{2\pi i}\partial_x\delta(x-y)\delta_{\nu,\nu'}. \tag{A.1.16}$$

We can safely get rid of the expectation value in the left-hand side since in the right-hand side there is no operator and the only known operator to yield unity as its expectation value in the vacuum state is the identity operator.

A.2 Calculations towards the bosonization identity

In the context of section 2.3.1, consider a system of M species of chiral fermions labeled by η. We begin by computing the following commutator using equations (2.2.12), (2.2.19) and (2.2.24).

$$[b_{p\eta}, \psi_{\eta'}(x)] = \left[-i\left(\frac{2\pi}{Lp}\right)^{1/2}\rho_{-p,\eta}, \frac{1}{\sqrt{L}}\sum_{k'} e^{i\alpha' k' x}c_{k'\eta'}\right] \tag{A.2.1}$$

$$= \left[-i\left(\frac{2\pi}{Lp}\right)^{1/2}\sum_{k}{:}c^\dagger_{k-p,\eta}c_{k\eta}{:}, \frac{1}{\sqrt{L}}\sum_{k'} e^{i\alpha' k' x}c_{k'\eta'}\right] \tag{A.2.2}$$

$$= -i\left(\frac{2\pi}{Lp}\right)^{1/2}\frac{1}{\sqrt{L}}\sum_{k,k'} e^{i\alpha' k' x}\left[{:}c^\dagger_{k-p,\eta}c_{k\eta}{:}, c_{k'\eta'}\right] \tag{A.2.3}$$

$$= i\left(\frac{2\pi}{Lp}\right)^{1/2}\frac{1}{\sqrt{L}}\sum_{k,k'} e^{i\alpha' k' x}\delta_{k',k-p}\delta_{\eta,\eta'}c_{k\eta} \tag{A.2.4}$$

$$= i\left(\frac{2\pi}{Lp}\right)^{1/2}\frac{1}{\sqrt{L}}e^{-i\alpha p x}\sum_{k} e^{i\alpha k x}c_{k\eta}\delta_{\eta,\eta'} \tag{A.2.5}$$

$$= i\left(\frac{2\pi}{Lp}\right)^{1/2} e^{-i\alpha p x}\psi_\eta(x)\delta_{\eta,\eta'}. \tag{A.2.6}$$

A.2.1 Operator identities

Now we prove the identities (2.3.20) and (2.3.21). Let's start with the former one.

$$[A,B] = DA \implies AB = DA + BA = (B+D)A \tag{A.2.7}$$

$$AB^2 = (B+D)AB = (B+D)(B+D)A = (B+D)^2A \tag{A.2.8}$$

$$AB^n = (B+D)^nA. \tag{A.2.9}$$

Now let us consider,

$$Af(B) = A\left(\sum_{n=0}^{\infty}\frac{1}{n!}f^{(n)}(a)\Big|_{a=0}B^n\right) = \sum_{n=0}^{\infty}\frac{1}{n!}f^{(n)}(a)\Big|_{a=0}AB^n \tag{A.2.10}$$

$$= \sum_{n=0}^{\infty}\frac{1}{n!}f^{(n)}(a)\Big|_{a=0}(B+D)^nA = \left[\sum_{n=0}^{\infty}\frac{1}{n!}f^{(n)}(a)\Big|_{a=0}(B+D)^n\right]A \tag{A.2.11}$$

$$= f(B+D)A. \tag{A.2.12}$$

To prove (2.3.21) we invoke the Baker-Campbell-Hausdorff formula [63]:

$$e^{-B}Ae^{B} = A + [A,B] + \frac{1}{2!}[[A,B],B] + \dots \tag{A.2.13}$$

Under the conditions $[A,B] = C$ with $[A,C] = [B,C] = 0$ the formula A.2.13 reduces to

$$e^{-B}Ae^{B} = A + C \implies (e^{-B}Ae^{B})(e^{-B}Ae^{B}) = (A+C)^2 \tag{A.2.14}$$

$$e^{-B}A^2e^{B} = (A+C)^2 \implies e^{-B}A^ne^{B} = (A+C)^n. \tag{A.2.15}$$

Finally,

$$e^{-B}f(A)e^{B} = \sum_{n=0}^{\infty}\frac{1}{n!}f^{(n)}(a)\Big|_{a=0}e^{-B}A^ne^{B} \tag{A.2.16}$$

$$= \sum_{n=0}^{\infty}\frac{1}{n!}f^{(n)}(a)\Big|_{a=0}(A+C)^n \tag{A.2.17}$$

$$= f(A+C). \tag{A.2.18}$$

A.3 Commutators of the dual bosonic fields

Our primary goal here is to prove that ϕ_γ and Π_γ, as defined in section 2.4, are canonical fields.

A.3.1 Preliminary commutators

We begin by defining the auxiliary field

$$\varphi_\eta(x) \equiv -\sum_{p>0}\frac{1}{\sqrt{Lp}}e^{-\lambda p/2}e^{i\alpha px}b_{p\eta}, \tag{A.3.1}$$

and looking for the commutator

$$[\varphi_\eta(x), \varphi^\dagger_{\eta'}(y)] = \sum_{p>0}\sum_{q>0}\frac{1}{\sqrt{Lp}\sqrt{Lq}}e^{-\lambda p/2}e^{-\lambda q/2}e^{i\alpha px}e^{-i\alpha' qy}\left[b_{p\eta}, b^\dagger_{q\eta'}\right] \tag{A.3.2}$$

$$= \sum_{p,q>0}e^{-\lambda p/2}e^{-\lambda q/2}e^{i\alpha px}e^{-i\alpha' qy}\delta_{p,q}\delta_{\eta,\eta'} \tag{A.3.3}$$

$$= \sum_{p>0} \frac{1}{Lp} e^{-\lambda p} e^{i\alpha p(x-y)} \delta_{\eta,\eta'}. \tag{A.3.4}$$

From this calculation it is clear that

$$[\varphi_\eta(x), \varphi_{\eta'}(y)] = 0 = \left[\varphi_\eta^\dagger(x), \varphi_{\eta'}^\dagger(y)\right]. \tag{A.3.5}$$

Also, we notice that the spatial dependence of $\varphi_\eta(x)$ is contained in a c-number, so then

$$[\varphi_\eta(x), \partial_y \varphi_{\eta'}^\dagger(y)] = \partial_y \left[\varphi_\eta(x), \varphi_{\eta'}^\dagger(y)\right] \tag{A.3.6}$$

$$= \partial_y \left(\sum_{p>0} \frac{1}{Lp} e^{-\lambda p} e^{i\alpha p(x-y)} \delta_{\eta,\eta'} \right) \tag{A.3.7}$$

$$= -\sum_{p>0} \frac{i\alpha}{L} e^{ip[\alpha(x-y)+i\lambda p]} \delta_{\eta,\eta'}. \tag{A.3.8}$$

As discussed in section 2.4.1, now we can choose periodic boundary conditions as long as at the end we take $L \to \infty$.

$$[\varphi_\eta(x), \partial_y \varphi_{\eta'}^\dagger(y)] = -\sum_{n>0} \frac{i\alpha}{L} \exp\left[i\frac{2\pi}{L} n(\alpha(x-y)+i\lambda)\right] \delta_{\eta,\eta'} \tag{A.3.9}$$

$$= -\frac{i\alpha}{L} \left[\sum_{n\geq 0} \exp\left[i\frac{2\pi n}{L}(\alpha(x-y)+i\lambda)\right] - 1 \right] \delta_{\eta,\eta'} \tag{A.3.10}$$

$$= -\frac{i\alpha}{L} \frac{1}{1-\exp\left[i\frac{2\pi\alpha}{L}(x-y+i\alpha\lambda)\right]} \delta_{\eta,\eta'} + \frac{i\alpha}{L} \delta_{\eta,\eta'}. \tag{A.3.11}$$

From this calculation we also realize that

$$[\varphi_\eta(x), \partial_y \varphi_{\eta'}(y)] = 0 = \left[\varphi_\eta^\dagger(x), \partial_y \varphi_{\eta'}^\dagger(y)\right]. \tag{A.3.12}$$

