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**On the tensorial structure of the Generalized Eigenstate
Thermalization Hypothesis**

**Sobre la estructura tensorial de la Hipótesis de
Termalización de Autoestados Generalizada**

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Contents

1	Introduction	1
1.1	Quantum thermalization	2
1.2	The eigenstate thermalization hypothesis	3
1.2.1	Thermalization and chaos in the complex SYK model	4
1.2.2	Thermalization in a nonrandom Hamiltonian of hard-core bosons	9
1.3	ETH and free probability theory	12
1.3.1	Free probability and freeness	12
1.3.2	The generalized eigenstate thermalization hypothesis	13
1.3.3	Subleading contributions	15
2	Subleading contributions in a system of random free fermions	17
2.1	Random free fermions	17
2.2	Gaussian orthogonal ensemble (GOE)	20
2.2.1	Correlators in eigenvectors	20
2.2.2	Thermal correlators	24
2.2.3	Energy average of the 2-point correlator for GOE	28
2.3	Gaussian unitary ensemble	35
2.3.1	Correlators in eigenvectors	35
2.3.2	Thermal correlators	36
2.3.3	Energy average of the 2-point correlator for GUE	36
2.4	Analytical results	37
2.5	Numerical solution	41
2.6	Conclusions and perspectives	43
	Appendices	44
A	Gaussian ensembles	44
A.1	Level density	45
A.2	Level spacing distributions	46
B	N -point correlation functions	46
C	Unfolding of the spectra	47
D	The Haar measure	48
E	Quantum channels	52
E.1	Completely positive trace preserving operators	52
E.2	Integration over the unitary group $U(d)$	53
E.3	Integration over the orthogonal group $O(d)$	54
F	Graph interpretation of free probability	57
	Bibliography	59

Resumen

En este trabajo, nos enfocamos en la Hipótesis de Termalización de Autoestados (ETH por sus siglas en inglés) [1] como una explicación clave de la aparición del comportamiento térmico en sistemas cuánticos de muchos cuerpos. Discutimos avances recientes en el campo, en particular el trabajo de Silvia, Laura y Jorge [2,3], quienes extendieron el ETH para incluir funciones de correlación térmicas de muchos puntos. Sus hallazgos revelaron una conexión directa entre estas correlaciones y los cumulantes de la probabilidad libre [4]. Esta conexión, junto con la condición de independencia libre, sugiere la existencia de contribuciones subdominantes que no están incluidas en la generalización del ETH ni en la probabilidad libre.

Para respaldar esta idea, calculamos analíticamente las contribuciones subdominantes para las funciones de correlación de dos puntos en un sistema de fermiones libres aleatorios; para ello, utilizamos herramientas de información cuántica y teoría de matrices aleatorias, incluyendo canales cuánticos, ensambles Gaussianos y promedios de Haar. Nuestros resultados confirmaron la hipótesis inicial: las contribuciones subdominantes contienen información relevante sobre la presencia o ausencia de simetría de inversión temporal en el sistema considerado.

Palabras clave: Hipótesis de termalización de autoestados, probabilidad libre, matriz aleatoria.

Abstract

In this work, we focus on the Eigenstate Thermalization Hypothesis (ETH) [1] as a leading explanation for the emergence of thermal behavior in many-body quantum systems. We discuss recent advances in the field, particularly the work of Silvia, Laura, and Jorge [2,3], who extended the ETH to include multipoint thermal correlation functions. Their findings revealed a direct connection between these correlations and the cumulants of free probability [4]. This connection, along with the free independence condition, suggests the existence of subleading contributions not captured by the generalized ETH and free probability.

To support this idea, we analytically computed the subleading contributions for the two-point correlation functions in a system of random free fermions; for that, we used quantum information and random matrix theory techniques that include quantum channels, Gaussian ensembles and Haar averages. Our results confirmed our initial hypothesis: the subleading contributions contain relevant information about the presence or absence of time-reversal symmetry in the system considered.

Keywords: Eigenstate thermalization hypothesis, free probability, random matrix.

Chapter 1

Introduction

Understanding the emergence of thermal behavior in closed many-body quantum systems has been a longstanding problem in physics [5, 6]. The early work of Boltzmann and Poincaré highlighted the apparent discrepancy between the reversible microscopic laws of physics and the irreversible behavior observed in macroscopic systems composed of a large number of particles. In an attempt to reconcile these descriptions, Boltzmann formulated the ergodic hypothesis, proposing that the emergence of thermal behavior could be explained by the random exploration of phase space by the system over time.

The ergodic hypothesis suggests that a system, given sufficient time, will explore all accessible states and exhibit behavior consistent with equilibrium thermodynamics. This hypothesis has played a crucial role in connecting microscopic dynamics with macroscopic thermodynamic properties but is not valid in all cases, especially in systems with complex dynamics or strong correlations. Although the applicability and limitations of the ergodic hypothesis in the quantum regime have been the subject of extensive research and debate [7, 8], a full understanding of the topic is still missing.

Our focus is on exploring the fundamentals of thermalization in closed many-body quantum systems, with a specific emphasis on the Eigenstate Thermalization Hypothesis (ETH). The ETH proposes that the individual eigenstates of a quantum system can exhibit properties similar to those observed in thermal equilibrium, despite the absence of an external heat bath [9, 10].

By investigating ETH and the mechanisms driving thermalization, our goal is to gain insights into how closed quantum systems transition from non-thermal to thermal states. We aim to understand the equilibration of observables and the emergence of statistical properties consistent with thermodynamics.

1.1 Quantum thermalization

When measuring macroscopic observables in isolated many-body quantum systems, empirical and numerical evidence shows that, under its own dynamics¹ after waiting for a sufficiently long period of time, the values of macroscopic observables tend to stabilize around equilibrium values predicted by the laws of thermodynamics, this process is called thermalization. It is reasonable that this happens only for local observables, since an isolated many-body quantum system initially in a pure state will remain pure after Schrödinger evolution, so the maximization of entropy in the thermal state could not occur globally as it does in classical statistical mechanics. The mechanism of this convergence is of great interest in the physics community and has been a topic of debate since the foundation of quantum mechanics [11].

In the fundamental formulation of classical statistical mechanics, an isolated system is in equilibrium if and only if all microstates subject to the macroscopic constraints are equally probable. If the system in question visits every region of available phase space and if, in the long time, the time spent in each region is proportional to its volume, then the temporal average of macroscopic observables can be substituted with ensemble averages. These ensembles are composed of all possible configurations compatible with the macroscopic constraints and assign the same probability to each of them, as required by the equilibrium condition. This ensemble is known as the Microcanonical Ensemble, and it is precisely the principle of equal probability of this ensemble that allows us to capture the long-term average of the observables.

Ergodic systems are closely related to chaotic systems. Classically, one may characterize a chaotic system by its sensitivity to initial conditions, which produces and exponentially fast separation of trajectories in phase space characterized by the Lyapunov exponent, so it is natural that all classical chaotic systems exhibit ergodicity and therefore thermalization.

Although this seems like a promising description, and in many simple cases it is, the ergodic hypothesis presents several weak points. To mention some of them, the time required for a system to visit all phase space grows exponentially with the number of degrees of freedom, making it inconsistent with the observation windows we consider (and impractical for analytical calculation). Furthermore, this description is weak in the sense that it captures the long-time average of observables, and not the value of observables at long times. We want to understand thermalization in the strong sense, in which the instantaneous values of observables approach the equilibrium value predicted by the microcanonical ensemble and remain close to it at almost all subsequent times [10]. Other alternatives to the ergodic hypothesis include typicality [12, 13], which is based on the assumption that from the point of view of macroscopic observables, almost all configurations are equivalent. In this description, thermalization occurs when the system reaches a typical configuration. Since this happens faster than the exploration of phase space, it solves the problem of the long window of time and also justifies the irreversibility of thermodynamics, since it has been shown that under this description the difficulty of passing from a typical to an atypical configuration increases with system size [10]. Some downsides are that the creation of typical configurations presents practical difficulties and it is limited to the measurement of macroscopic quantities.

The reconstruction of the Gibbs description in quantum mechanics also faces several problems. If one aims to apply the ergodic hypothesis, a definition of a phase space that is consistent with

¹That is, without the presence of a heat reservoir.

the Heisenberg uncertainty principle is required. Furthermore, since chaos implies ergodicity in the classical case, if one aims to define ergodicity in terms of chaos other difficulties appear, for instance, since Schrödinger's equation is linear, the key ingredient leading to chaos in classical systems, which is the nonlinearity in the equations of motion, is absent in quantum mechanics. For all these reasons, it seems that the process of thermalization and chaos in isolated many-body quantum systems should be described by new features. A description that has proven to work is the Eigenstate Thermalization Hypothesis (ETH), a mathematical ansatz that emerges under the umbrella of different but related areas, including random matrix theory (RMT), quantum information and high-energy physics. To proceed, let us define what we mean by thermalization in quantum mechanics.

Definition 1 (Quantum thermalization). *For a quantum system, initially prepared in a state $|\psi\rangle$ and subject to a temporal evolution determined by a Hamiltonian $\hat{H}$ with eigenstates $|m\rangle$, the expected value of an arbitrary observable $\hat{O}$ on the initial state is given by*

$$O(t) = \langle \psi(t) | \hat{O} | \psi(t) \rangle = \sum_m |c_m|^2 O_{mm} + \sum_{m,n \neq m} c_m^* c_n e^{i(E_m - E_n)t} O_{mn}. \quad (1.1)$$

where $O_{mn} = \langle m | \hat{O} | n \rangle$ are the matrix elements of the observable on the basis of eigenstates, and $c_m = \langle m | \psi \rangle$ is the transition amplitude from the initial state to the m th-eigenstate. In analogy with the classical case, we say that the observable $\hat{O}$ thermalizes if [14]:

1) The temporal average $\overline{O}$ of the expected value coincides with the statistical average $\langle O \rangle_{\text{Micro}}$ predicted by the microcanonical ensemble, i.e.

$$\overline{O} = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T O(t) dt \approx \langle O \rangle_{\text{Micro}}. \quad (1.2)$$

2) The temporal fluctuations around the microcanonical average $\overline{O^2} - \overline{O}^2$ decay with the inverse of the Hilbert space dimension.

For many-body quantum systems, the temporal average of the second term in (1.1) is typically zero [14], which implies that for the observable to thermalize, the sum of the diagonal elements must coincide with the microcanonical average, i.e.

$$\overline{O} = \sum_m |c_m|^2 \overline{O_{mm}} \approx \langle O \rangle_{\text{Micro}}. \quad (1.3)$$

The above condition is immediately satisfied for chaotic many-body quantum systems in which the system's Hamiltonian can be modeled using a random matrix ensemble [15]. This connection between quantum chaotic systems and the behavior predicted by statistical mechanics, mediated by RMT [16–18], is interpreted as the quantum analogue of the relationship between classical chaotic systems and classical statistical mechanics mediated by the ergodic hypothesis.

