

A Monte-Carlo approach to the effect of noise on local stability in polynomial difference equations

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Abstract

We present an analysis of the stability behaviour of a class of one-step difference equations describing an iterated polynomial mapping. Such equations are commonly used to model population dynamics in discrete time. We use Monte-Carlo methods to investigate the effect of a state-dependent random perturbation on the local stability of such equations. In particular we focus on the probability of stability in transitional initial-value regions; regions where a switch in the qualitative behaviour of the deterministic equation is observed.

Keywords: Local stability; Discrete stochastic process; Difference equation; Monte-Carlo simulation

MSC(2000): 37H10, 39A11, 65C05, 65Q05, 65C20

1 Introduction

We seek to describe the effect of a state dependent stochastic perturbation on the asymptotic stability of solutions of difference equations of the form

$$x_{n+1} = x_n(1 - \mu x_n^{2\nu}), \quad n \in \mathbb{N}, \quad x_0 \in \mathbb{R}, \quad (1)$$

where $\mu \in \mathbb{R}$ and $\nu \in \mathbb{N}$. (1) is a generalisation of the logistic equation, widely used in population modelling (see, for example Chapter 1 in Elaydi [3]). The dynamics of the solutions of (1) are well known, and are summarized in Theorem 2.2 of Section 2.

Recall that solutions of (1) are asymptotically stable in some basin of attraction $\mathcal{S} \subseteq \mathbb{R}$ if, for every $x_0 \in \mathcal{S}$, solutions of (1) satisfy

$$\lim_{n \rightarrow \infty} x_n = 0.$$

Our approach is to use Monte-Carlo methods to examine the probability of asymptotic stability of solutions of the stochastically perturbed equation

$$X_{n+1} = X_n(1 - \mu X_n^{2\nu} + \sigma \xi_{n+1}), \quad n \in \mathbb{N}, \quad X_0 \in \mathbb{R}, \quad (2)$$

where $\{\xi_{n+1}\}_{n \geq 0}$ is a sequence of independent, identically distributed random variables with zero mean and unit variance, and continuous symmetric distribution supported on $\mathbb{R}$. Since the dynamics of solutions of (1) are similar regardless of

the choice of $\nu \in \mathbb{N}$, the dynamics of (2) with $\nu = 1$ can reasonably be taken as a guide to the general dynamics.

The local asymptotic stability of stochastically perturbed equations of the form (2), with *bounded* perturbation, has been considered elsewhere. For example see the paper by Fraser et al [4], which describes the effect of a bounded, state independent, stochastic perturbation on an equation that includes (1) as a special case. More recently, local stability of general nonlinear equations under vanishing, state-independent, and *unbounded* perturbation has been considered in [1]. However, the effect of an *unbounded, state-dependent* perturbation of the form seen in (2) has not been addressed, to the best of our knowledge. This is a significant gap in the literature, since unbounded Gaussian perturbations arise naturally from an Euler-Maruyama discretisation of a stochastic differential equation of Itô type, and state-dependent stochastic perturbations are a natural choice for describing intra-system effects like, for example, birth-rate fluctuations in population modeling.

The Monte-Carlo approach requires that we design a numerical test to determine whether or not an individual path of a solution is asymptotically stable. This test must ensure that paths are correctly identified as stable or unstable. Fortunately, the special structure of the mapping defined in (1) allows us to develop a test that is robust in this sense.

The results of the simulation allow us to observe the transition from asymptotic stability to instability of solutions of (2) with $\nu = 1$ as the initial value changes. Significantly, the results of the simulation highlight the role that σ , the intensity of the noise perturbation, plays in this transition.

We anticipate that these numerical results will guide us in the search for a theoretical description of the dynamics of (2).

In Section 2, we describe the dynamics of solutions of the deterministic equation (1). In Section 3 we design a numerical test for the asymptotic stability of individual paths of the same special case of (2), and apply it to determine the behaviour of the probability of asymptotic stability across a range of values of X_0 . We also use this test to illustrate the role of σ in the probability of stability. In Section 4 we present conclusions.