Finally, we can exploit again the spatial dependence in the operators, namely, that position enters into them through a c-number, to find

$$\left[\varphi_\eta(x), \partial_y \varphi_{\eta'}^\dagger(y)\right]^\dagger = \left(\varphi_\eta(x)\partial_y \varphi_{\eta'}^\dagger(y) - \partial_y \varphi_{\eta'}^\dagger(y)\varphi_\eta(x)\right)^\dagger \tag{A.3.13}$$

$$= \partial_y \varphi_{\eta'}(y)\varphi_\eta^\dagger(x) - \varphi_\eta^\dagger(x)\partial_y \varphi_{\eta'}(y) \tag{A.3.14}$$

$$= [\partial_y \varphi_{\eta'}(y), \varphi_\eta^\dagger(x)]. \tag{A.3.15}$$

A.3.2 Dual bosonic fields

The reason to introduce the auxiliary field $\varphi_\eta(x)$ is that the actual bosonic fields $\phi_\eta(x)$ defined in (2.4.1) are just the Hermitian combination $\phi_\eta(x) = \varphi_\eta(x) + \varphi_\eta^\dagger(x)$. Therefore, the commutators (A.3.11), (A.3.12), (A.3.15) are the building blocks to determine the commutators for the bosonic fields:

$$[\phi_\eta(x), \partial_y \phi_{\eta'}(y)] = \left[\varphi_\eta(x) + \varphi_\eta^\dagger(x), \partial_y \varphi_{\eta'}(y) + \partial_y \varphi_{\eta'}^\dagger(y)\right] \tag{A.3.16}$$

$$= \left[\varphi_\eta(x), \partial_y \varphi^\dagger_{\eta'}(y)\right] + \left[\varphi^\dagger_\eta(x), \partial_y \varphi^\dagger_{\eta'}(y)\right] \tag{A.3.17}$$

$$= \left[\varphi_\eta(x), \partial_y \varphi^\dagger_{\eta'}(y)\right] - \left[\varphi_\eta(x), \partial_y \varphi^\dagger_{\eta'}(y)\right]^\dagger \tag{A.3.18}$$

$$= 2i\,\mathrm{Im}\left\{\left[\varphi_\eta(x), \partial_y \varphi^\dagger_{\eta'}(y)\right]\right\} \tag{A.3.19}$$

$$= 2i\,\mathrm{Im}\left\{-i\frac{\alpha}{L}\delta_{\eta,\eta'}\frac{1}{1-\left[1+i\frac{2\pi\alpha}{L}(x-y+i\alpha\lambda)\right]} + i\frac{\alpha}{L}\delta_{\eta,\eta'}\right\} \tag{A.3.20}$$

$$= -i\alpha\frac{1}{\pi}\frac{\lambda}{(x-y)^2+\lambda^2}\delta_{\eta,\eta'} + i\frac{2\alpha}{L}\delta_{\eta,\eta'} \tag{A.3.21}$$

$$= -i\alpha\delta(x-y)\delta_{\eta,\eta'} + \frac{i2\alpha}{L}\delta_{\eta,\eta'}. \tag{A.3.22}$$

In the line (A.3.20) we used (A.3.11), and expanded the exponential up to first order in λ and $x-y$, as $\lambda \to 0^+$ and $L \to \infty$. Finally in (A.3.22) we used again the representation of the Dirac delta function, equation (A.1.15).

Ultimately, we can show that ϕ_γ and Π_γ are canonical in the limit $L \to \infty$,

$$[\phi_\gamma(x), \Pi_{\gamma'}(y)] = \frac{1}{2}[\phi_{L\gamma}(x) - \phi_{R\gamma}(x), \partial_y(\phi_{L\gamma'}(y) + \phi_{R\gamma'}(y))] \tag{A.3.23}$$

$$= \frac{1}{2}([\phi_{L\gamma}(x), \partial_y\phi_{L\gamma'}(y)] - [\phi_{R\gamma}(x), \partial_y\phi_{R\gamma'}(y)]) \tag{A.3.24}$$

$$= \frac{1}{2}\left[i\delta(x-y) - i\frac{2}{L} - \left(-i\delta(x-y) + i\frac{2}{L}\right)\right]\delta_{\gamma,\gamma'} \tag{A.3.25}$$

$$= i\delta(x-y)\delta_{\gamma,\gamma'}. \tag{A.3.26}$$

A.3.3 Chiral bosonic fields

Upon implementing the bosonization identity (2.4.10) the commutator between chiral bosonic fields is required. This can be computed from the commutator of the auxiliary fields φ_η.

$$[\phi_\eta(x), \phi_{\eta'}(y)] = \left[\varphi_\eta(x), \varphi_{\eta'}(y)^\dagger\right] + \left[\varphi^\dagger_\eta(x), \varphi_{\eta'}(y)\right] \tag{A.3.27}$$

$$= \sum_{p>0}\frac{1}{Lp}e^{-\lambda p}e^{i\alpha p(x-y)}\delta_{\eta,\eta'} + \sum_{p>0}\frac{-1}{Lp}e^{-\lambda p}e^{-i\alpha p(x-y)}\delta_{\eta,\eta'} \tag{A.3.28}$$

$$= \sum_{n>0}\frac{1}{2\pi}\left[\frac{e^{i\alpha 2\pi n(x-y+i\alpha\lambda)/L} - e^{-i\alpha 2\pi n(x-y-i\alpha\lambda)/L}}{n}\right]\delta_{\eta,\eta'}. \tag{A.3.29}$$

In equation (A.3.29) we chose periodic boundary conditions since at the end we plan to take the thermodynamic limit $L \to \infty$, so the choice is immaterial. Now we use the so-called Mercator series for complex x [101][2],

$$\ln(1-x) = -\sum_{n>0}\frac{x^n}{n}. \tag{A.3.30}$$

[2]This series converges for $|x| < 1$ after choosing a branch for the logarithm.

Therefore, the commutator reads,

$$[\phi_\eta(x),\phi_{\eta'}(y)] = \frac{1}{2\pi}\left\{-\ln\left[1-e^{i\alpha 2\pi(x-y+i\alpha\lambda)/L}\right]+\ln\left[1-e^{-i\alpha 2\pi(x-y-i\alpha\lambda)/L}\right]\right\}\delta_{\eta,\eta'} \tag{A.3.31}$$

$$= \frac{1}{2\pi}\ln\left[\frac{1-e^{-i\alpha 2\pi(x-y-i\alpha\lambda)}}{1-e^{i\alpha 2\pi(x-y+i\alpha\lambda)}}\right]\delta_{\eta,\eta'} \tag{A.3.32}$$

$$= \frac{1}{2\pi}\ln\left[\frac{1-\left(1-i\alpha\frac{2\pi}{L}(x-y-i\alpha\lambda)\right)}{1-\left(1+i\alpha\frac{2\pi}{L}(x-y+i\alpha\lambda)\right)}\right]\delta_{\eta,\eta'} \tag{A.3.33}$$

$$= \frac{1}{2\pi}\ln\left\{\frac{\lambda-[-i\alpha(x-y)]}{\lambda+[-i\alpha(x-y)]}\right\}\delta_{\eta,\eta'}. \tag{A.3.34}$$

In equation (A.3.33) we expanded the exponentials up to first order in λ due to the fact that at some point we will take $\lambda \to 0^+$ and $L \to \infty$. Finally equation (A.3.34) has a suitable form to invoke the following limit [42]:

$$\lim_{\lambda\to 0^+}\ln\left[\frac{\lambda-i(x-y)}{\lambda+i(x-y)}\right] = -i\pi\,\mathrm{sgn}(x-y), \tag{A.3.35}$$

with the sign function defined as

$$\mathrm{sgn}(x) = \begin{cases} 1, & x>0 \\ 0, & x=0 \\ -1, & x<0. \end{cases} \tag{A.3.36}$$

Therefore, we find the commutator

$$[\phi_\eta(x),\phi_{\eta'}(y)] = +\frac{i\alpha}{2}\delta_{\eta,\eta'}\mathrm{sgn}(x-y). \tag{A.3.37}$$

Using the last result we can find the commutator between the ϕ_γ and θ_γ fields.