1.2 The eigenstate thermalization hypothesis

In more general quantum systems, apart from those described by random matrix ensembles, the expected values of observables depend on the system's temperature, and the relaxation

times depend on the specific observable. Therefore, there is additional information in the matrix entries Equation (1.1) that cannot be described by random matrix ensembles. In a series of works in 1999 [19–21], Mark Srednicki, using semiclassical arguments, generalized the prediction of RMT to describe these physical observables. This generalization is known as the Eigenstate Thermalization Hypothesis (ETH). The ETH is an ansatz on the matrix elements of observables in the eigenstate basis of the Hamiltonian [8]:

$$\langle i | \hat{O} | j \rangle = O_{ij} = \mathcal{O}(E_{ii}^+) \delta_{ij} + F_{E_{ij}^+}(\omega_{ij}) e^{-\frac{S(E_{ij}^+)}{2}} R_{ij}, \quad (1.4)$$

where $E_{ij}^+ = \frac{E_i + E_j}{2}$, $\omega_{ij} = E_i - E_j$ denote the average and energy difference between the eigenstates $|i\rangle$ and $|j\rangle$. $S(E)$ is the thermodynamic entropy (Boltzmann entropy) evaluated at energy E , $\mathcal{O}(E_{ii}^+)$ is equal to the expected value of the microcanonical ensemble at energy E_i , $F_{E_{ij}^+}(\omega_{ij})$ is a smooth function of the energy difference, and R_{ij} is a random variable with zero mean and unit variance. It is expected that all local observables for which statistical mechanics applies will satisfy this hypothesis. In the next sections, we present examples of different physical systems that exhibit thermalization and chaotic behavior.

1.2.1 Thermalization and chaos in the complex SYK model

Due to holographic duality [22], understanding the mechanism of thermalization in unitary quantum field theories enhances the description of black hole formation and thermalization, since black holes are described by thermal states. The study of thermalization and chaos in the framework of this duality has been of increasing interest in recent years [23], and a discussion of this duality is given in [24].

One of the simplest models for studying thermalization in the context of holography is the Sachdev-Ye-Kitaev (SYK) model [23, 25], a disordered quantum system with all-to-all couplings. Consider the version of the SYK model given by the following Hamiltonian:

$$\hat{H} = \sum_{i,j,k,l}^N J_{ij;kl} \hat{c}_i^\dagger \hat{c}_j^\dagger \hat{c}_k \hat{c}_l, \quad (1.5)$$

where $J_{ij;kl}$ are complex coupling constants, and we define the creation and annihilation operators, satisfying the anti-commutator relations;

$$\{\hat{c}_i, \hat{c}_j^\dagger\} = \delta_{ij} \quad \{\hat{c}_i, \hat{c}_j\} = \{\hat{c}_i^\dagger, \hat{c}_j^\dagger\} = 0. \quad (1.6)$$

The second equality in Equation (1.6) implies that $J_{ij;kl} = -J_{ji;kl}$ and $J_{ij;kl} = -J_{ij;lk}$, and the hermiticity of the Hamiltonian implies that $J_{ij;kl} = J_{kl;ij}^*$. The remaining independent couplings are taken from a Gaussian distribution with mean zero and variance given by:

$$\overline{|J_{ij;kl}|^2} = \frac{3!J^2}{N^3},$$

where J is a scaling constant.

Given the simplicity of this model, it is possible to solve it using the technique of exact diagonalization. For this purpose, we used the Python QuSpin library.

Thermalization in the SYK model

According to Equation (1.4), the diagonal elements of the expectation value matrix of macroscopic observables are smooth functions of the average energy, while the off-diagonal values are suppressed by the exponential factor $e^{-\frac{S(\bar{E})}{2}}$. To illustrate this, we consider the local operator $\hat{n}_s = c_s^\dagger c_s$, which is the s th term in the total number operator,

$$\hat{n}_F = \sum_i^N c_i^\dagger c_i. \quad (1.7)$$

This operator commutes with the Hamiltonian, allowing us to work in sectors with a fixed number of particles. This system can be thought of as a system with N sites, where a fraction $\nu = n_F/N$ is occupied by fermions. We will work with the system at half-filling, which means that $\nu = 1/2$. It is important to emphasize that interpreting this in terms of sites on a lattice is not physically accurate, since the system does not have a defined topology according to the Hamiltonian (1.5).

The behavior of the expectation value matrix is shown in Figure 1.1. It is observed that the off-diagonal part is suppressed, while the diagonal part takes non-zero values. This is evident even for relatively small values of N and continues to hold for large N as expected.

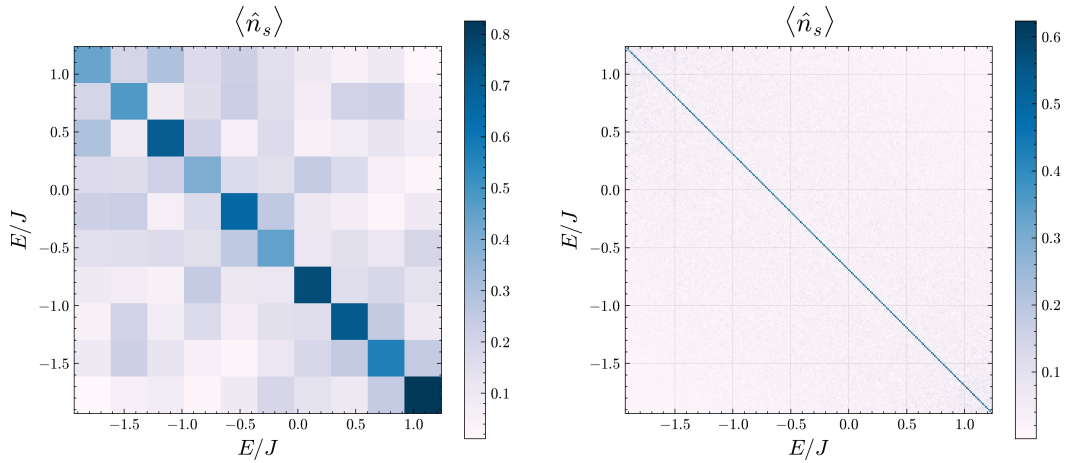


Figure 1.1: The absolute value of the expected values of the number operator at site s , $|\langle n | \hat{n}_s | n \rangle|$, as a function of energy, E/J , with $J = 1$ for a single realization. **Left:** With $N = 5$. **Right:** With $N = 10$. Reproduced from [26].

As N increases, the diagonal terms are expected to converge to a smooth function of the average energy $\bar{E}$. To demonstrate this, we consider the diagonal terms of the expected value matrix of $\hat{n}_s$ as a function of E/J , for a single realization, and we also consider the average over 1000 realizations. The results are shown in Figures 1.2a and 1.2b. Since in both scenarios (for 1 realization and for the average over many realizations), we observe the convergence to a smooth function as N increases, we can affirm that the diagonal part of the ETH for this model is self-averaging.

Similarly, according to the ETH, Equation (1.4), the off-diagonal part should converge to a smooth function of the energy difference and its distribution should be normal. This is evidenced in Figures

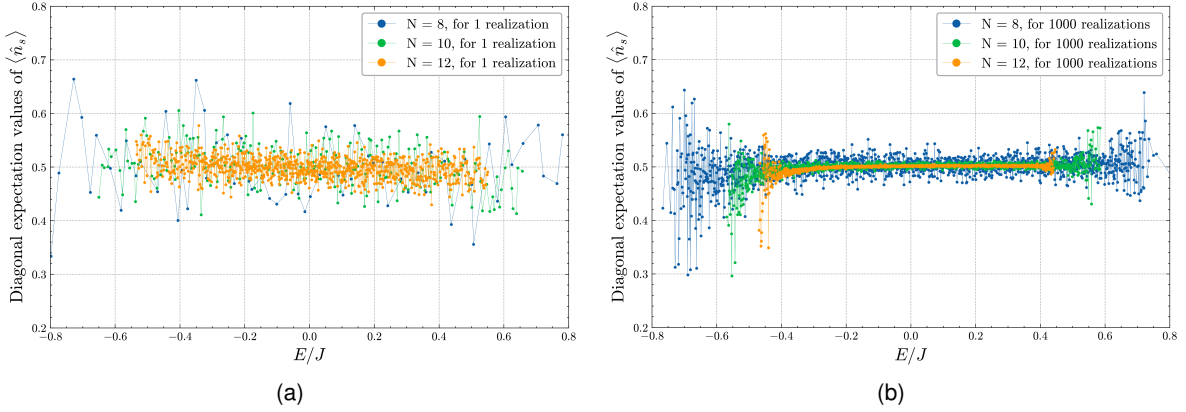


Figure 1.2: The absolute value of the diagonal terms of the expected value matrix of the number operator at site s , $|\langle n | \hat{n}_s | n \rangle|$. **(a)** For one realization and different values of N . **(b)** The average over 1000 realizations for different values of N . Reproduced from [26].

1.3a and 1.3b.

Note that for a pure random Hamiltonian, the off-diagonal expectation values behave like a constant function [8, 26]. In Figure 1.3a we can see that the Hamiltonian (1.5) behaves like a random matrix only in the bulk of energies. The histogram in 1.3b of the off-diagonal expectation values shows not only a zero mean, which is consistent with Figure 1.1, but also a decrease in the variance for high N , suggesting a correction to the microcanonical average that is sub-leading in the thermodynamic limit.

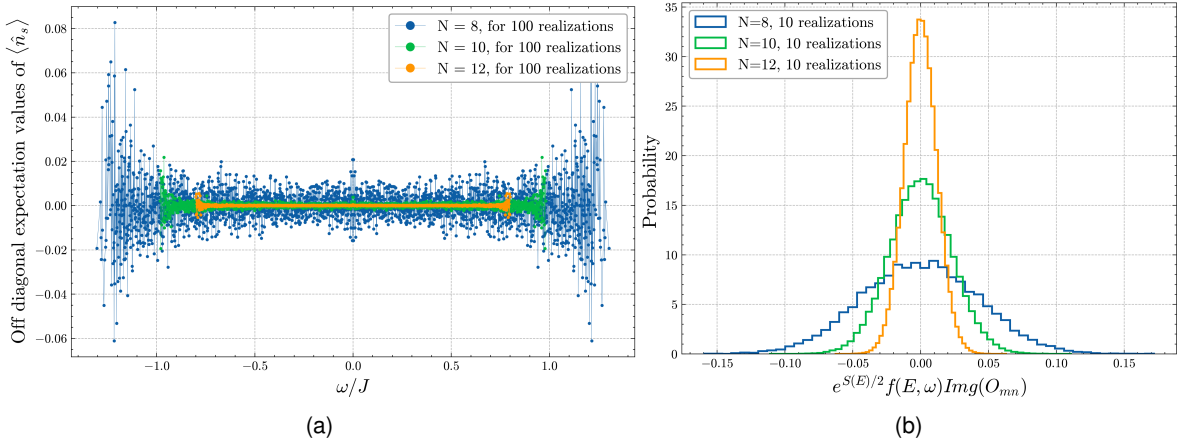


Figure 1.3: **(a)** Average of the absolute value of the expected values of the number operator at site s , $|\langle n | \hat{n}_s | n \rangle|$, as a function of energy differences ω , for 100 realizations at different values of N and $J = 1$. **(b)** Average histogram over 10 realizations of the imaginary off-diagonal part of the expected values of $\langle \hat{n}_s \rangle$ for different values of N . Reproduced from [26].