2 Theoretical Background

2.1 Mathematical preliminaries

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n \in \mathbb{N}}, \mathbb{P})$ be a complete probability space with filtration $\{\mathcal{F}_n\}_{n \in \mathbb{N}}$. Let $\{\xi_n\}_{n \in \mathbb{N}}$ be a sequence of independent random variables defined on the probability space satisfying $\mathbb{E}\xi_n = 0$, $\mathbb{E}\xi_n^2 = 1$. Since we will consider equations with deterministic initial values we set $\mathcal{F}_n = \sigma\{\xi_i : 1 \leq i \leq n\}$, for $n \in \mathbb{N}$.

We must generalise the notion of asymptotic stability so that it applies to stochastic processes.

Definition 2.1. *Solutions of (2) are asymptotically stable with probability p in some region $\mathcal{R} \subseteq \mathbb{R}$ if, for all $X_0 \in \mathcal{R}$,*

$$\mathbb{P} \left[\lim_{n \rightarrow \infty} X_n = 0 \right] = p.$$

2.2 Theoretical analysis of the deterministic equation

We give a full description of the dynamics of the deterministic equation (1).

Theorem 2.2. *Let the sequence $\{x_n\}_{n \geq 0}$ be a solution of (1). Then we can classify the stability behaviour of $\{x_n\}$ according to the initial value x_0 as follows:*

1. *If $|x_0| < \sqrt[2\nu]{\frac{2}{\mu}}$ then $\lim_{n \rightarrow \infty} x_n = 0$. Furthermore,*
 - (a) *If $|x_0| < \sqrt[2\nu]{\frac{1}{\mu}}$ then the convergence is monotone.*
 - (b) *If $|x_0| = \sqrt[2\nu]{\frac{1}{\mu}}$ then $x_n = 0$ for every $n \geq 1$.*
 - (c) *If $\sqrt[2\nu]{\frac{1}{\mu}} < |x_0| < \sqrt[2\nu]{\frac{2}{\mu}}$ then the solution changes sign finitely many times, before becoming monotone.*
2. *The set $\left\{ \pm \sqrt[2\nu]{\frac{2}{\mu}} \right\}$ is a 2-cycle of (1).*
3. *If $|x_0| > \sqrt[2\nu]{\frac{2}{\mu}}$ then $\lim_{n \rightarrow \infty} |x_n| = \infty$, and $x_{n+1}x_n < 0$ for each $n \geq 0$.*

Proof. Part (i) is a special case of Theorem 1 in Appleby, Guzowska & Rodkina [2]. Parts (ii) and (iii) are proved by applying Theorems 1.12 and 7.9 in Elaydi [3]. $\square$

We seek to develop the fullest possible description of the dynamics of (2).

3 Numerical Analysis of the Solutions of (2)

3.1 Setting up a test equation

In this section we address the main theme of the paper. We use numerical simulation to explore the dynamics of the solutions of (2) with $\nu = 1$. For various values of σ , we use

$$X_{n+1} = X_n (1 - 8X_n^2 + \sigma\xi_{n+1}), \quad n \in \mathbb{N}, \quad X_0 \in \mathbb{R}, \quad (3)$$

as a test equation, where each ξ_n is an i.i.d standard Normal random variable. Thus we see that, to three decimal places, $\sqrt{1/\mu} = 0.354$, and $\sqrt{2/\mu} = 0.5$.

In Figure 1, we show four path simulations of (3) with $\sigma = 0.1$, each with a different initial value, chosen to illustrate the dynamics described in the statement of Theorem 2.2.