$$[\phi_\gamma(x),\theta_{\gamma'}(y)] = \frac{1}{2}[\phi_{L\gamma}(x)-\phi_{R\gamma}(x),\phi_{L\gamma'}(y)+\phi_{R\gamma'}(y)] \tag{A.3.38}$$

$$= \frac{1}{2}([\phi_{L\gamma}(x),\phi_{L\gamma'}(y)]-[\phi_{R\gamma}(x),\phi_{R\gamma'}(y)]) \tag{A.3.39}$$

$$= -\frac{i}{2}\mathrm{sgn}(x-y)\delta_{\gamma,\gamma'}. \tag{A.3.40}$$

A.3.4 Density operator

We conclude this appendix by deriving an expression for the density $\rho_\eta(x)$ in terms of the chiral bosonic operators $\phi_\eta(x)$. We rewrite (2.2.18) as

$$\rho_\eta(x) = \frac{N_\eta}{L}+\frac{1}{L}\sum_{p>0}\left(e^{-i\alpha px}\rho_{p\eta}+e^{i\alpha px}\rho_{-p,\eta}\right) \tag{A.3.41}$$

$$= \frac{N_\eta}{L} + \frac{1}{L}\sum_{p>0}\left[(-i)\left(\frac{Lp}{2\pi}\right)^{1/2} e^{-i\alpha px} b^\dagger_{p\eta} + i\left(\frac{Lp}{2\pi}\right)^{1/2} e^{i\alpha px} b_{p\eta}\right] \tag{A.3.42}$$

$$= \frac{N_\eta}{L} - \frac{\alpha}{\sqrt{2\pi}}\sum_{p>0} -i\alpha p \frac{1}{\sqrt{Lp}}\left(e^{i\alpha px} b_{p\eta} - e^{-i\alpha px} b^\dagger_{p\eta}\right) \tag{A.3.43}$$

$$= \frac{N_\eta}{L} - \frac{\alpha}{\sqrt{2\pi}}\partial_x \phi_\eta(x). \tag{A.3.44}$$

Appendix B

Functional integration

In this appendix we present a (very) short introduction to the functional integration formalism employed to perform the computations of this work. We shall state the relevant results for field theory after having discussed (but not proved) their point-particle counterpart. Although along the main text we have adopted natural units in which $\hbar = 1$, in this appendix we keep $\hbar$ to tacitly appreciate some wonders of this framework regarding classical and semi-classical limits. All the developments are performed for one-dimensional bosonic systems, as is the case in the main text, since there is no new physics that shall emerge in higher dimensions, other than a change in the dimensions of the integrals and the Dirac delta function. Finally, analytic proofs of the results we are about to present and their extension to fermionic systems can be found in any of the references provided along the way.

B.1 Particles

The path integral formulation of quantum mechanics is renowned because it highlights the physical differences between classical and quantum systems. It is also very illuminating to understand semi-classical and classical limits of quantum systems where at the limit $\hbar \to 0$ the only non-zero contribution comes from the path that makes the action stationary, that is, the classical solution to the equations of motion. This formulation is obtained upon writing as an integral over paths the transition amplitude for a particle being at x_i at time t_i to be found at a later time t_f at a position x_f[1].

$$\langle x_f, t_f | x_i, t_i \rangle = \int \mathcal{D}p(t)\mathcal{D}x(t) \exp\left\{ \frac{i}{\hbar} \int_{t_i}^{t_f} dt \left[p(t)\dot{x}(t) - H(p(t), x(t)) \right] \right\}. \tag{B.1.1}$$

In the left-hand side we have the transition amplitude from the state $|x_i, t_i\rangle$ to the state $|x_f, t_f\rangle$, both in the Heisenberg picture, while in the right-hand side we have the position and momentum eigenvalues at time t, $x(t)$ and $p(t)$, respectively, and the Hamilton function evaluated at these eigenvalues. The notation $\int \mathcal{D}p(t)\mathcal{D}x(t)$ means that we are integrating over all possible functions $x(t)$ and $p(t)$ with boundary conditions $x(t_i) = x_i$ and $x(t_f) = x_f$,

[1] For a derivation of this result see e.g. [63, 82, 102–104].

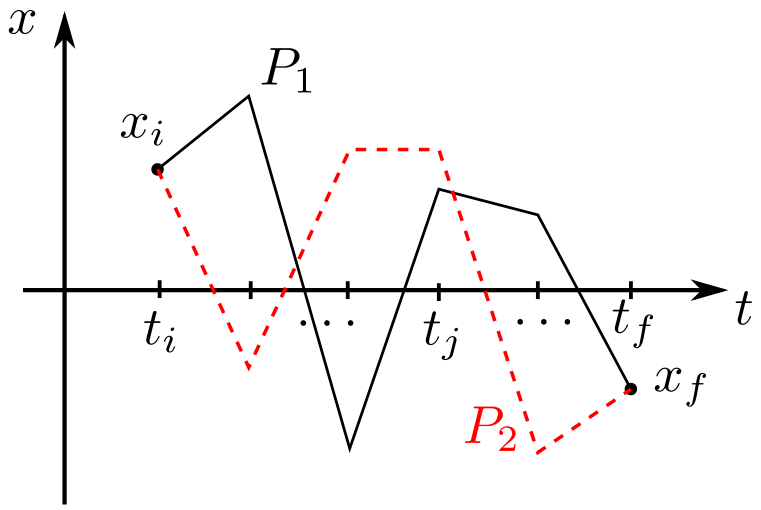

Figure B.1: Functional integration as a sum over the different paths that the particle can follow between the fixed points $x(t_i) = x_i$ and $x(t_f) = x_f$.

that is, *over all possible configurations* in phase space. Since we are integrating over functions and not numbers as usual, this operation bears a special name, *functional integration*.

The idea behind functional integration is illustrated in figure B.1. We construct different paths by keeping the endpoints fixed while varying the value of x at intermediate instants of time t_j. Notice that these are not classical trajectories since each x_j can be anything, regardless of $x_{j\pm1}$. So, even after taking $t_{j+1} - t_j \to 0$ there is no well-defined derivative, though the trajectory is continuous. This is a dramatic departure from classical physics, these are *quantum paths*. Each of these paths defines a configuration $(x(t), p(t))$ in phase space, thus functional integration $\int \mathcal{D}p(t)\mathcal{D}x(t)$ can be regarded as a sum over paths.

The most outstanding feature of this formulation is that in the left-hand side of (B.1.1) we have a purely quantum mechanical quantity, while in the right-hand side we have *classical functions*, not operators. Thus, we can reach quantum mechanics from classical physics as long as we visit all the possible paths.

If H is quadratic in p and Weyl-ordered[2] we can use standard Gaussian integrals to reach the path integral formulation in configuration space [82, 102],

$$\langle x_f, t_f | x_i, t_i \rangle = \mathcal{N} \int \mathcal{D}x(t) \exp\left(\frac{i}{h} S[x(t)]\right) = \mathcal{N} \int \mathcal{D}x(t) \exp\left(\frac{i}{h} \int_{t_i}^{t_f} dt\, L(x(t), \dot{x}(t))\right), \tag{B.1.2}$$

where S is the classical action of the system, L the classical Lagrangian, and $\mathcal{N}$ is strictly speaking a divergent constant but it shall not have any physical relevance. The notation $S[x(t)]$ means that S depends on the whole function $x(t)$, not on a particular value for a given t.

[2]That is, the expression must be symmetric in the operators x and p.

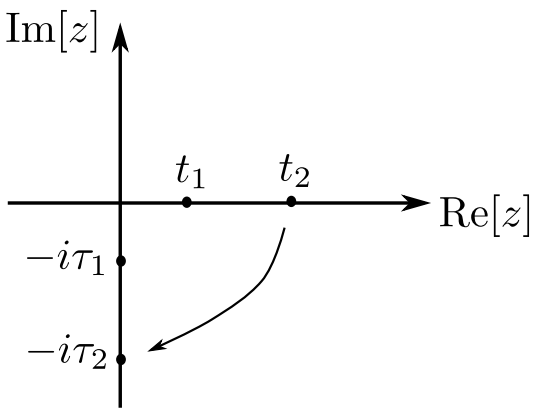

Figure B.2: Wick rotation as a complex analytic continuation.

B.1.1 Wick rotation

We just mentioned that after performing the Gaussian integrals in (B.1.1) we can get (B.1.2). Actually Gaussian integrals are essential tools upon performing exact calculations in the functional integral formalism. Nevertheless, there is a caveat about their scope. Strictly speaking, Gaussian integrals are well-defined only for real functions, while in (B.1.1) (and in any other functional integral of relevance to compute physical quantities) there is a complex argument in the exponential.