The last numerical results show the validity of the ETH and that the ansatz gets better and better as the number of degrees of freedom increases. The presence of randomness in the Hamiltonian is intimately related to quantum chaos. Actually, the SYK model is well known to be a highly chaotic

system and chaotic systems always thermalize in the sense of ETH (1.4). The converse implication is not true, as we will see in Chapter 2, where a system that is not chaotic presents thermalization. In fact, thermalization and chaos are not the only relevant factors. In certain systems, the role of entanglement has been shown to be key for the onset of thermalization [27].

The definition of chaos is a subject of debate in the physics community these days. All we can do is to test the system in the different properties that a chaotic system must have, such as the similarity of the level spacing statistics to that of RMT [28], or the exponential decay of the out-of-time-ordered correlator (OTOC) functions.

Chaos in the SYK model

In many-body quantum systems, important chaos indicators are the statistics of the energy level spacings and the behavior of the four-point correlation functions. In the first case, numerical studies of physical systems have shown a behavior that seems to follow a pattern, which is condensed in what is known as the Berry Tabor and the BGS conjectures. These conjectures state that for a many-body quantum system in the thermodynamic limit, if its classical counterpart is integrable, then the statistics of level spacings are Poissonian, and if its classical chaotic counterpart is chaotic, the level spacing distribution follows a Wigner distribution, which corresponds to the statistics of a random matrix taken from one of the three classical Gaussian ensembles (see Appendix A).

There is no formal proof of this statement. However, it has been shown to serve as a guidance and indicator of quantum chaos even for systems without a classical counterpart, such as the SYK model considered in this section.

For the SYK model, the statistics of level spacings shown in Figure 1.4, exhibit a clear convergence to a Wigner-Dyson distribution corresponding to $\beta = 2$, (see Appendix A), which indicates that the spectral properties of the Hamiltonian (1.5) are equivalent to those of a random matrix taken from the Gaussian Unitary Ensemble (GUE). Therefore, we can affirm that in the bulk of energies, the state of the system is maximally chaotic and there is a lack of time-reversal symmetry. For any other system, the level spacing statistics would lie between the two extremes defined by a Poisson and a Wigner distribution [9].

For the second indicator of chaos, we consider the hopping operator between sites $N - 1$ and N , defined as

$$\hat{h}_{N-1,N} = \hat{c}_{N-1}^\dagger \hat{c}_N + \hat{c}_N^\dagger \hat{c}_{N-1}.$$

We compute the two-point correlation functions of the form $\langle \hat{h}(t) \hat{h}(0) \rangle$ and the four-point correlation function $F(t)$ is defined as

$$F(t) = \frac{\langle \hat{W}(t) \hat{V}(0) \hat{W}(t) \hat{V}(0) \rangle + \langle \hat{V}(t) \hat{W}(0) \hat{V}(t) \hat{W}(0) \rangle}{2 \langle \hat{W}(0) \hat{W}(0) \hat{V}(0) \hat{V}(0) \rangle}. \quad (1.8)$$

Here, we take $\hat{W}(t) = \hat{h}_{N-1,N}$ and $\hat{V}(t) = \hat{h}_{N-3,N-2}$.

The expected value $\langle . \rangle$ is taken to be that of the canonical ensemble, in quantum mechanics. The expected value in the microcanonical ensemble is considered. In the case of the two-point correlator these expectation values are defined as follows:

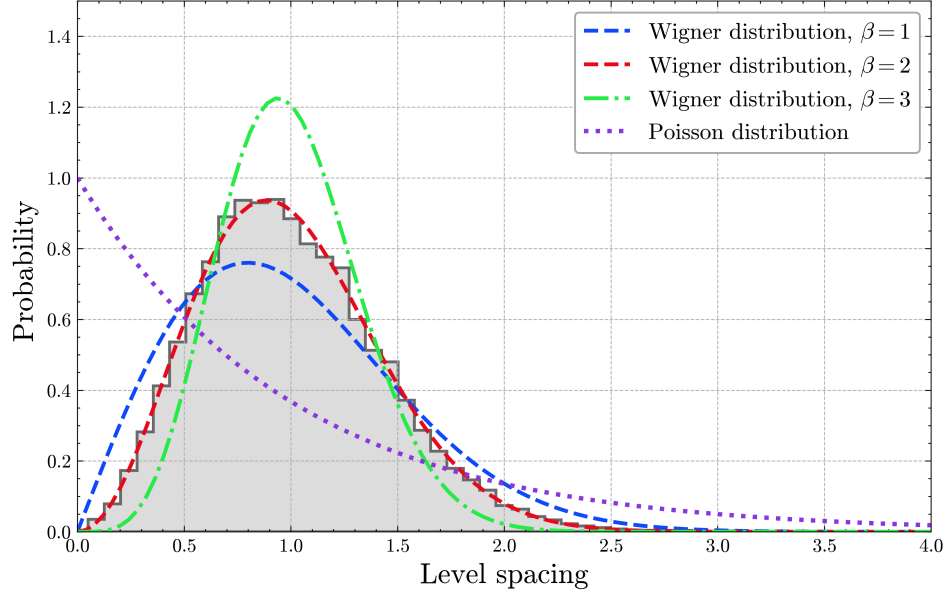


Figure 1.4: Statistics of the average energy level spacing distribution for the SYK model (1.5), averaged over 100 realizations with $N = 10$ and $J = 1$, ignoring the 10% of states at both low and high energy extremes.

$$G^\beta(t) = \frac{1}{Z(\beta)} \text{Tr} \left(e^{-\beta H} h_{ij}(t) h_{ij} \right), \quad Z(\beta) = \sum_n e^{-\beta E_n}. \quad (1.9)$$

where $Z(\beta)$ is the partition function. The expectation value in a single energy state is given by

$$G^n(t) = \langle n | h_{ij}(t) h_{ij} | n \rangle. \quad (1.10)$$

In the microcanonical ensemble, the expected value is defined by the average over states within an energy window Δ around $\bar{E}$,

$$G^{\bar{E}}(t) = \frac{1}{N_{\bar{E}}} \text{Tr} (h_{ij}(t) h_{ij}), \quad (1.11)$$

where $N_{\bar{E}}$ is the number of states in the energy window.

To compute the thermal expectation (1.9), we found the temperature $\beta = 1/T$ for which the following relation holds,

$$\bar{E} = \frac{1}{Z} \text{Tr}(e^{-\beta \hat{H}} \hat{H}),$$

where $\bar{E}$ is the average of the spectrum of the SYK Hamiltonian (1.5) obtained by exact diagonalization. The same relation is used to compute the quantum expectation (1.10) but this time by finding the state $|n\rangle$ associated with the energy E_n belonging to the bulk of the spectrum and that is closer to the mean energy $\bar{E}$, so that

$$E_n \approx \frac{1}{Z} \text{Tr}(e^{-\beta \hat{H}} \hat{H}).$$

The window of energy in the microcanonical average is arbitrary, it is taken around the mean energy $\bar{E}$, and the trace in (1.11) is taken over the states in this window.

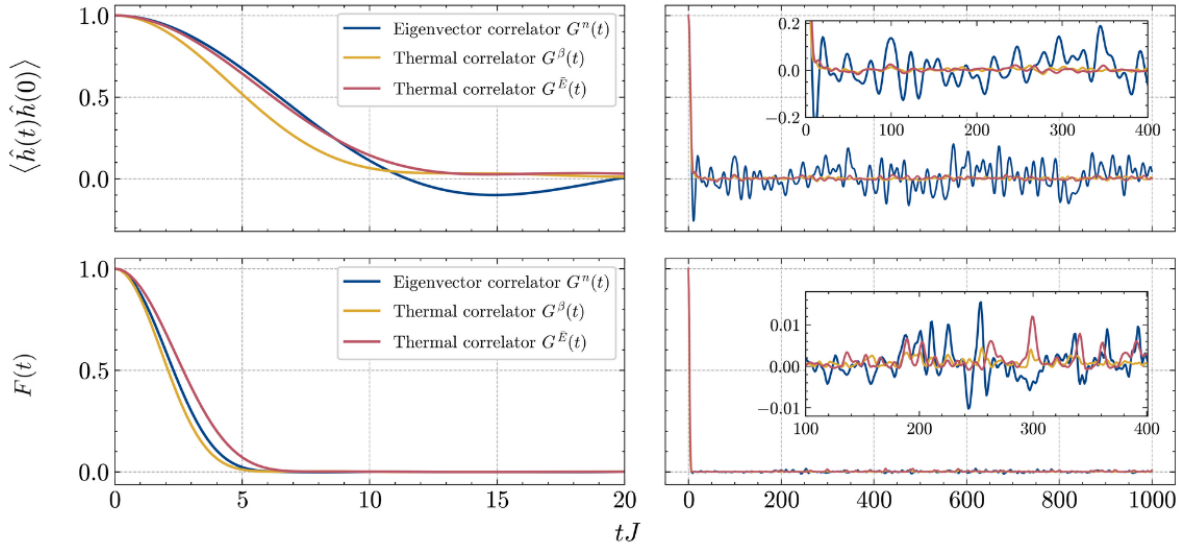


Figure 1.5: **Top panel:** Two-point correlation functions for the hopping operator $\hat{h}_{N-1,N}$ for one realization. **Left:** Decay at early times. **Right:** Decay at long times. **Bottom panel:** Four-point correlation functions for the hopping operators $\hat{h}_{N-1,N}$ and $\hat{h}_{N-3,N-2}$ for one realization. **Left:** Decay at early times. **Right:** Decay at long times. Reproduced from [26].

In chaotic quantum systems, the OTOC exhibits non-trivial behavior of the form $F(t) \approx F_0 - \alpha e^{\lambda_L t}$ for some coefficients α and λ_L . This exponential decay is the quantum analog of the exponentially fast separation of trajectories in classical chaotic systems, and λ_L is the quantum Lyapunov exponent, which is bounded by the temperature $\lambda_L \leq \frac{2\pi}{\beta}$.

From Figures in 1.5, it is observed that the two- and four-point correlation functions in the eigenvectors are qualitatively similar to thermal correlations with some fluctuations that are observed to decrease for larger values of N and for larger numbers of realizations. This indicates the equivalence between equilibrium statistics and eigenvector statistics, which is a sign that the system exhibits thermalization, as confirmed in the last section. Furthermore, the exponential decay of the OTOC (1.8) and the average level spacing statistics obtained in (1.4) confirm the chaotic nature of the system.