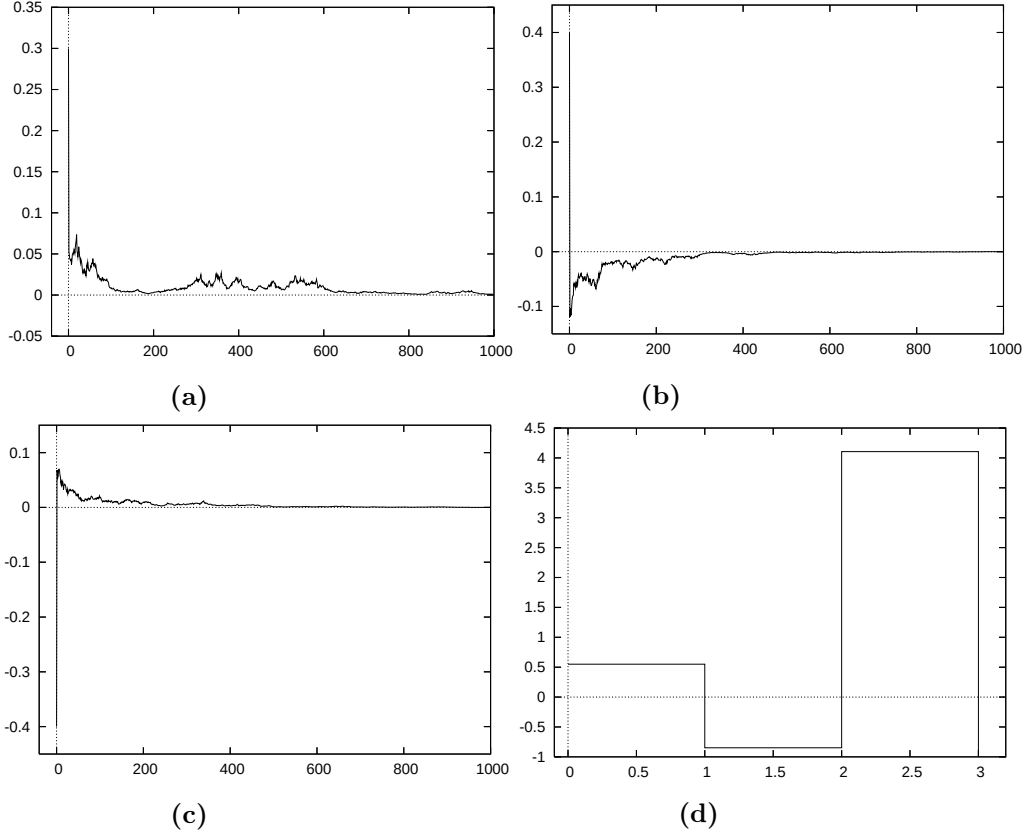


Figure 1: Four simulated paths of (3) with $\sigma = 0.1$. In (a), $X_0 = 0.3$. In (b), $X_0 = 0.4$. In (c), $X_0 = -0.4$. In (d), $X_0 = 0.55$, and the beginning of an unbounded oscillation can be seen. Indeed from inspection of the simulation data we see that $X_3 = -549.2372$ (not shown due to scale), and the path exceeds the overflow bound of a 64-bit floating point number in 8 timesteps.

3.2 A numerical test for stability

Since we are exclusively concerned with the phenomenon of asymptotic stability, we must develop a numerical test for the stability of individual paths of solutions of (3). For a particular initial value X_0 , the basic design of the test is as follows:

1. Initialise a counter `numStabPaths` to zero;
2. Simulate a path of length $N = 1000$;
3. If $|X_{999}| < \delta$, for some $\delta > 0$ then increment `numStabPaths`. Otherwise, do nothing;
4. Repeat 100 times with independent sets of random numbers;

5. The quantity `numStabPaths/100` represents the *Monte-Carlo approximation of the probability of asymptotic stability*.