To overcome this drawback we set up and analytic continuation that will allow us to get exponentials of the form $e^{-S/\hbar}$ instead of $e^{iS/\hbar}$. This in turn brings a fascinating connection with statistical mechanics. Let us begin by considering a time-independent Hamiltonian H with positive eigenvalues E_n, $H\,|n\rangle = E_n\,|n\rangle$, using the relationship between Heisenberg and Schrödinger pictures $|x,t\rangle = e^{iHt/\hbar}\,|x\rangle$ the transition amplitude $\langle x',t'|x,t\rangle$ is given by

$$\langle x',t'|x,t\rangle = \sum_n e^{-iE_n(t'-t)/\hbar}\,\langle x'|n\rangle\,\langle n|x\rangle\,. \tag{B.1.3}$$

Since the last expression is time translational invariant we can set $t=0$. In particular, let us specialize in the case that the particle arrives at the same initial point, so that $x'=x$, and consider *all* possible such paths, that is, we integrate over x.

$$\int dx\,\langle x,t'|x,0\rangle = \int dx \sum_n e^{-iE_nt'/\hbar}\,\langle x|n\rangle\,\langle n|x\rangle\,. \tag{B.1.4}$$

We perform an analytic continuation of t to the complex numbers, rotating it $\pi/2$ clockwise in the complex plane, as indicated in figure B.2. A clockwise instead of a counterclockwise rotation guarantees a finite time-evolution operator, and hence a convergent transition amplitude. The transformation $t \to -i\tau$ is known as Wick rotation, and τ is referred to as Euclidean time. Implementing the rotation $t' \to -i\tau$ in (B.1.4) we get

$$\int dx\,\langle x,-i\tau|x,0\rangle = \int dx \sum_n e^{-E_n\tau/\hbar}\,\langle x|n\rangle\,\langle n|x\rangle = \sum_n \langle n|e^{-H\tau/\hbar}|n\rangle = \mathrm{Tr}\left[e^{-H\tau/\hbar}\right]. \tag{B.1.5}$$

This is an outstanding result, for if we identify $\tau/\hbar$ with $\beta = 1/k_B T$ then we get the partition function for the canonical ensemble in statistical mechanics. Thus, beginning with a single-particle propagator and considering all the possible closed paths we arrive to an analogue statistical quantity.

Now we can translate this result to the path integral formalism. Consider a typical Lagrangian of the form

$$L(x, \dot{x}) = \frac{1}{2}\dot{x}^2 - V(x) \qquad (m = 1), \tag{B.1.6}$$

so that the action reads

$$S[x] = \int_{t_i}^{t_f} dt \left[\frac{1}{2}\dot{x}^2 - V(x)\right]. \tag{B.1.7}$$

After the Wick rotation $t \to -i\tau$, S reads

$$S[x(-i\tau)] = i\int_{\tau_i}^{\tau_f} d\tau \left[\frac{1}{2}\left(\frac{dx}{d\tau}\right)^2 + V(x(-i\tau))\right] \equiv iS_E[x], \tag{B.1.8}$$

the subscript E stands for Euclidean after reasons that become evident in quantum field theory, see Ref. [102] for reference.

Let us implement this into (B.1.2):

$$\langle x_f, -i\tau_f | x_i, -i\tau_i\rangle = \int \mathcal{D}x(-i\tau) \exp\left(\frac{i}{\hbar}S[x(-i\tau)]\right) = \int \mathcal{D}x(-i\tau) \exp\left(-\frac{1}{\hbar}S_E[x]\right). \tag{B.1.9}$$

In what follows we shall write the time variable just as τ instead of $-i\tau$, but we should bear in mind that we are working in Euclidean time. Notice that through the Wick rotation we get an exponential function with a real argument, S_E, making it possible to employ Gaussian integrals in the computations, which was our primary motivation to introduce this analytic continuation.

Writing equation (B.1.5) as a functional integral leads us to the famous Feynmann-Kac formula,

$$\int dx\, \langle x, \tau | x, 0\rangle = \int dx \int_{x(\tau)=x(0)} \mathcal{D}x(\tau') \exp\left(-\frac{1}{\hbar}S_E[x(\tau')]\right) \tag{B.1.10}$$

$$\mathrm{Tr}\left[e^{-\tau H/\hbar}\right] = \int_{\text{periodic } x(\tau')} \mathcal{D}x(\tau') \exp\left(-\frac{1}{\hbar}S_E[x(\tau')]\right). \tag{B.1.11}$$

The idea behind this integral is depicted in figure B.3. Feynman-Kac formula is completely unexpected but very important, for it allows us to use functional integration in statistical mechanics. As it is known, given the partition function of the system, we can find all the thermodynamic quantities associated to it. It is important to remark, though, that we started with $\langle x, t | x, 0\rangle$ which is a quantum mechanical quantity for a single-particle system, there is no heat bath, nor an ensemble of particles. This is just one example of several exciting dualities that appear in modern physics and that show an intrinsic connection between very distinct realms of physics.

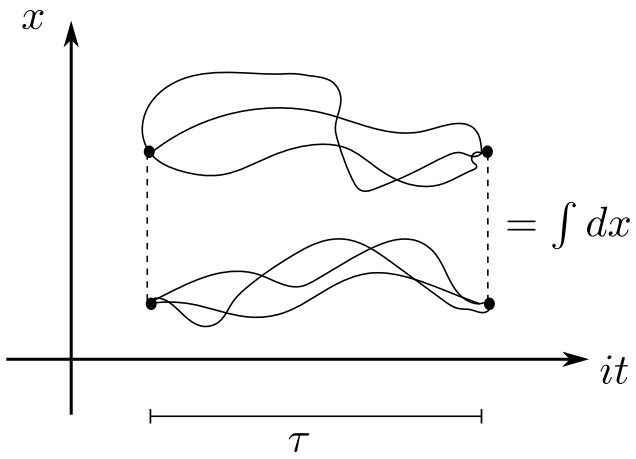

Figure B.3: For a given $x(0) = x(\tau)$ we perform a functional integral summing over all possible paths. Then, we repeat the summation for all possible $x(0)$, which would be the integral over x in (B.1.10).

B.1.2 Correlation functions

After proving equation (B.1.2) it is straightforward to show that (see Ref. [102] for reference)

$$\langle x_f, t_f | x(t_2)x(t_1) | x_i, t_i \rangle = \mathcal{N} \int \mathcal{D}x(t) \exp\left(\frac{i}{\hbar} S[x(t)]\right) x(t_2)x(t_1), \text{ for } t_f > t_2 > t_1 > t_i. \tag{B.1.12}$$

The condition on the instants of time arises from the fact that when writing the transition amplitude $\langle x_f, t_f | x_i, t_i \rangle$ as a functional integral we construct the different paths in an *ordered* manner, starting at t_i and ending at t_f (as shown in figure B.1). Thus, if $t_1 > t_2$ when we evaluate the operator $x(t_1)$ on the state $|x_1, t_1\rangle$ we have already summed over all the possible states $|x_2, t_2\rangle$, so that it would be impossible to evaluate the operator $x(t_2)$ later. This is a major subtlety that shows up in the left-hand side only, for in the right-hand side we are dealing with functions and $x(t_2)x(t_1) = x(t_1)x(t_2)$. Thus, we are able to write $\langle x_f, t_f | x(t_2)x(t_1) | x_i, t_i \rangle$ as a functional integral *if and only if the operators are time-ordered.*

Thus it is natural to introduce the time ordering operator $T\{\ldots\}$. In general, the action of $T\{\ldots\}$ on any product of operators $\prod_j \Lambda_j(t_j)$ is to place the operators at earlier times at the right of operators at later times. For instance, for two operators we would have

$$T\{\Lambda_1(t_1)\Lambda_2(t_2)\} \equiv \Lambda_1(t_1)\Lambda_2(t_2)\theta(t_1 - t_2) + \Lambda_2(t_2)\Lambda_1(t_1)\theta(t_2 - t_1). \tag{B.1.13}$$

Therefore the left-hand side of (B.1.12) can be written as $\langle x_f, t_f | T\{x(t_1)x(t_2)\} | x_i, t_i \rangle$, and it is generalized to the case of N position operators, defining the N-point correlation function as

$$\langle x_f, t_f | T\{x(t_1)x(t_2)\ldots x(t_N)\} | x_i, t_i \rangle = \mathcal{N} \int \mathcal{D}x(t) \exp\left(\frac{i}{\hbar} S[x(t)]\right) x(t_1)\ldots x(t_N). \tag{B.1.14}$$

As mentioned before, strictly speaking one has to perform a Wick rotation to properly handle with the exponential as a Gaussian integral. However, some technicalities arise related to how to handle with $T\{\ldots\}$ in such a case. Even when working in Euclidean time, the time ordering operator is still defined in real time, that is, the operators are ordered with respect to $-i\tau_j$. We could ignore the operator $T\{\ldots\}$ by imposing periodic boundary conditions on $x(\tau')$, as done for deriving the Feynman-Kac formula, for if $-i\tau_1 < -i\tau_2$ then we simply consider $-i(\tau_1 + \tau) > -i\tau_2$ so that the order does not really matters. Then a rigorous expression for the N-point correlation function is

$$\langle x_f, \tau_f | x(\tau_1)x(\tau_2)\ldots x(\tau_N)|x_i, \tau_i\rangle = \mathcal{N}\int_{\text{periodic } x(\tau')} \mathcal{D}x(\tau')\exp\left(-\frac{1}{\hbar}S_E[x(\tau')]\right)x(\tau_1)\ldots x(\tau_N). \tag{B.1.15}$$

Upon doing this, however, our system is mapped into a canonical ensemble with inverse temperature $\beta = \tau/\hbar$. Once found the correlation function, we should go back to the original system first by taking $\beta \to \infty$, which would be the same as setting the temperature to zero, then by undoing the Wick rotation setting $\tau_j \to it_j$.