1.2.2 Thermalization in a nonrandom Hamiltonian of hard-core bosons

Over the past decade, significant effort has been devoted to studying ultra-cold quantum gases. Given that these systems are nearly isolated, it is worth questioning whether equilibrium statistical mechanics applies after relaxation. Numerous studies have demonstrated that thermalization is expected in generic nonintegrable systems within the gapless phase. In this section, following [29], we examine a system of hard-core bosons where thermalization occurs in the gapped phase. This finding supports the notion that non-integrability is a sufficient condition for thermalization, regardless of the presence or absence of gaps in the spectrum. Furthermore, in this system, thermalization occurs deeper in this phase as the dimension increases. The Hamiltonian describing the one-dimensional hard-core boson (HCB) model is given by;

$$\hat{H}_b = \sum_{i=1}^L \left\{ -t \left(\hat{b}_i^\dagger \hat{b}_{i+1} + \text{H.c.} \right) + V \left(\hat{n}_i^b - \frac{1}{2} \right) \left(\hat{n}_{i+1}^b - \frac{1}{2} \right) + V' \left(\hat{n}_i^b - \frac{1}{2} \right) \left(\hat{n}_{i+2}^b - \frac{1}{2} \right) \right\}, \quad (1.12)$$

where t is the nearest-neighbor hopping, V is the nearest-neighbor interaction, and V' is the next-nearest-neighbor interaction. This system is chaotic [29,30]; the chaotic to integrable transition occurs when the next-nearest-neighbor interaction is omitted at $V' = 0$. In addition, there exists a critical value $V'_c = 3$, such that for $V' < V'_c$ the ground state is a gapless superfluid, while for $V' \geq V'_c$ it is a gapped insulator. We work on a lattice at $1/3$ filling, $V = 6$ and V' is varied in the interval $[0, 9]$ following [29].

To study indicators of quantum chaos and thermalization in both the gapless and gapped phases, we analyze the level spacing distribution and the level number variance as we change the parameter V' that defines these phases.

The level spacing distribution may be fitted to the Brody distribution, which in turn is parameterized by β^2 ;

$$P_\beta(s) = (\beta + 1) b s^\beta e^{-b s^{\beta+1}}, \quad b = \left[\gamma \left(\frac{\beta + 2}{\beta + 1} \right) \right]^{\beta+1}.$$

When $\beta = 0$ the system is integrable, since we obtain the Poisson distribution $P_0(s) = e^{-s}$, while $\beta = 1$ gives the Wigner-Dyson distribution for a random matrix taken from the GOE ensemble; $P_1(s) = \frac{\pi}{2} s e^{-\frac{\pi}{4} s^2}$. In Figure 1.6a we show the values of β averaged over all momentum sectors by fitting the Brody distribution to the average histogram of level spacings, calculated with the unfolded spectrum. The unfolding procedure used in this case is given by the approximation of the peaks of energy by Gaussian functions (see Appendix C). We can see how the parameter β changes from zero to one as we move from the integrable gapless region to the chaotic gapped region by increasing V' , and a strong dependence on the system size is observed. We can see that a larger system's size implies a wider region for chaos, that is, smaller values of V' are required for the first integrable-chaotic transition and larger values of V' for the departure of chaos.

The variance of the number of levels measures long-range correlations and is defined as

$$\Sigma^2(l) = \langle N(l, \epsilon)^2 \rangle - \langle N(l, \epsilon) \rangle^2,$$

where $N(l, \epsilon)$ is the number of states contained in the interval $[\epsilon, \epsilon + l]$ for some interval length l , and $\langle \rangle$ denotes an average over different initial values of energy ϵ . When the distribution is Poissonian, the level number variance grows linearly with l , $\Sigma_P^2(l) = l$, which means that the spectrum is distributed uniformly. This absence of local fluctuations is a characteristic of integrable systems. For highly chaotic systems, the behavior is not linear; for the spectrum of a random matrix taken from the GOE ensemble, it takes the form;

$$\Sigma_{GOE}^2(l) = 2 \frac{\ln(2\pi l) + \gamma + 1 - \pi^2/8}{\pi^2},$$

where γ is the Euler constant.

²Here, β is a parameter that characterizes the different statistics of level spacings, not to be confused with the inverse temperature of a thermal state.

Figure 1.6b shows the level number variance for different values of V' , we work with the unfolded spectrum by fitting the staircase function with polynomials of degree 7 (see Appendix C). Here we can evidence the existence of an overlap between the chaotic and gapped regimes. For $V' = 0$, the system starts near the integrable regime in which the level number variance grows linearly with l . As we increase the next-nearest neighbor interactions, the integrable-chaotic transition is evident as the level number variance starts to behave like the result for *GOE*. Furthermore, for larger values of V' , the system departs from chaoticity, as is evident in the behavior of the curves for $V' = 7$ and $V' = 9$.

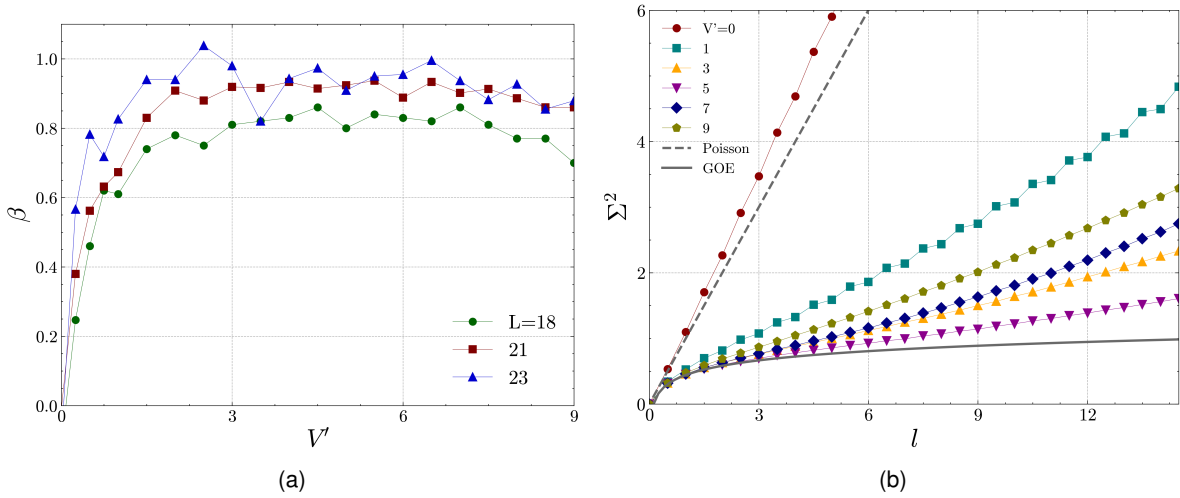


Figure 1.6: We show chaos indicators for different values of V' , evidencing the existence of an integrable-chaotic transition. Reproduced from [29]. **(a)** The parameter β used to fit the Brody's distribution to the average histogram over all momentum sectors as a function of V' . **(b)** The average level number variance over the same k sectors for $L = 23$ as a function of the interval length l .

Thermalization is intimately related to the statistics of the eigenstates. To verify ETH, we analyze the expectation values of one-body observables, the kinetic energy $K = \sum_i -t \left(\hat{b}_i^\dagger \hat{b}_{i+1} + \text{H.c.} \right)$ and the number operator at site s (for a fixed momentum sector) on the basis of the Hamiltonian, as one moves from superfluid states to insulator states. Figure 1.7a shows that for small values of V' , the expected values exhibit high fluctuations, while these fluctuations decrease as V' increases near the critical value. If we further increase V' too far from the critical value, the system departs from chaoticity as it becomes localized in the momentum space [29] and therefore also departs from the thermal state. ETH holds for intermediate values of V' where the fluctuations in the center of the spectrum are minimal, for the values of V' considered this happens at $V' = 5$. The same behavior is observed in Figure 1.7b for the number operator at site s . These results are consistent with the chaos indicators presented before, showing the existence of thermal behavior and its close relation to quantum chaos even for systems in a gapped phase and with a Hamiltonian that does not contain any random element.

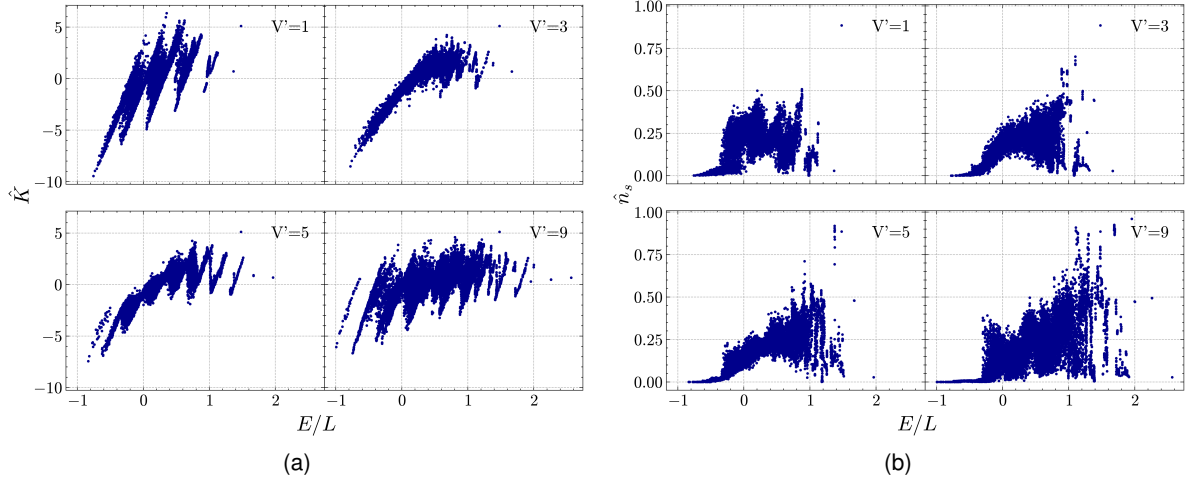


Figure 1.7: We show thermalization indicators for different values of V' , evidencing the relation between the integrable-chaotic transition and the validity of ETH. Reproduced from [29]. **(a)** The average over all k sectors of the expected value of the kinetic energy operator for $L = 20$. **(b)** The average over all k sectors of the expected value of the number operator at site s for $L = 20$.

1.3 ETH and free probability theory

1.3.1 Free probability and freeness

Free probability is a theory of probability for non-commutative random variables. Geometrically, there is an analogy between the moment-cumulant formula for the calculation of the n th moment of a random variable X , and the number of simple undirected graphs that we can form from a set of n vertices. The type of connectivity that we consider to define in this set of nodes will define the kind of probability theory. The classical probability for commutative random variables would correspond to the type of connectivity that joins two vertices by a line, while free probability theory also considers the geometric connectivity; see Appendix F.