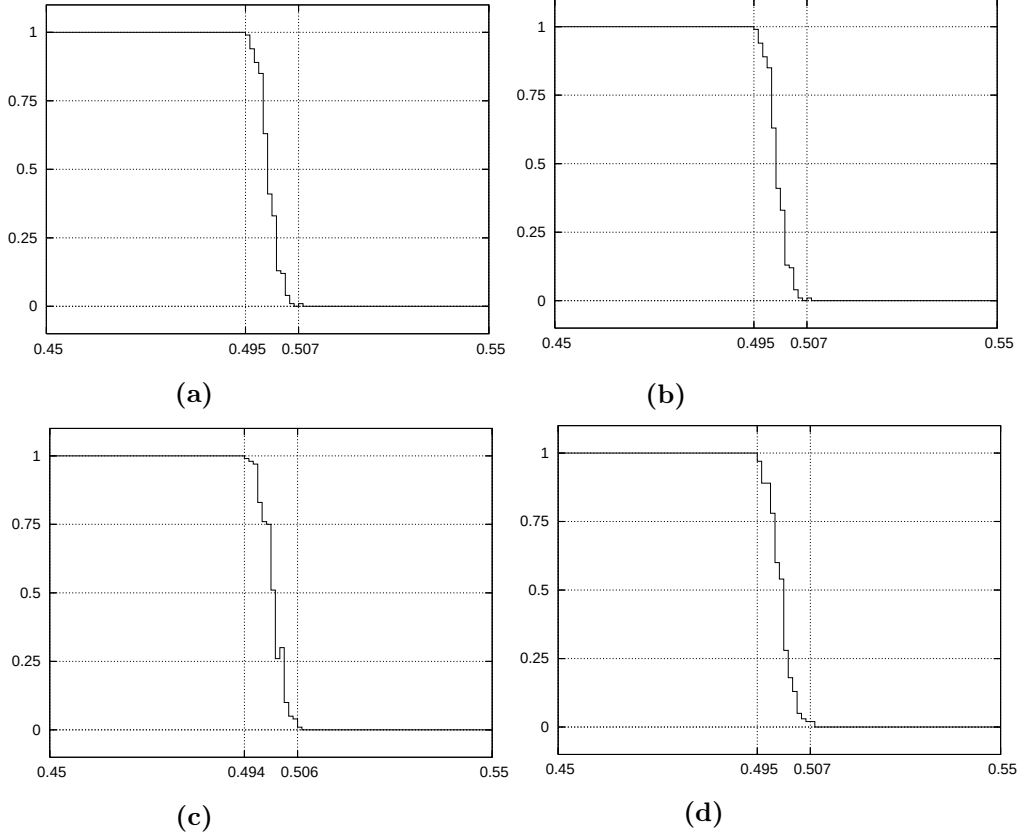


Figure 2: Monte-Carlo approximations for four values of δ of the probability of stability of solutions of (3) with $\sigma = 0.1$, across a set of initial values, uniformly spaced 0.001 apart on the real line. In (a), $\delta = 0.01$. In (b), $\delta = 0.1$. In (c), $\delta = 0.6$. In (d), $\delta = 1.79769313486231570 \times 10^{308}$, the overflow bound of a 64-bit IEEE-754 floating point number.

By applying this procedure across a range of initial values, we can build up a graph of probabilities which will illustrate the nature of the transition from stability instability in solutions of (3). Since the mapping in (1) has point symmetry through the origin, we can restrict our examination to positive initial values without loss of generality. However, we must first determine the effect of the choice of δ on the Monte-Carlo approximation of the probability of asymptotic stability.

3.3 Sensitivity of the stability test to δ

First, we observe that δ must be sufficiently large that the magnitude of a stable path may reasonably be expected drop below it in 1000 timesteps. To ensure that

stable paths are detected using this method, it is desirable to have δ as large as possible. However, this raises the possibility that, if the value of δ is too large, unstable paths will be counted as stable. We examine this possibility in this subsection, and show that the stability test is in fact robust to increasing values of δ .