B.2 Fields

The passage from point-particle quantum mechanics to quantum field theory is somewhat straightforward at the level of the formulas, but not so at the conceptual level. The breakthrough insight is to downgrade position operator x to a parameter, at the same footing as time t, and regard the wavefunction $\psi(x,t)$ as a (field) operator instead of the single-particle element $\langle x,t|\psi\rangle$. The physical picture changes entirely: We are no longer dealing with a localized particle, but with an object that spreads out over all space and time. Thus we pass from working with a single degree of freedom to work with an *infinite number of degrees of freedom*, one at each value of (x,t).

Since position is downgraded to a parameter, so is the momentum p. Our dynamical variable is going to be some bosonic field $\phi(x,t)$ (such as the bosonic fields we have defined in the main text), and so it is its conjugate canonical variable denoted $\Pi(x,t)$, which satisfy the bosonic algebra $[\phi(x,t),\Pi(y,t)] = i\hbar\delta(x-y)$. So let us think on how we would generalize equation (B.1.1), in the right-hand side the immediate replacements are $p(t) \to \Pi(x,t)$ and $x(t) \to \phi(x,t)$. The Hamiltonian $H(p,x)$ has to be replaced accordingly with a Hamiltonian density $\mathcal{H}(\phi(x,t),\Pi(x,t))$. It is called density because now it can be thought of as energy distributed over all space. Finally, if x and t have to be at the same footing then we should also integrate over x. However, opposite to what happens with t, we should not integrate between two given positions x_i and x_f, but over the whole space, for the field is not to be found at a given point of space but it is, at any time, spread over all the space. Therefore, the right-hand side of equation (B.1.1) is generalized into

$$\int \mathcal{D}\Pi(x,t)\mathcal{D}\phi(x,t)\ \exp\left\{\frac{i}{\hbar}\int_{t_i}^{t_f}dt\int_{-\infty}^{\infty}dx\,[\Pi(x,t)\partial_t\phi(x,t) - \mathcal{H}(\phi,\Pi)]\right\}, \tag{B.2.1}$$

where likewise as in the point-particle case, we are dealing with *classical fields*, not operators, and the functional integral is performed by exploring all the configuration of these fields. Space and time, as parameters, are the background on which these configurations take place.

It is much more tricky to think on the generalization of the left-hand side of equation (B.1.1), for one thing, it does not make sense anymore to ask about the transition amplitude for the field being at x_i at t_i and then at x_f at t_f. Now the field spreads out over all space at any time! The question we should raise now is about the transition amplitude for the field being at its *ground state* at t_i and then again at its ground state at t_f.

The simplest system we can consider is that of a non-interacting scalar field $\phi(x,t)$. It turns out that such a system can be regarded as an infinite collection of decoupled harmonic oscillators, one for each value of p at some time t [99, 102]. Fixed p and t we can describe the harmonic oscillators as in elementary quantum mechanics through creation and annihilation operators [63], then the ground state of the system $|0\rangle$ is given by

$$a_p\,|0\rangle = 0 \quad \text{for all } p, \tag{B.2.2}$$

and it shall be called *vacuum state* after the possibility of *interpreting* the nth excitation of each harmonic oscillator with momentum p as a state comprised of n particles of momentum p and energy $E = \hbar\omega_p = \hbar|p|$ [99, 102].

Consequently, the field generalization of equation (B.1.1) is given by

$$\langle 0, t_f|0, t_i\rangle = \int \mathcal{D}\Pi(x,t)\mathcal{D}\phi(x,t)\ \exp\left\{\frac{i}{\hbar}\int_{t_i}^{t_f} dt \int dx\,[\Pi(x,t)\partial_t\phi(x,t) - \mathcal{H}(\phi,\Pi)]\right\}. \tag{B.2.3}$$

Likewise the case of particles, to get expressions we could handle with as Gaussian integrals we must work in Euclidean time, through a Wick rotation. In the case of fields t keeps being a parameter so the Wick rotation works in the same fashion. Then the transition amplitude reads

$$\langle 0, \tau_f|0, \tau_i\rangle = \int \mathcal{D}\Pi(x,\tau)\mathcal{D}\phi(x,\tau)\exp\left\{\frac{1}{\hbar}\int_{\tau_i}^{\tau_f} d\tau \int dx\,[i\Pi(x,\tau)\partial_\tau\phi(x,\tau) - \mathcal{H}(\phi,\Pi)]\right\}. \tag{B.2.4}$$

B.2.1 Correlation functions

In what follows we shall lighten a bit the notation by writing the functional integral measure just as $\mathcal{D}\Pi\mathcal{D}\phi$ since regardless of the independent variables the integration runs over all the configurations of the fields. Now, it is straightforward to write down the field version of equation (B.1.15),

$$\begin{aligned}&\langle 0, \tau_f|\phi(x_1,\tau_1)\ldots\phi(x_N,\tau_N)|0,\tau_i\rangle \\ &= \int_{\text{periodic }\phi(x,\tau')} \mathcal{D}\Pi\mathcal{D}\phi\ \phi(x_1,\tau_1)\ldots\phi(x_N,\tau_N)\end{aligned}$$

$$\times \exp\left\{\frac{1}{\hbar}\int_{\tau_i}^{\tau_i+\tau} d\tau' \int dx\,[i\Pi(x,\tau')\phi(x,\tau') - \mathcal{H}(\phi,\Pi)]\right\}, \tag{B.2.5}$$

the periodicity is taken over Euclidean time $\phi(x,\tau') = \phi(x,\tau'+\tau)$. Due to this fact, we have got rid of the time ordered product $T\{\ldots\}$ in the left-hand side.

From the arguments invoked upon deriving equation (B.1.15) (see Refs. [99,102] for reference) it is clear that nothing would change in the derivation if we considered products of functions of the field operators $f(\phi(x_j,\tau_j),\Pi(x_j,\tau_j))$ instead of the product $\phi(x_1,\tau_1)\ldots\phi(x_N,\tau_N)$, as long as the Hamiltonian density $\mathcal{H}$ is Weyl-ordered. Therefore,

$$\begin{aligned}&\langle 0,\tau_f|f(\phi(x_1,\tau_1),\Pi(x_1,\tau_1))\,g(\phi(x_2,\tau_2),\Pi(x_2,\tau_2))|0,\tau_i\rangle\\ &= \int_{\text{periodic }\phi(x,\tau')} \mathcal{D}\Pi\mathcal{D}\phi\; f(x_1,\tau_1)g(x_2,\tau_2)\exp\left\{\frac{1}{\hbar}\int_{\tau_i}^{\tau_i+\tau} d\tau' \int dx[i\Pi(x,\tau')\phi(x,\tau') - \mathcal{H}(\phi,\Pi)]\right\}.\end{aligned} \tag{B.2.6}$$

It is worthwhile to emphasize that all the expressions discussed hitherto make reference to a transition from the vacuum state to the very same state for a *non-interacting* field theory, for only in that case we can regard the system as an infinite collection of quantum harmonic oscillators. Yet we have not said what we understand by a non-interacting field theory. After reasons that become clear upon being familiar with the standard computational tools of quantum field theory, we say that a theory is non-interacting if the Hamiltonian density is quadratic in the canonical fields. For instance, the last term in equation (3.0.13) makes the effective Hamiltonian for the FFLO system to be describing an interacting field theory.