Of important interest to us is the concept of independence of two non-commutative random variables, called free independence. The definition of free independence has its origins in the work of Voiculescu on non-commutative algebras [31, 32]. In general terms, we can say that freeness is a rule that allows us to calculate joint moments of free independent random variables. To be precise:

If $\mathcal{A}_1, \dots, \mathcal{A}_k$ are free subalgebras of $(\mathcal{A}, \varphi)$, composed of a unital associative algebra $\mathcal{A}$ and a unital linear functional $\varphi : \mathcal{A} \rightarrow \mathbb{C}$, the following condition must hold [31]

$$\varphi[(a_1 - \varphi(a_1))(a_2 - \varphi(a_2)) \dots (a_k - \varphi(a_k))] = 0. \quad (1.13)$$

From this equation, it is easy to verify that if $k = 2$ we recover the property of independence of two random variables a_1 and a_2 in classical probability theory: $\varphi(a_1 a_2) = \varphi(a_1) \varphi(a_2)$.

The transition from free independence in non-commutative subalgebras to subgroups is shown in proposition 1.3 of [32], and it has been shown that we can get free independence in the context

of random matrix theory by the application of rotations. See lecture 23 in [31] about Haar unitary random matrices for a complete treatment, and Appendix D about the Haar measure. To be precise, we are interested in the following theorem:

Theorem 1 (Random matrix model of an arbitrary free pair). *Let $(A_N)_{N \in \mathbb{N}}$ and $(B_N)_{N \in \mathbb{N}}$ be sequences of constant matrices with a limit distribution such as $N \rightarrow \infty$. And let $(U_N)_{N \in \mathbb{N}}$ be a sequence of Haar unitary $N \times N$ random matrices. Then $U_N A_N U_N^\dagger$ and B_N are asymptotically free.*

This theorem is also applicable to the orthogonal group $O(N)$.

The importance of this theorem is that, since $U_N A_N U_N^\dagger$ and B_N are free in the large N limit, we might apply condition (1.13) for the alternated products;

$$\langle (U A U^\dagger - \langle A \rangle)(B - \langle B \rangle)(U A U^\dagger - \langle A \rangle)(B - \langle B \rangle) \rangle = 0, \quad (1.14)$$

where $\langle A \rangle = \frac{1}{N} \text{Tr}(A)$, and for a random matrix O taken from the orthogonal group $O(N)$ it is defined in terms of the Haar average; $\langle O A O^\top \rangle = \frac{1}{N} \int [O^\top dO] \text{Tr}(O A O^\top)$, where $[O^\top dO]$ is the Haar measure of the orthogonal group (see Appendix D about the Haar measure).³ Direct simplification of this condition gives the result for $\langle B U A U^\dagger B U A U^\dagger \rangle$,

$$\langle B U A U^\dagger B U A U^\dagger \rangle = \langle A^2 \rangle \langle B \rangle^2 + \langle A \rangle^2 \langle B^2 \rangle - \langle A \rangle^2 \langle B \rangle^2. \quad (1.15)$$

Even more interesting is the fact that this result is closely related to ETH. Recent studies [2, 3] have shown that the leading contributions to the thermal moments of local operators are in one-to-one correspondence with the cumulants of free probability.

1.3.2 The generalized eigenstate thermalization hypothesis

The multipoint correlations of the matrix entries of the operators describing macroscopic observables are important in various areas of many-body physics. According to the ETH hypothesis given in Equation (1.4), the correlations between three points are described as follows:

$$\overline{O_{lm} O_{ks} O_{pq}} = e^{-\frac{S(E_{lm}^+)}{2}} e^{-\frac{S(E_{ks}^+)}{2}} e^{-\frac{S(E_{pq}^+)}{2}} F_{E_{lm}^+}(\omega_{lm}) F_{E_{ks}^+}(\omega_{ks}) F_{E_{pq}^+}(\omega_{pq}) \overline{R_{lm} R_{ks} R_{pq}}.$$

Since R_{ij} are random variables, the average of the product in the equation tends to zero⁴, which implies that $\overline{O_{lm} O_{ks} O_{pq}} = 0$. This demonstrates that the ETH is inconsistent with multipoint correlations.

A new version of ETH proposed by Laura Foini and Jorge Kurchan [3], using typicality arguments [33, 34], states that the average of the products of matrix entries with different indices $i_1 \neq i_2 \dots \neq i_q$ is given by;

³The same definition holds for the unitary group by replacing O by a random matrix $U \in U(N)$. In general, it can be defined for any compact topological group.

⁴This is because they are taken from a Gaussian distribution with zero mean and unit variance, as discussed in Section 1.2.

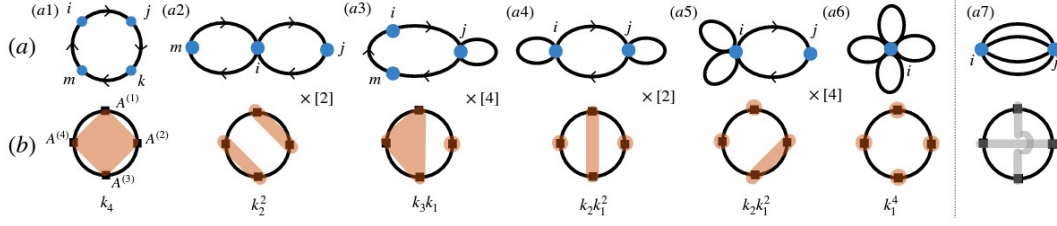


Figure 1.8: In (a), the diagrammatic structure of the matrix elements is shown, where (a1) is a cycle diagram, and the diagrams from (a2) to (a6) are cactus-type diagrams. These types of diagrams contribute to the ETH correlators. In (b), non-crossing partitions are shown, where each block contributes a free cumulant k_n . In (a7), a non-cactus type diagram and its corresponding intersecting partition are shown, which do not contribute to the ETH correlators. Image taken from [2].

$$\overline{O_{i_1 i_2} O_{i_2 i_3} \dots O_{i_q i_1}} = e^{-(q-1)S(E^+)} F_{E^+}^{(q)}(\omega_{i_1 i_2}, \dots, \omega_{i_{q-1} i_q}). \quad (1.16)$$

If the indices are repeated more than twice, then the expected values can be factored into the product

$$\overline{O_{i_1 i_2} \dots O_{i_{k-1} i_1} O_{i_1, i_{k+1}} \dots O_{i_q i_1}} = (\overline{O_{i_1 i_2} \dots O_{i_{k-1} i_1}}) (\overline{O_{i_1, i_{k+1}} \dots O_{i_q i_1}}), \quad (1.17)$$

where $E^+ = (E_{i_1} + \dots + E_{i_q})/q$ is the average energy and $F_{E^+}^{(q)}(\vec{\omega})$ is a smooth function of E^+ and $\vec{\omega}$, where $\vec{\omega} = (\omega_{i_1 i_2}, \dots, \omega_{i_{q-1} i_q})$ is a vector of energy separations $\omega_{ij} = E_i - E_j$.

This version of ETH not only provides a general form for the multipoint correlation functions that contain the original version of ETH (1.4), but also introduces a diagrammatic description of the matrix entries that relate thermal averages and free probability. To be precise, the products of matrix entries A_{ij} can be interpreted diagrammatically through directed connections between pairs of points i and j . Different diagrams correspond to all possible ways that two points can be identified (or index contractions). Diagrams are classified into cycles, cactus-type diagrams, and non-cactus type diagrams (1.8a). In October 2022, Silvia Pappalardi, Laura Foini, and Jorge Kurchan [2] demonstrated that there is a one-to-one correspondence between cactus-type diagrams and non-crossing partitions described in the Appendix F. Similarly, non-cactus type diagrams are related to intersecting partitions. As illustrated in Figure 1.8.

Additionally, the connection between ETH and free probability allows for the diagrammatic calculation of thermal correlation functions for an observable at different times.

$$S_{\beta}^{(q)}(\vec{t}) = \langle \hat{O}(t_1) \hat{O}(t_2) \dots \hat{O}(t_{q-1}) \hat{O}(0) \rangle_{\beta}, \quad (1.18)$$

where the average is defined in terms of the trace, that is, $\langle \hat{O} \rangle_{\beta} = \text{Tr}(\rho \hat{O})$, with $\rho = \frac{1}{Z} e^{-\beta H}$, $Z = \text{Tr}(e^{-\beta H})$, and $\hat{O}(t) = e^{i\hat{H}t} \hat{O} e^{-i\hat{H}t}$ is the observable in the Heisenberg representation. Since these correlators are sums over the matrix elements of $\hat{O}$, and given the relationship between these and non-crossing partitions, the thermal free cumulants $k_q^{\beta}(\vec{t})$ are implicitly defined as

$$S_{\beta}^{(q)}(\vec{t}) = \sum_{\pi \in NC(q)} \kappa_{\pi}^{\beta}(\hat{O}(t_1)\hat{O}(t_2)\dots\hat{O}(t_{q-1})\hat{O}(0)). \quad (1.19)$$

In other words, the Generalized Thermalization Hypothesis allows for the calculation of thermal correlation functions through the free cumulants, where the terms in the sum are obtained by analyzing the diagrammatic structure of non-crossing partitions.

Equation (1.19) is essentially an implication of (74) in Appendix F. The striking fact is that the ETH and free probability are equivalent in the thermodynamic limit, where the leading contribution to the moments comes only from the non-crossing partitions and formula (1.19) is valid.

1.3.3 Subleading contributions

The last two subsections presented results for the calculation of the moments of the n-point correlation functions. The main conclusion is that, in the limit of a large Hilbert space dimension, the generalized ETH is equivalent to free probability. That is, the thermal expectation values of the form (1.18) must be equivalent to those obtained by the condition of free independence (1.13). Furthermore, since theorem 1 is valid for both unitary and orthogonal rotations, one finds that $\langle BU AU^{\dagger} BU AU^{\dagger} \rangle = \langle BO AO^{\top} BO AO^{\top} \rangle$, holds for large matrices with the conditions of theorem 1. For arbitrary matrix dimensions, we might compute the moments in an exact form using quantum channels (see Appendix E on quantum channels). We are interested in the calculation of the average (1.15), which by definition is given by

$$\langle BU AU^{\dagger} BU AU^{\dagger} \rangle = \frac{1}{N} \int [U^{\dagger} dU] \text{Tr}(BU AU^{\dagger} BU AU^{\dagger}), \quad (1.20)$$

where $[U^{\dagger} dU]$ is the Haar measure of the unitary group (see Appendix D about Haar measures). For our purposes, equation (1.20) is easier to analyze in tensor representation, where operators are represented by boxes, subspaces by wires to the right, and dual spaces by wires to the left; that is,

$$\begin{aligned} \langle BU AU^{\dagger} BU AU^{\dagger} \rangle &= \frac{1}{N} \int [U^{\dagger} dU] \text{---} B - U - A - U^{\dagger} \text{---} B - U - A - U^{\dagger} \text{---} \\ &= \frac{1}{N} \int [U^{\dagger} dU] \text{---} \begin{array}{c} B - U - A - U^{\dagger} \\ B - U - A - U^{\dagger} \end{array} \text{---} \\ &= \frac{1}{N} \text{---} \left(\int [U^{\dagger} dU] \text{---} \begin{array}{c} B - U - A - U^{\dagger} \\ B - U - A - U^{\dagger} \end{array} \text{---} \right) \text{---} \end{aligned}$$