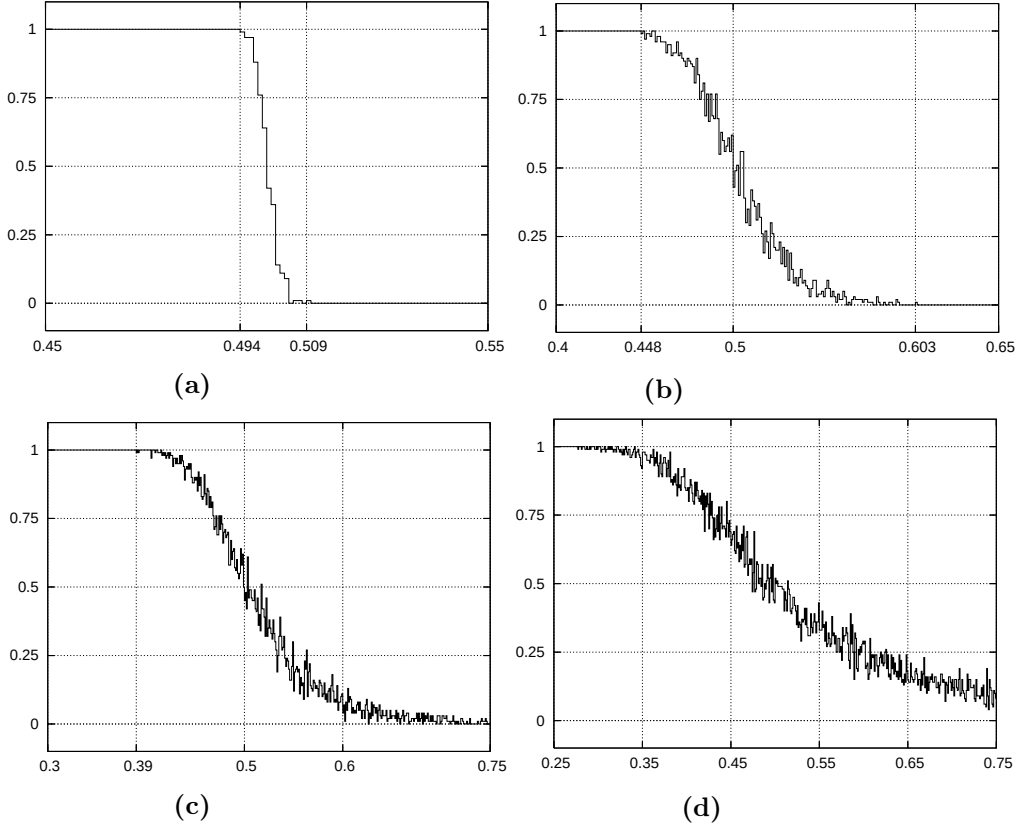


Figure 3: Monte-Carlo approximations of the probability of stability of solutions of (3), across a set of initial values, uniformly spaced 0.001 apart on the real line. In (a), $\sigma = 0.1$. In (b), $\sigma = 1$. In (c), $\sigma = 2$. In (d), $\sigma = 5$. Note the increasing scale on the horizontal axis of successive graphs.

We apply the stability test described in Subsection 3.2 to solutions of (3) with $\sigma = 0.1$, for increasing values of δ . The results are shown in Figure 2.

We see that the shape of the resulting graph of the Monte-Carlo approximation of the probability of asymptotic stability against initial value is unaffected by the size of δ . In fact we can increase the value of δ up to the overflow bound of our floating point representation without affecting the overall shape.

In particular, we observe that the intervals over which the transition from a.s. asymptotic stability to instability takes place in each Monte Carlo simulation are insensitive to the choice of δ . For Parts (a)–(d), the intervals are $[0.494, 0.507]$,

$[0.495, 0.507]$, $[0.494, 0.506]$, $[0.495, 0.507]$ respectively.

A possible cause of the insensitivity of the test to the choice of δ may be seen in Figure 1, Part (d); when a path moves into the instability region of the equation, it grows extremely quickly without bound, exceeding the overflow limit within a few timesteps. Therefore, unstable paths are unlikely to be mistaken for stable paths.

For the remainder of the paper, we set $\delta = 1.79769313486231570 \times 10^{308}$, which represents the largest rational number that can be represented under the 64 bit IEEE-754 floating point standard.

3.4 A numerical stability analysis.

Next, we observe the effect of varying the value of the intensity of the noise, σ , on the probability curve produced using the technique developed in the previous subsection.

From Figure 3, we see that the Monte-Carlo approximation of the probability of asymptotic stability transitions smoothly from one to zero over a finite interval of non-zero length. In (a) and (b) this interval is marked on the horizontal axis of the graph. As σ increases, the length of this interval increases.

3.5 Implementation of the simulation

All simulations in this chapter are direct machine implementations of (3) in the Java programming language, where real numbers are rationally approximated with 64-bit floating-point numbers satisfying the IEEE-754 standard. The sequence of Gaussian random numbers in (2) has been replaced in (3) with a sequence of pseudo-random Gaussian numbers generated using the `nextGaussian()` method of the `java.util.Random()` class. This method implements the polar form of the Box-Muller-Marsaglia transformation to generate independent pairs of Normally Distributed pseudo-random numbers from pseudo-random numbers uniformly distributed over the interval $[0, 1]$. A full description can be found, for example, in Section 3.4.1, Subsection C of Knuth [5].

4 Conclusions and Future Work

Using Monte-Carlo methods, it is possible to develop an indicative description of the stability behaviour of solutions of a special case of (2). This description indicates that the intensity of the noise perturbation plays a significant role in the transition from stable to unstable behaviour.

The special structure of the unperturbed equation (1) allows us to develop a robust numerical test for pathwise stability in (2), and apply it to develop a Monte-Carlo approximation of the probability of asymptotic stability for any given initial value. By examining these probabilities across a range of initial values we see that the probability of stability appears to transition smoothly from

approximately one to approximately zero over a finite interval centred on $\sqrt{2/\mu}$, the point of transition from stability to instability in (1). Increasing the stochastic perturbation increases the length of the interval over which the transition occurs.

We anticipate that the numerical analysis in this paper will guide the further theoretical analysis of the solutions of the stochastic equation.

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