It is an elementary fact in quantum mechanics that the ground state of a non-interacting system changes in the presence of interactions [63], and likewise happens in any quantum field theory [105]. Let us denote the vacuum of the interacting theory as $|\Omega\rangle$, then the relevant physical quantities are associated to the transition amplitude $\langle\Omega,t_f|\Omega,t_i\rangle$. Expressions as (B.2.6) should be corrected accordingly, and the way to do it is requiring the interacting vacuum to remain as the vacuum, $\langle\Omega|\Omega\rangle = 1$ [99]. Therefore, by defining,

$$Z \equiv \int_{\text{periodic }\phi(x,\tau')} \mathcal{D}\Pi\mathcal{D}\phi\,\exp\left\{\frac{1}{\hbar}\int_{\tau_i}^{\tau_i+\tau} d\tau' \int dx[i\Pi(x,\tau')\phi(x,\tau') - \mathcal{H}(\phi,\Pi)]\right\}, \tag{B.2.7}$$

then equation (B.2.6) for an interacting quantum field theory reads

$$\begin{aligned}&\langle\Omega,\tau_f|f(\phi(x_1,\tau_1),\Pi(x_1,\tau_1))\,g(\phi(x_2,\tau_2),\Pi(x_2,\tau_2))|\Omega,\tau_i\rangle\\ &= \frac{1}{Z}\int_{\text{periodic }\phi(x,\tau')} \mathcal{D}\Pi\mathcal{D}\phi\; f(x_1,\tau_1)g(x_2,\tau_2)\exp\left\{\frac{1}{\hbar}\int_{\tau_i}^{\tau_i+\tau} d\tau' \int dx[i\Pi(x,\tau')\phi(x,\tau') - \mathcal{H}(\phi,\Pi)]\right\}.\end{aligned} \tag{B.2.8}$$

There is a much more elegant way to justify the last formula. Consider the field version of equation (B.1.11),

$$\text{Tr}\left[e^{-\tau H/\hbar}\right] = \int_{\text{periodic }\phi(x,\tau')} \mathcal{D}\Pi\mathcal{D}\phi\;\exp\left\{\frac{1}{\hbar}\int_{\tau_i}^{\tau_i+\tau} d\tau' \int dx[i\Pi(x,\tau')\phi(x,\tau') - \mathcal{H}(\phi,\Pi)]\right\}. \tag{B.2.9}$$

Setting $\beta = \tau/\hbar$ we have a duality with (field) statistical mechanics, and now it makes sense to have called (B.2.7) as Z, for it is the partition function of the system. Then, regardless of whether the field theory is interacting or not, we can find the expectation value of the operator $f(\phi(x_1,\tau_1),\Pi(x_1,\tau_1))\, g(\phi(x_2,\tau_2),\Pi(x_2,\tau_2))$ through the density matrix

$$\langle f(x_1,\tau_1)\, g(x_2,\tau_2)\rangle = \mathrm{Tr}\left[\rho\, f(\phi(x_1,\tau_1),\Pi(x_1,\tau_1))\, g(\phi(x_2,\tau_2),\Pi(x_2,\tau_2))\right] \tag{B.2.10}$$

$$= \frac{1}{Z}\,\mathrm{Tr}\left[e^{-\beta H} f(\phi(x_1,\tau_1),\Pi(x_1,\tau_1))\, g(\phi(x_2,\tau_2),\Pi(x_2,\tau_2))\right]. \tag{B.2.11}$$

Likewise we expressed the trace (B.1.11) as a functional integral, we can write the trace of $f(x_1,\tau_1)g(x_2,\tau_2)e^{-\beta H}$ as

$$\mathrm{Tr}\left[f(x_1,\tau_1)g(x_2,\tau_2)e^{-\beta H}\right] = \int_{\text{periodic }\phi(x,\tau')} \mathcal{D}\Pi\mathcal{D}\phi\, f(x_1,\tau_1)g(x_2,\tau_2) \times \exp\left\{\frac{1}{\hbar}\int_{\tau_i}^{\tau_i+\hbar\beta} d\tau' \int dx[i\Pi(x,\tau')\phi(x,\tau') - \mathcal{H}(\phi,\Pi)]\right\}, \tag{B.2.12}$$

therefore (B.2.8) coincides with (B.2.11), and we have reinforced previous key claims concerning quantum statistical mechanics, for instance, why it makes sense to neglect the time ordering operator when working in Euclidean time.

Equation (B.2.8) is the chief result of this appendix, it dictates a rigorous, elegant framework to compute correlation functions using functional integration. However, we have not shown *how to* actually perform any of these integrals. As claimed several times along this appendix, the standard method is to use Gaussian integrals, which is the subject of the next appendix.

B.2.2 Matsubara frequencies

Upon performing the Wick rotation, imposing periodicity for the fields $\phi(x,\tau') = \phi(x,\tau'+\tau)$, and identifying $\tau/\hbar$ with β, we can regard our fields $\phi(x,\tau)$[3] as defined over the whole Euclidean time axis, and periodic on an interval of length β, say $[0,\beta]$. As such, we can expand them as a Fourier series in the time variable,

$$\phi(x,\tau) = \frac{1}{\sqrt{\beta}}\sum_n e^{-i\omega_n\tau}\phi(x,\omega_n), \qquad \phi(x,\omega_n) = \frac{1}{\sqrt{\beta}}\int_0^\beta d\tau\, e^{i\omega_n\tau}\phi(x,\tau), \tag{B.2.13}$$

where

$$\omega_n = \frac{2\pi n}{\beta}, \quad n\in\mathbb{Z}, \tag{B.2.14}$$

[3] Tricky tongue twister between τ and τ'. Hereinafter the period is given by β and the independent variable is going to be τ.

are known as (*bosonic*) Matsubara frequencies[4].

We would like to set periodic boundary conditions in the position variable as well, for then the derivatives in both space and (Euclidean) time in (B.2.8) become algebraic operations. We can actually manage to do this: We split space into boxes of length L and make the physics of the system to be the same in each box, that is, $\phi(x,\tau) = \phi(x+L,\tau)$. Therefore, we can also perform a Fourier expansion in the position variable leading us to

$$\phi(x,\tau) = \frac{1}{\sqrt{L\beta}} \sum_{k,n} e^{i(kx-\omega_n\tau)}\, \phi(k,\omega_n), \quad k = \frac{2\pi m}{L}, \quad m \in \mathbb{Z}. \tag{B.2.15}$$

Since this discretization of space is artificial, after doing the computations we shall take the limit $L \to \infty$ to recover continuous space. As is known, differential operators in the (x,τ) space are now algebraic in the (k,ω_n)-space,

$$\partial_x \phi(x,\tau) = \frac{1}{\sqrt{L\beta}} \sum_{k,n} (ik) e^{i(kx-\omega_n\tau)}\, \phi(k,\omega_n) \tag{B.2.16}$$

$$\partial_\tau \phi(x,\tau) = \frac{1}{\sqrt{L\beta}} \sum_{k,n} (-i\omega_n) e^{i(kx-\omega_n\tau)}\, \phi(k,\omega_n). \tag{B.2.17}$$

[4]Recall that the entire field formulation holds for a bosonic field $\phi(x,t)$. Fermionic fields display different features as elements of Grassmann algebras, in particular, they must satisfy anti-periodic instead of periodic boundary conditions to reach a duality with statistical mechanics. Therefore, even though they can be decomposed in Fourier space, the values of the Matsubara frequencies shall be different to the ones stated. For more reference see Refs. [82, 83].

Appendix C

Gaussian integrals

In this appendix we derive the relevant formulas to perform the functional integrals present in equation (B.2.8). At the end, we shall realize that the appeal of non-interacting field theories is that, being the Hamiltonian density quadratic in the fields, we are able to use the formulas of this appendix to calculate the *exact* value of the integrals.

The basic Gaussian integral is

$$I = \int_{-\infty}^{\infty} dx\, e^{-ax^2/2} = \sqrt{\frac{2\pi}{a}}, \quad a > 0. \tag{C.0.1}$$

Taking the square of this formula,

$$I^2 = \int dxdy\, e^{-a(x^2+y^2)/2} = \int_0^{2\pi} d\theta \int_0^{\infty} dr\; re^{-ar^2/2} = \frac{2\pi}{a}. \tag{C.0.2}$$

This can be generalized to any quadratic form in the exponent, $\mathbf{u}^T\mathbf{A}\mathbf{u}$, with $\mathbf{A}$ a positive definite real symmetric N-dimensional matrix, and $\mathbf{u}$ a N-dimensional real vector.

$$\int d^N u\, \exp\left(-\frac{1}{2}\mathbf{u}^T\mathbf{A}\mathbf{u}\right) = (2\pi)^{N/2}(\det \mathbf{A})^{-1/2}, \tag{C.0.3}$$

which can be proved by noting that $\mathbf{A}$ can be diagonalized as it is symmetric, and that $\det \mathbf{A} = \prod_j a_j$ with a_j the eigenvalues of $\mathbf{A}$. The square root is well-defined provided that $\mathbf{A}$ is positive definite and then $a_j > 0$.