This can be written as

$$\begin{aligned}
\langle BU AU^\dagger BU AU^\dagger \rangle &= \frac{1}{N} \text{Tr} \left(\int [U^\dagger dU] (B \otimes B) (U \otimes U) (A \otimes A) (U^\dagger \otimes U^\dagger) \text{SWAP} \right) \\
&= \frac{1}{N} \text{Tr} \left(\text{SWAP} (B \otimes B) \int [U^\dagger dU] (U \otimes U) (A \otimes A) (U^\dagger \otimes U^\dagger) \right) \\
&= \frac{1}{N} \text{Tr} \left(\text{SWAP} (B \otimes B) \Phi_2^{\text{Haar}} (A \otimes A) \right).
\end{aligned}$$

Here, we have identified the two-fold quantum channel for $U(N)$. The result of this channel is known from representation theory and is deduced in Appendix E.2. The result is given by Equation (64). After evaluation of $A \otimes A$ in the channel and taking the trace, we get the final result

$$\begin{aligned}
\langle BU AU^\dagger BU AU^\dagger \rangle &= \frac{1}{N(N^2 - 1)} \left(-\frac{1}{N} \text{Tr}(A^2) \text{Tr}(B^2) + \text{Tr}(A^2) \text{Tr}(B)^2 \right. \\
&\quad \left. + \text{Tr}(A)^2 \text{Tr}(B^2) - \frac{1}{N} \text{Tr}(A)^2 \text{Tr}(B)^2 \right). \quad (1.21)
\end{aligned}$$

A similar calculation using the result of Equation (70) in Appendix E.3 gives the correlator for $O(N)$,

$$\begin{aligned}
\langle BOAO^\top BOAO^\top \rangle &= \frac{1}{N(N^2 + N - 2)} \left(\left(1 - \frac{2}{N}\right) \text{Tr}(A^2) \text{Tr}(B^2) + \text{Tr}(A^2) \text{Tr}(B)^2 \right. \\
&\quad \left. + \text{Tr}(A)^2 \text{Tr}(B^2) - \frac{1}{N} \text{Tr}(A)^2 \text{Tr}(B)^2 \right). \quad (1.22)
\end{aligned}$$

Comparing these results with Equation (1.15), we conclude that the exact calculation contains the additional term, $\langle A^2 \rangle \langle B^2 \rangle \sim \text{Tr}(A^2) \text{Tr}(B^2)$. This shows that there are contributions to the multipoint correlators that are lost in the large- N limit for which theorem 1 is valid. Furthermore, due to the equivalence between the generalized ETH and free probability in the sense of (1.19), a loss of information is also expected in the calculation of thermal correlators by the use of the generalized ETH. The additional contributions play an important role when we are interested in the thermalization of systems in nontrivial thermodynamic limits. Although the recovery of these subleading contributions by exact calculation is not always possible due to the complexity of many-body Hamiltonians, in the next chapter we show the explicit form of these contributions for a system of random free fermions. In the system considered, the subleading contributions play an important role in defining the presence or absence of time-reversal symmetry.

Chapter 2

Subleading contributions in a system of random free fermions

2.1 Random free fermions

As mentioned before, chaos is not fundamental for quantum thermalization to take place. Here we focus on a family of Gaussian Hamiltonians that describe noninteracting free fermions, where every oscillator is connected to every other democratically. Again, as in the case of the SYK model in Section 1.2.1, these types of systems, although they may seem unnatural due to the lack of local structure, they serve as toy models for black hole physics, and therefore understanding the mechanism of thermalization in them is a fundamental question in the context of the holographic duality [22] nowadays. The Hamiltonian considered in this chapter is given by

$$\hat{H} = \gamma \sum_{i=1}^N \hat{c}_i^\dagger \hat{c}_i + \sum_{i,j=1}^N \hat{c}_i^\dagger V_{ij} \hat{c}_j. \quad (2.1)$$

Each term in the Hamiltonian is the sum over all states of single-particle operators, due to the fact that the particles are non-interacting. The first term corresponds to the total rest mass energy, with γ being the average rest mass energy per fermion. The second term is the kinetic energy term that depends on the one-particle potential terms V_{ij} , these parameters are independent and taken from a Gaussian distribution with zero mean and variance $1/N$. $\hat{c}_i^\dagger$ and $\hat{c}_i$ are the creation and annihilation operators of spinless complex fermions, with the same anticommutator relations in (1.6).

We consider a dimensionless Hamiltonian, so that γ is a dimensionless parameter that fixes the scale of energy. We can always obtain such a Hamiltonian by dividing the whole equation by a constant with units of energy. This model has an effective description through the exact diagonalization of the matrix $(V)_{ij} = V_{ij}$. Let ψ^a be the eigenvectors of this matrix, with $a = 1, \dots, N$, and ϵ_a be its eigenvalues (dimensionless). The eigenvector decomposition takes the form

$$\sum_{j=1}^N V_{ij} \psi_j^a = \epsilon_a \psi_i^a. \quad (2.2)$$

Moreover, since the matrix vectors are orthogonal,

$$\sum_{a=1}^N \psi_i^{a*} \psi_j^a = \delta_{ij},$$

then the first term in the Hamiltonian (2.1) becomes

$$\begin{aligned} \gamma \sum_{i=1}^N \hat{c}_i^\dagger \hat{c}_i &= \gamma \sum_{i,j=1}^N \hat{c}_i^\dagger \hat{c}_j \delta_{ij} = \gamma \sum_{i,j=1}^N \hat{c}_i^\dagger \hat{c}_j \left(\sum_{a=1}^N \psi_i^{a*} \psi_j^a \right) \\ &= \gamma \sum_{a=1}^N \left(\sum_{i=1}^N \hat{c}_i^\dagger \psi_i^{a*} \right) \left(\sum_{j=1}^N \psi_j^a \hat{c}_j \right) = \gamma \sum_{a=1}^N \hat{d}_a^\dagger \hat{d}_a. \end{aligned} \quad (2.3)$$

Here, we have defined new creation and annihilation operators as linear combinations of the original creation and annihilation operators, given by

$$\hat{d}_a = \sum_{i=1}^N \psi_i^a \hat{c}_i. \quad (2.4)$$

The operators $\hat{d}_a$ and $\hat{d}_a^\dagger$ define collective excitations that can be understood as single particles (quasiparticles) or modes; see Figure 2.1.

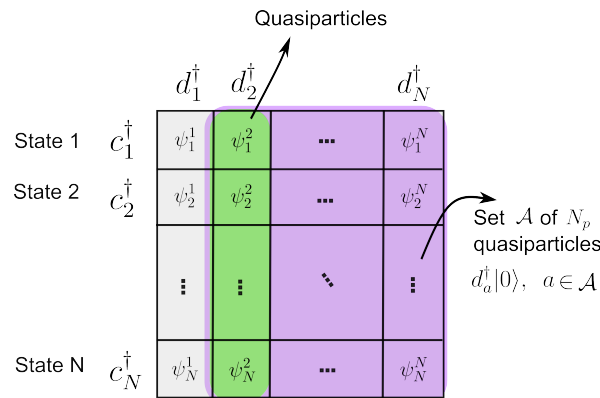


Figure 2.1: Graphical representation of the collective excitations. The matrix of eigenvectors is a random matrix Haar distributed over the orthogonal group.

Now, from Equation (2.2) we have

$$\begin{aligned}
 \sum_{j=1}^N V_{ij} \psi_j^a &= \epsilon_a \psi_i^a \longrightarrow \sum_{a=1}^N \sum_{j=1}^N V_{ij} \psi_j^{a*} \psi_k^a = \sum_{a=1}^N \epsilon_a \psi_i^{a*} \psi_k^a \\
 &= \sum_{j=1}^N V_{ij} \left(\sum_{a=1}^N \psi_j^{a*} \psi_k^a \right) = \sum_{j=1}^N V_{ij} \delta_{jk} = \sum_{a=1}^N \epsilon_a \psi_i^{a*} \psi_k^a.
 \end{aligned} \tag{2.5}$$

Therefore, $V_{ij} = \sum_{a=1}^N \epsilon_a \psi_i^{a*} \psi_j^a$ and the second term in the Hamiltonian (2.1) becomes;

$$\sum_{i,j=1}^N \hat{c}_i^\dagger V_{ij} \hat{c}_j = \sum_{ij} \hat{c}_i^\dagger \left(\sum_{a=1}^N \epsilon_a \psi_i^{a*} \psi_j^a \right) \hat{c}_j = \sum_{a=1}^N \epsilon_a \left(\sum_{i=1}^N \hat{c}_i^\dagger \psi_i^{a*} \right) \left(\sum_{j=1}^N \hat{c}_j \psi_j^a \right) = \sum_{a=1}^N \epsilon_a \hat{d}_a^\dagger \hat{d}_a,$$

and the Hamiltonian (2.1) is written as

$$\hat{H} = \sum_{a=1}^N (\gamma + \epsilon_a) \hat{d}_a^\dagger \hat{d}_a = \sum_{a=1}^N E_a \hat{d}_a^\dagger \hat{d}_a. \tag{2.6}$$

That is, in the effective description, the system behaves like a noninteracting Fermi gas whose energies follow a shifted Wigner semicircle law (see Appendix A about Gaussian ensembles). The eigenstates associated with this system are obtained by choosing a set $\mathcal{A}$ containing N_p quasiparticles and filling up the vacuum state

$$|\Psi^{N_p}\rangle = \prod_{a \in \mathcal{A}} \hat{d}_a^\dagger |0\rangle.$$

Note that there are $\binom{N}{N_p}$ independent ways to choose the states for a fixed N_p , then the states are degenerate. Due to particle-hole symmetry, we will focus on the range $N_p < N/2$.

The objective of this chapter is to study the thermalization process in this system by analytically calculating the expectation values of the number operator at the site s and its variance. Furthermore, we will study the onset of thermalization for ensembles that belong to the Wigner-Dyson classification of autonomous systems based on their anti-unitary symmetries. This defines the three universality classes: orthogonal, unitary, and symplectic.

It is well known that the Hamiltonian of a physical system must be represented by Hermitian matrices. Furthermore, the presence of unitary symmetries in the system (for instance, invariance under rotations), or anti-unitary symmetries such as time-reversal symmetry, is reflected in the structure of the Hamiltonian matrix. Assuming that there is no time-reversal symmetry (and no unitary symmetries), H is a permissible Hamiltonian if H is Hermitian, but so is $U H U^\dagger$ for any unitary matrix U . That is, the unitary group is the set of canonical transformations for the class of systems without time-reversal symmetry. Actually, the elements of the unitary group are precisely the time evolution operators generated by this set of Hermitian matrices (the Lie algebra). For this and other reasons (see [35] Section 2.4), complex Hermitian Hamiltonians are said to form the unitary symmetry class.

On the other hand, in the presence of time-reversal symmetry modeled by a time-reversal operator T , which by definition is such that $[H, T] = 0$ and $T^2 = 1$ (the last condition is required to preserve inner products; we might also have $T^2 = -1$), the permissible Hamiltonians are the set of real and symmetric matrices, and the orthogonal group constitutes the set of canonical transformations. These Hamiltonians form the orthogonal symmetry class.

For Hamiltonians of the form (2.1), where one parameter does not have a fixed value but is a random variable taken from a given probability distribution; we consider an ensemble of systems, consisting of different realizations of the random variable, and compute the observables averaged over this ensemble. In this and next section, we consider ensembles of Hamiltonians taken from the Wigner ensembles, GOE and GUE. In the first case, the Hamiltonian is a real symmetric matrix, and in the second case, it is a complex Hermitian matrix. They differ by the presence or absence of time-reversal symmetry in the system, respectively.

2.2 Gaussian orthogonal ensemble (GOE)

2.2.1 Correlators in eigenvectors

According to equation (2.4) and following [36], the 2-point correlation function with respect to the operators $\hat{c}_i$ and $\hat{c}_i^\dagger$, is given by;

$$C_{ij}^\Psi = \langle \Psi^{N_p} | \hat{c}_i^\dagger \hat{c}_j | \Psi^{N_p} \rangle = \sum_{a,b=1}^N \psi_i^a \psi_j^b \langle \Psi^{N_p} | \hat{d}_a^\dagger \hat{d}_b | \Psi^{N_p} \rangle = \sum_{a \in \mathcal{A}} \psi_i^a \psi_j^a$$

Note that when $i = j$ we obtain the expected value of the operator $\hat{c}_i^\dagger \hat{c}_i$, which is the number of particles at site i . This is a local operator and therefore, if the system thermalizes, we would expect this observable to behave in a similar way as for the SYK model in Section 1.2.1. Here, we explicitly compute the expectation value in both the quantum state $|\Psi^{N_p}\rangle$ and in a thermal state with temperature β^1 in order to see in which scenarios both averages coincide.

Since the ψ_i^a are the entries of the eigenvectors of a random matrix taken from the GOE, we might arrange these vectors in a matrix and assume no structure for it. The most generic form of this matrix is obtained when we sample it uniformly, in this case, from the orthogonal group, from now on we rename the entries ψ_i^a as O_{ai} for clarity. In other words, its entries are Haar distributed. By Haar distributed, we mean that all ensemble averages will be computed using the Haar measure of the orthogonal group. Note that since $O(N)$ is a compact topological group, the existence and uniqueness of the Haar measure is a fact (see Appendix D about the Haar measure). Therefore, if we denote by $[C_{ij}^\Psi]$ the average over the ensemble for different realizations of the matrix in Figure 2.1, we get

$$[C_{ij}^\Psi] = \sum_{a \in \mathcal{A}} [O_{ai} O_{aj}],$$

with $[O_{ai} O_{aj}]$ the Haar average.

¹We work in natural units, so that β is just the inverse of the temperature, $\beta = \frac{1}{T}$.

The striking property of the Haar measure is its invariance under left and right action of the group elements, which is useful in the calculation of averages that involve integrals, since in this case for some function f it is true that

$$\int [O^T dO] = 1, \quad \int [O^T dO] f(VO) = \int [O^T dO] f(OV) = \int [O^T dO] f(O), \quad \forall V \in O(\setminus),$$

where $[O^T dO]$ denotes the Haar measure.

Therefore, it is convenient to switch from notation in terms of indices O_{ai} , to notation in which the two-point correlator is written as a function of the entire matrix O .

This is easily done as

$$C_{ij}^\Psi = \sum_{a \in \mathcal{A}} O_{ai} O_{aj} = \sum_{a \in \mathcal{A}} O_{ia}^\top O_{aj} = \sum_{a \in \mathcal{A}} \langle i | O^\top | a \rangle \langle a | O | j \rangle = \langle i | O^\top \mathbb{1}_{\mathcal{A}} O | j \rangle,$$

where $\mathbb{1}_{\mathcal{A}} = \sum_{a \in \mathcal{A}} |a\rangle \langle a|$ is the identity operator acting on the subspace of states in $\mathcal{A}$. With this notation, it is now clear what we mean by a Haar average

$$[C_{ij}^\Psi] = \sum_{a \in \mathcal{A}} [O_{ai} O_{aj}] \rightarrow [C_{ij}^\Psi] = \langle i | [O^\top \mathbb{1}_{\mathcal{A}} O] | j \rangle,$$

where $[O^\top \mathbb{1}_{\mathcal{A}} O] = \int [dO^T O] O^\top \mathbb{1}_{\mathcal{A}} O$ is the integral average of the operator $O^\top \mathbb{1}_{\mathcal{A}} O$ using the Haar measure. From now on, we use the tensor notation convention in which transformations are represented by boxes. Wires to the right of these boxes represent subspaces, and wires to the left correspond to dual spaces, that is

$$\langle i | O^\top \mathbb{1}_{\mathcal{A}} O | j \rangle = \langle i | - O^\top - \mathbb{1}_{\mathcal{A}} - O - | j \rangle,$$

and

$$\begin{aligned} [C_{ij}^\Psi] &= \int [O^\top dO] \langle i | - O^\top - \mathbb{1}_{\mathcal{A}} - O - | j \rangle = \langle i | - \left(\int [O^\top dO] \langle i | - O^\top - \mathbb{1}_{\mathcal{A}} - O - | j \rangle \right) - | j \rangle \\ &= \langle i | - \boxed{\Phi_1^{\text{Haar}}(\mathbb{1}_{\mathcal{A}})} - | j \rangle. \end{aligned}$$

In the last equation, we realize that this type of integral is actually the one-fold quantum channel for the operator $\mathbb{1}_{\mathcal{A}}$, (see Appendix E on quantum channels)

$$\Phi_1^{\text{Haar}}(\mathbb{1}_{\mathcal{A}}) = \int [O^\top dO] O^\top \mathbb{1}_{\mathcal{A}} O. \quad (2.7)$$

The result of this channel is obtained from representation theory

$$\begin{aligned}
 \langle i | \boxed{\Phi_1^{\text{Haar}}(\mathbb{1}_{\mathcal{A}})} | j \rangle &= \langle i | \left(\frac{1}{N} \text{Tr}(\mathbb{1}_{\mathcal{A}}) \right) | j \rangle = \langle i | \left(\frac{1}{N} \sum_{k=1}^N \sum_{a \in \mathcal{A}} \langle k | a \rangle \langle a | k \rangle \right) | j \rangle \\
 &= \langle i | \left(\frac{1}{N} \sum_{a \in \mathcal{A}} \right) | j \rangle = \langle i | j \rangle \frac{N_p}{N} .
 \end{aligned}$$

Then

$$[C_{ij}^{\Psi}] = \frac{N_p}{N} \delta_{ij} . \quad (2.8)$$

Now, the two-point correlators in eigenvectors in the effective description are given by

$$\begin{aligned}
 C_{ij}^{\Psi} C_{kl}^{\Psi} &= \left(\sum_{a \in \mathcal{A}} O_{ai} O_{aj} \right) \left(\sum_{b \in \mathcal{A}} O_{bk} O_{bl} \right) = \sum_{a, b \in \mathcal{A}} O_{ia}^{\top} O_{aj} O_{kb}^{\top} O_{bl} \\
 &= \sum_{a, b \in \mathcal{A}} \langle i | O^{\top} | a \rangle \langle a | O | j \rangle \langle k | O^{\top} | b \rangle \langle b | O | l \rangle \\
 &= \langle i | O^{\top} \left(\sum_{a \in \mathcal{A}} | a \rangle \langle a | \right) O | j \rangle \langle k | O^{\top} \left(\sum_{b \in \mathcal{A}} | b \rangle \langle b | \right) O | l \rangle \\
 &= \langle i | O^{\top} \mathbb{1}_{\mathcal{A}} O | j \rangle \langle k | O^{\top} \mathbb{1}_{\mathcal{A}} O | l \rangle .
 \end{aligned}$$

In tensor notation,

$$C_{ij}^{\Psi} C_{kl}^{\Psi} = \langle i | \text{---} O^{\top} \text{---} \mathbb{1}_{\mathcal{A}} \text{---} O \text{---} | j \rangle \langle k | \text{---} O^{\top} \text{---} \mathbb{1}_{\mathcal{A}} \text{---} O \text{---} | l \rangle .$$

And by averaging over the orthogonal group we identify the two-fold channel,

$$[C_{ij}^{\Psi} C_{kl}^{\Psi}] = \langle i | \text{---} \left(\int [O^{\top} O] \text{---} O^{\top} \text{---} \mathbb{1}_{\mathcal{A}} \text{---} O \text{---} \right) | j \rangle \langle k | \text{---} \boxed{\Phi_2^{\text{Haar}}(\mathbb{1}_{\mathcal{A}} \otimes \mathbb{1}_{\mathcal{A}})} | l \rangle . \quad (2.9)$$

$$[C_{ij}^{\Psi} C_{kl}^{\Psi}] = \langle i | \text{---} \boxed{\Phi_2^{\text{Haar}}(\mathbb{1}_{\mathcal{A}} \otimes \mathbb{1}_{\mathcal{A}})} | l \rangle . \quad (2.10)$$

The result for the two-fold quantum channel over $O(N)$ can also be obtained from representation theory. In Appendix E.3 we provide an alternative and easy-to-follow deduction. The final result given by Equation (70), reads

$$\Phi_2^{\text{Haar}}(\mathbb{1}_{\mathcal{A}} \otimes \mathbb{1}_{\mathcal{A}}) = \alpha(\mathbb{1}_{\mathcal{A}} \otimes \mathbb{1}_{\mathcal{A}}) \text{---} + \beta(\mathbb{1}_{\mathcal{A}} \otimes \mathbb{1}_{\mathcal{A}}) \text{---} + \gamma(\mathbb{1}_{\mathcal{A}} \otimes \mathbb{1}_{\mathcal{A}}) \text{---} , \quad (2.11)$$

where

$$\alpha(\mathbb{1}_{\mathcal{A}} \otimes \mathbb{1}_{\mathcal{A}}) = \frac{1}{N^2 + N - 2} \left[\frac{1+N}{N} \text{Tr} \left(\mathbb{1}_{\mathcal{A}} \otimes \mathbb{1}_{\mathcal{A}} \overline{\quad} \right) - \frac{1}{N} \text{Tr} \left(\mathbb{1}_{\mathcal{A}} \otimes \mathbb{1}_{\mathcal{A}} \times \right) - \frac{1}{N} \text{Tr} \left(\mathbb{1}_{\mathcal{A}} \otimes \mathbb{1}_{\mathcal{A}} \bigcirc \right) \right], \quad (2.12)$$

$$\beta(\mathbb{1}_{\mathcal{A}} \otimes \mathbb{1}_{\mathcal{A}}) = \frac{1}{N^2 + N - 2} \left[-\frac{1}{N} \text{Tr} \left(\mathbb{1}_{\mathcal{A}} \otimes \mathbb{1}_{\mathcal{A}} \overline{\quad} \right) + \frac{1+N}{N} \text{Tr} \left(\mathbb{1}_{\mathcal{A}} \otimes \mathbb{1}_{\mathcal{A}} \times \right) - \frac{1}{N} \text{Tr} \left(\mathbb{1}_{\mathcal{A}} \otimes \mathbb{1}_{\mathcal{A}} \bigcirc \right) \right], \quad (2.13)$$

$$\gamma(\mathbb{1}_{\mathcal{A}} \otimes \mathbb{1}_{\mathcal{A}}) = \frac{1}{N^2 + N - 2} \left[-\frac{1}{N} \text{Tr} \left(\mathbb{1}_{\mathcal{A}} \otimes \mathbb{1}_{\mathcal{A}} \overline{\quad} \right) - \frac{1}{N} \text{Tr} \left(\mathbb{1}_{\mathcal{A}} \otimes \mathbb{1}_{\mathcal{A}} \times \right) + \frac{1+N}{N} \text{Tr} \left(\mathbb{1}_{\mathcal{A}} \otimes \mathbb{1}_{\mathcal{A}} \bigcirc \right) \right]. \quad (2.14)$$