By implementing a clever change of variables, one can similarly show that (see Ref. [82] for reference),

$$\int d^N u\, \exp\left(-\frac{1}{2}\mathbf{u}^\dagger \cdot \mathbf{A}\mathbf{u} + \mathbf{a}^\dagger \cdot \mathbf{u}\right) = (2\pi)^{N/2}(\det \mathbf{A})^{-1/2} \exp\left(\frac{1}{2}\mathbf{a}^\dagger \cdot \mathbf{A}^{-1}\mathbf{a}\right). \tag{C.0.4}$$

Notice that these are multidimensional integrals. To get its functional counterpart consider $N \to \infty$ so that $\mathbf{u}$ becomes an infinite dimensional array in which we can store the information of a function, say $u(k, \omega_n)$. Thus $\int d^N u$ becomes a summation over all the possible

configurations of the function itself, that is, $\int d^N u \to \int \mathcal{D}\mathbf{u}$. One question arises straightaway: How is this step supposed to make sense if the factor $(2\pi)^{N/2}$ clearly diverges for $N \to \infty$, and so does the quantity $\det \mathbf{A}$, for the matrix has infinite dimension? Strictly speaking, the factor $\mathcal{N} \equiv \lim_{N\to\infty}(2\pi)^{N/2}(\det \mathbf{A})^{-1/2}$ diverges, however, it turns out that, very surprisingly, it shall cancel out in all the calculations of physical quantities. Notice that this is not the first time we face such a constant, see section B.1.

Acknowledging that quantum field theory is far from mathematical rigor but full of experimental validation, we stick with this $\mathcal{N}$ and write,

$$G \equiv \int \mathcal{D}\mathbf{u} \exp\left(-\frac{1}{2}\mathbf{u}^\dagger \cdot \mathbf{A}\mathbf{u} + \mathbf{a}^\dagger \cdot \mathbf{u}\right) = \mathcal{N} \exp\left(\frac{1}{2}\mathbf{a}^\dagger \cdot \mathbf{A}^{-1}\mathbf{a}\right). \tag{C.0.5}$$

In the last two expressions we have included some sort of scalar product $\cdot$ between vectors and dual vectors. This can be more general than just the usual Euclidean scalar product, it actually denotes any contraction between the vectors. In particular, we are interested in the following one

$$\mathbf{u}^\dagger \cdot \mathbf{A}\mathbf{u} \equiv \sum_{k,\omega_n} \mathbf{u}^\dagger(k,\omega_n)\mathbf{A}(k,\omega_n)\mathbf{u}(k,\omega_n), \tag{C.0.6}$$

where in the right-hand side matrix product is understood, and the functional dependence (k,ω_n) makes reference to the Fourier expansion described in section B.2.2.

The integral (C.0.5) is not only important on its own right, but it also serves as a "generator" of other useful integral identities. For instance, according to equation (B.2.8) we are interested in integrals of the form

$$\int \mathcal{D}\mathbf{u}\, u_j(q,\omega_m)u_i(k,\omega_n) \exp\left(-\frac{1}{2}\mathbf{u}^\dagger \cdot \mathbf{A}\mathbf{u}\right), \tag{C.0.7}$$

which can be derived from G in (C.0.5). In order to do so we need to introduce functional derivatives, we shall do it in a rather light fashion, we refer the reader to any textbook of calculus of variations to further reference, see e.g. [106–108].

C.1 Functional derivatives

Smooth functions defined on a manifold M are denoted as $C^\infty(M)$, and therefore a functional F is a map

$$\begin{aligned} F : C^\infty(M) &\mapsto \mathbb{R} \\ g &\mapsto F[g], \end{aligned} \tag{C.1.1}$$

that is, a map whose domain is the set of functions. It is common to refer to functionals as "functions of functions", which is a rather sloppy way to regard them, for a function of functions would look like $f(x(t))$ which is not the same as the functional $F[x(t)]$. The difference relies on the fact that the input for $f(x(t))$ is the number (or set of numbers) $x(t)$

for some value of t, while the input of $F[x(t)]$ is the function $x(t)$ itself. In other words, $F[x(t)]$ requires knowledge of the entire function $x(t)$, and not only its value at a given point. Hence, functionals are non-local objects.

Now we provide a somewhat formal definition of what a functional derivative is. Since functionals are non-local objects, it is customary to define them in terms of definite integrals as

$$F[x(t)] \equiv \int dt' \; g(x(t'), \dot{x}(t'), \ldots), \tag{C.1.2}$$

as is the case of the action which is defined as an integral of the Lagrangian.

Consider a 1-parameter family of variations of F given by replacing $x(t')$ by

$$x_n(\varepsilon, t') \equiv x(t') + \varepsilon h_n(t, t'), \tag{C.1.3}$$

with each h_n being a test function, i.e. it vanishes outside some compact region, and whose limit as a sequence is the Dirac delta function

$$\lim_{n\to\infty} h_n(t, t') = \delta(t - t'). \tag{C.1.4}$$

Then the functional derivative is defined by

$$\frac{\delta F[x(t)]}{\delta x(t)} \equiv \lim_{n\to\infty} \frac{d}{d\varepsilon} \; F[x(t') + \varepsilon h_n(t, t')]\Big|_{\varepsilon=0}. \tag{C.1.5}$$

Let us exemplify how this definition works. Consider the functional

$$G[f] \equiv \int dx \; (f'(x))^2, \tag{C.1.6}$$

so that

$$\begin{aligned}
\frac{\delta G[f]}{\delta f(y)} &= \lim_{n\to\infty} \frac{d}{d\varepsilon} \; G[f(x) + \varepsilon h_n(x, y)]\Big|_{\varepsilon=0} && \text{(C.1.7)}\\
&= \lim_{n\to\infty} \int dx \frac{d}{d\varepsilon} \; (f'(x) + \varepsilon h_n'(x, y))^2\Big|_{\varepsilon=0} && \text{(C.1.8)}\\
&= \lim_{n\to\infty} \int dx \; 2(f'(x) + \varepsilon h_n'(x, y)) h_n'(x, y)\Big|_{\varepsilon=0} && \text{(C.1.9)}\\
&= \lim_{n\to\infty} \int dx 2 f'(x) h_n'(x, y) && \text{(C.1.10)}\\
&= \lim_{n\to\infty} \int dx \left\{ 2\frac{d}{dx}[f(x)h_n(x, y)] - 2f''(x)h_n(x, y) \right\} && \text{(C.1.11)}\\
&= \lim_{n\to\infty} -2 \int dx f''(x) h_n(x, y) && \text{(C.1.12)}\\
&= -2f''(y), && \text{(C.1.13)}
\end{aligned}$$

where in (C.1.12) we used the fact that the $h_n(x,y)$'s vanish out of some compact region.

We look for shortcuts to compute these derivatives more efficiently. For our purposes, it suffices with the following two, which can be proved using the definition (C.1.5).

$$\frac{\delta x(t')}{\delta x(t)} \equiv \delta(t'-t) \tag{C.1.14}$$

$$\frac{\delta g(x(t))}{\delta x(t')} \equiv \frac{\partial g}{\partial x(t)}\frac{\delta x(t)}{\delta x(t')}. \tag{C.1.15}$$

The advantage of expressing functional derivatives in terms of partial derivatives is that we regain all the known properties of differentiation.

C.2 Application

Recalling (C.0.5) let us compute,

$$\frac{\delta}{\delta a_i^*(k,\omega_n)}\frac{\delta}{\delta a_j^*(-q,-\omega_m)} G\Big|_{\mathbf{a}=0}$$

$$= \frac{\delta}{\delta a_i^*(k,\omega_n)}\frac{\delta}{\delta a_j^*(-q,-\omega_m)} \int \mathcal{D}\mathbf{u}\ \exp\left(-\frac{1}{2}\mathbf{u}^\dagger\cdot\mathbf{A}\mathbf{u}+\mathbf{a}^\dagger\cdot\mathbf{u}\right)\Big|_{\mathbf{a}=0} \tag{C.2.1}$$

$$= \frac{\delta}{\delta a_i^*(k,\omega_n)} \int \mathcal{D}\mathbf{u}\ \exp\left(-\frac{1}{2}\mathbf{u}^\dagger\cdot\mathbf{A}\mathbf{u}+\mathbf{a}^\dagger\cdot\mathbf{u}\right)\frac{\delta}{\delta a_j^*(-q,-\omega_m)}\sum_{p,l,r} a_r^*(p,\omega_l)u_r(p,\omega_l)\Big|_{\mathbf{a}=0} \tag{C.2.2}$$

$$= \frac{\delta}{\delta a_i^*(k,\omega_n)} \int \mathcal{D}\mathbf{u}\ \exp\left(-\frac{1}{2}\mathbf{u}^\dagger\cdot\mathbf{A}\mathbf{u}+\mathbf{a}^\dagger\cdot\mathbf{u}\right)\sum_{p,l,r}\delta_{r,j}\delta_{p,-q}\delta_{l,-m}u_r(p,\omega_l)\Big|_{\mathbf{a}=0} \tag{C.2.3}$$

$$= \frac{\delta}{\delta a_i^*(k,\omega_n)} \int \mathcal{D}\mathbf{u}\ \exp\left(-\frac{1}{2}\mathbf{u}^\dagger\cdot\mathbf{A}\mathbf{u}+\mathbf{a}^\dagger\cdot\mathbf{u}\right)u_j(-q,-\omega_m)\Big|_{\mathbf{a}=0} \tag{C.2.4}$$

$$= \int \mathcal{D}\mathbf{u}\exp\left(-\frac{1}{2}\mathbf{u}^\dagger\cdot\mathbf{A}\mathbf{u}+\mathbf{a}^\dagger\cdot\mathbf{u}\right)u_j(-q,-\omega_m)\frac{\delta}{\delta a_i^*(k,\omega_n)}\sum_{p,l,r} a_r^*(p,\omega_l)u_r(p,\omega_l)\Big|_{\mathbf{a}=0} \tag{C.2.5}$$

$$= \int \mathcal{D}\mathbf{u}\exp\left(-\frac{1}{2}\mathbf{u}^\dagger\cdot\mathbf{A}\mathbf{u}+\mathbf{a}^\dagger\cdot\mathbf{u}\right)u_j(-q,-\omega_m)\sum_{p,l,r}\delta_{r,i}\delta_{p,k}\delta_{l,n}u_r(p,\omega_l)\Big|_{\mathbf{a}=0} \tag{C.2.6}$$

$$= \int \mathcal{D}\mathbf{u}\ u_j^*(q,\omega_m)u_i(k,\omega_n)\exp\left(-\frac{1}{2}\mathbf{u}^\dagger\cdot\mathbf{A}\mathbf{u}\right), \tag{C.2.7}$$

where in the last line we have used the fact that $\mathbf{u}$ is real and therefore its decomposition in the (k,ω_n)-space verifies $u_j^*(k,\omega_n) = u_j(-k,-\omega_n)$.

On the other hand, we can perform the functional integral of G first and then take the functional derivatives,

$$\frac{\delta}{\delta a_i^*(k,\omega_n)}\frac{\delta}{\delta a_j^*(-q,-\omega_m)} G\Big|_{\mathbf{a}=0}$$

$$= \frac{\delta}{\delta a_i^*(k,\omega_n)} \frac{\delta}{\delta a_j^*(-q,-\omega_m)} \mathcal{N} \exp\left(\frac{1}{2}\mathbf{a}^\dagger \cdot \mathbf{A}^{-1}\mathbf{a}\right)\Big|_{\mathbf{a}=0} \tag{C.2.8}$$

$$= \frac{\delta}{\delta a_i^*(k,\omega_n)} \mathcal{N} \exp\left(\frac{1}{2}\mathbf{a}^\dagger \cdot \mathbf{A}^{-1}\mathbf{a}\right) \frac{\delta}{\delta a_j^*(-q,-\omega_m)} \frac{1}{2} \sum_{p,l,r,s} a_r^*(p,\omega_l) A_{rs}^{-1}(p,\omega_l) a_s(p,\omega_l)\Big|_{\mathbf{a}=0} \tag{C.2.9}$$

$$= \frac{\delta}{\delta a_i^*(k,\omega_n)} \mathcal{N} \exp\left(\frac{1}{2}\mathbf{a}^\dagger \cdot \mathbf{A}^{-1}\mathbf{a}\right) \frac{1}{2} \sum_{p,l,r,s} \delta_{r,j}\delta_{p,-q}\delta_{l,-m} A_{rs}^{-1}(p,\omega_l) a_s(p,\omega_l\Big|_{\mathbf{a}=0}$$
$$+ \frac{\delta}{\delta a_i^*(k,\omega_n)} \mathcal{N} \exp\left(\frac{1}{2}\mathbf{a}^\dagger \cdot \mathbf{A}^{-1}\mathbf{a}\right) \frac{1}{2} \sum_{p,l,r,s} a_r^*(p,\omega_l) A_{rs}^{-1}(p,\omega_l) \delta_{j,s}\delta_{p,q}\delta_{l,m}\Big|_{\mathbf{a}=0} \tag{C.2.10}$$

$$= \frac{\delta}{\delta a_i^*(k,\omega_n)} \mathcal{N} \exp\left(\frac{1}{2}\mathbf{a}^\dagger \cdot \mathbf{A}^{-1}\mathbf{a}\right) \frac{1}{2} \sum_{s} A_{js}^{-1}(-q,-\omega_m) a_s(-q,-\omega_m)\Big|_{\mathbf{a}=0}$$
$$+ \frac{\delta}{\delta a_i^*(k,\omega_n)} \mathcal{N} \exp\left(\frac{1}{2}\mathbf{a}^\dagger \cdot \mathbf{A}^{-1}\mathbf{a}\right) \frac{1}{2} \sum_{r} a_r^*(q,\omega_m) A_{rj}^{1}(q,\omega_m)\Big|_{\mathbf{a}=0} \tag{C.2.11}$$

$$= \mathcal{N} \exp\left(\frac{1}{2}\mathbf{a}^\dagger \cdot \mathbf{A}^{-1}\mathbf{a}\right) \frac{\delta}{\delta a_i^*(k,\omega_n)} \sum_{s} \frac{1}{2} A_{js}^{-1}(-q,-\omega_m) as(-q,-\omega_m)\Big|_{\mathbf{a}=0}$$
$$+ \mathcal{N} \exp\left(\frac{1}{2}\mathbf{a}^\dagger \cdot \mathbf{A}^{-1}\mathbf{a}\right) \frac{\delta}{\delta a_i^*(k,\omega_n)} \sum_{r} \frac{1}{2} a_r^*(q,\omega_m) A_{rj}^{-1}(q,\omega_m)\Big|_{\mathbf{a}=0} \tag{C.2.12}$$

$$= \frac{\mathcal{N}}{2} A_{ji}^{-1}(-q,-\omega_m)\delta_{q,k}\delta_{m,n} + \frac{\mathcal{N}}{2} A_{ij}^{-1}(q,\omega_m)\delta_{q,k}\delta_{m,n} \tag{C.2.13}$$

$$= \frac{\mathcal{N}}{2}\delta_{q,k}\delta_{m,n}\big(A_{ij}^{-1}(-q,-\omega_m) + A_{ij}^{-1}(q,\omega_m)\big). \tag{C.2.14}$$

Thus, equations (C.2.7) and (C.2.14) show that,

$$\int \mathcal{D}\mathbf{u}\; u_j^*(q,\omega_m) u_i(k,\omega_n) \exp\left(-\frac{1}{2}\mathbf{u}^\dagger \cdot \mathbf{A}\mathbf{u}\right) = \frac{\mathcal{N}}{2}\delta_{q,k}\delta_{m,n}\big(A_{ij}^{-1}(-q,-\omega_m) + A_{ij}^{-1}(q,\omega_m)\big) \tag{C.2.15}$$

$$\int \mathcal{D}\mathbf{u}\; \exp\left(-\frac{1}{2}\mathbf{u}^\dagger \cdot \mathbf{A}\mathbf{u}\right) = \mathcal{N}. \tag{C.2.16}$$

These two results are all we need to compute the correlation functions as established in appendix B.

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