Explicit evaluation of the traces in these expressions gives

$$\text{Tr} \left(\mathbb{1}_{\mathcal{A}} \otimes \mathbb{1}_{\mathcal{A}} \overline{\quad} \right) = \begin{array}{c} \text{---} \mathbb{1}_{\mathcal{A}} \text{---} \\ \text{---} \mathbb{1}_{\mathcal{A}} \text{---} \end{array} = N_p^2, \quad (2.15)$$

$$\text{Tr} \left(\mathbb{1}_{\mathcal{A}} \otimes \mathbb{1}_{\mathcal{A}} \times \right) = \begin{array}{c} \text{---} \mathbb{1}_{\mathcal{A}} \text{---} \\ \text{---} \mathbb{1}_{\mathcal{A}} \text{---} \end{array} \times = \begin{array}{c} \text{---} \mathbb{1}_{\mathcal{A}} \text{---} \end{array} = N_p, \quad (2.16)$$

$$\text{Tr} \left(\mathbb{1}_{\mathcal{A}} \otimes \mathbb{1}_{\mathcal{A}} \bigcirc \right) = \begin{array}{c} \text{---} \mathbb{1}_{\mathcal{A}} \text{---} \\ \text{---} \mathbb{1}_{\mathcal{A}} \text{---} \end{array} \bigcirc = \begin{array}{c} \text{---} \mathbb{1}_{\mathcal{A}} \text{---} \end{array} = N_p. \quad (2.17)$$

Substituting these results into the last equations, we get

$$[C_{ij}^{\Psi} C_{kl}^{\Psi}] = \frac{1}{N^2 + N - 2} \left[\left(\frac{(1+N)}{N} N_p^2 - 2 \frac{N_p}{N} \right) \delta_{ij} \delta_{kl} + \left(N_p - \frac{N_p^2}{N} \right) \delta_{il} \delta_{kj} + \left(N_p - \frac{N_p^2}{N} \right) \delta_{ik} \delta_{jl} \right]. \quad (2.18)$$

This, together with Equation (2.8), gives the result for the variance,

$$[C_{ij}^{\Psi} C_{kl}^{\Psi}] - [C_{ij}^{\Psi}][C_{kl}^{\Psi}] = \frac{1}{N^2 + N - 2} \left(1 - \frac{N_p}{N} \right) \left(-2 \frac{N_p}{N} \delta_{ij} \delta_{kl} + N_p \delta_{il} \delta_{kj} + N_p \delta_{ik} \delta_{jl} \right) \quad (2.19)$$

We now consider two limiting cases:

- The limit in which the density $\zeta = \frac{N_p}{N}$ is fixed while the system's size N increases,
- The same limit, where now $N \gg N_p$ such that $\zeta \rightarrow 0$.

The result of the first limit is given by the expression

$$[C_{ij}^\Psi C_{kl}^\Psi] - [C_{ij}^\Psi][C_{kl}^\Psi] = \frac{\zeta}{N} (1 - \zeta) (\delta_{il}\delta_{kj} + \delta_{ik}\delta_{jl}), \quad (2.20)$$

and the result of the second limit is obtained by ignoring the second-order term in ζ in the last expression, obtaining

$$[C_{ij}^\Psi C_{kl}^\Psi] - [C_{ij}^\Psi][C_{kl}^\Psi] = \frac{N_p}{N^2} (\delta_{il}\delta_{kj} + \delta_{ik}\delta_{jl}).$$

From these results, we obtain the following expression for the exact two-point correlator in eigenvectors in the thermodynamic limit

$$C_{ij}^\Psi \approx \frac{N_p}{N} \delta_{ij} \pm \frac{\zeta}{N} (1 - \zeta) (\delta_{il}\delta_{kj} + \delta_{ik}\delta_{jl}). \quad (2.21)$$

Next, we assume that our system can be observed in some thermal state at temperature β , and calculate the last expression in this thermal state. If thermalization occurs, we must be able to find a temperature β , for which the description of the system with the thermal expectation values is equivalent to (2.21).

2.2.2 Thermal correlators

The two-point thermal correlation function is given by

$$\begin{aligned} C_{ij}^\beta &= \frac{1}{Z} \text{Tr} \left(e^{-\beta H} \hat{c}_i^\dagger \hat{c}_j \right) = \frac{1}{Z} \sum_{a,b=1}^N O_{ai} O_{bj} \text{Tr} \left(e^{-\beta H} \hat{d}_a^\dagger \hat{d}_b \right) = \sum_{a=1}^N O_{ai} O_{aj} n_\beta^{E_a} \\ &= \sum_{a=1}^N \langle i | O^\top | a \rangle \langle a | O | j \rangle n_\beta^{E_a} = \langle i | O^\top \left(\sum_{a=1}^N | a \rangle \langle a | n_\beta^{E_a} \right) O | j \rangle = \langle i | O^\top \mathbb{1}_\beta O | j \rangle, \end{aligned}$$

where $n_\beta^{E_a} = 1 / (e^{\beta E_a} + 1)$ is the Fermi-Dirac distribution for the mean number of particles at energy E_a . We have also defined the operator $\mathbb{1}_\beta = \sum_{a=1}^N | a \rangle \langle a | n_\beta^{E_a}$. Taking the average over the orthogonal group, we identify the one-fold quantum channel

$$\begin{aligned}
 [C_{ij}^\beta] &= \int [O^\top dO] \langle i | -O^\top - \mathbb{1}_\beta - O | j \rangle = \langle i | - \left(\int [O^\top dO] \langle i | -O^\top - \mathbb{1}_\beta - O | j \rangle \right) | j \rangle \\
 &= \langle i | - \boxed{\Phi_1^{\text{Haar}}(\mathbb{1}_\beta)} | j \rangle = \langle i | j \rangle \frac{1}{N} \text{Tr}(\mathbb{1}_\beta) = \frac{1}{N} \delta_{ij} \sum_{k=1}^N \left(\sum_{a=1}^N \langle k | a \rangle \langle a | n_\beta^{E_a} | k \rangle \right) \\
 &= \frac{1}{N} \delta_{ij} \sum_{k,a=1}^N \delta_{ak} n_\beta^{E_a} = \frac{1}{N} \delta_{ij} \sum_{k=1}^N n_\beta^{E_k} .
 \end{aligned} \tag{2.22}$$

In the last expression, the term $\sum_{k=1}^N n_\beta^{E_k}$ is a sum over the average number of particles with energy E_k , since the particles are indistinguishable, $n_\beta^{E_k}$ must be independent of k , and we can replace it by an average over the distribution $P(\lambda)$ ² of eigenvalues:

$$n_\beta^{E_k} \rightarrow [n_\beta^{E_k}] = [n_\beta^{\gamma+\lambda}] = [n_\beta] = \int_{-R}^R P(\lambda) n_\beta^{\gamma+\lambda} .$$

So, $\sum_{k=1}^N n_\beta^{E_k} = N[n_\beta]$. Thus, the average of the correlation matrix is³,

$$[C_{ij}^\beta] = [n_\beta] \delta_{ij} . \tag{2.23}$$

On the other hand, the two-point thermal correlation function in the effective description is

$$\begin{aligned}
 C_{ij}^\beta C_{kl}^\beta &= \left(\frac{1}{Z} \sum_{a=1}^N O_{ai} O_{aj} e^{-\beta E_a} \langle a | \hat{n}_a | a \rangle \right) \left(\frac{1}{Z} \sum_{b=1}^N O_{bk} O_{bl} e^{-\beta E_b} \langle b | \hat{n}_b | b \rangle \right) \\
 &= \frac{1}{Z^2} \sum_{a,b=1}^N O_{ai} O_{aj} O_{bk} O_{bl} e^{-\beta(E_a+E_b)} \langle a | \hat{n}_a | a \rangle \langle b | \hat{n}_b | b \rangle \\
 &= \langle i | O^\top \left(\sum_{a=1}^N |a\rangle \langle a| \frac{1}{Z} e^{-\beta E_a} n_a \right) O | j \rangle \langle k | O^\top \left(\sum_{b=1}^N |b\rangle \langle b| \frac{1}{Z} e^{-\beta E_b} n_b \right) O | l \rangle \\
 &= \langle i | O^\top \left(\sum_{a=1}^N |a\rangle \langle a| n_\beta^{E_a} \right) O | j \rangle \langle k | O^\top \left(\sum_{b=1}^N |b\rangle \langle b| n_\beta^{E_b} \right) O | l \rangle \\
 &= \langle i | O^\top \mathbb{1}_\beta O | j \rangle \langle k | O^\top \mathbb{1}_\beta O | l \rangle .
 \end{aligned}$$

In tensor notation

$$C_{ij}^\beta C_{kl}^\beta = \langle i | -O^\top - \mathbb{1}_\beta - O | j \rangle \langle k | -O^\top - \mathbb{1}_\beta - O | l \rangle .$$

²By RMT theory, we know that the distribution of eigenvalues of an $N \times N$ random matrix taken from a Gaussian ensemble converges to the Wigner semicircle law for large N , (see Appendix A on Gaussian ensembles).

³Note that here, $[n_\beta]$ is the first moment of the eigenvalues taking into account their distribution and not an average over the ensemble of matrices, such average was already taken in Equation (2.22).

Furthermore, note that

$$\mathbb{1}_\beta \otimes \mathbb{1}_\beta = \left(\sum_{a=1}^N |a\rangle \langle a| n_\beta^{E_a} \right) \otimes \left(\sum_{b=1}^N |b\rangle \langle b| n_\beta^{E_b} \right) = \sum_{a,b=1}^N |a\rangle \langle a| \otimes |b\rangle \langle b| n_\beta^{E_a} n_\beta^{E_b}.$$

Then, the average over the orthogonal group is given by

$$\begin{aligned} [C_{ij}^\beta C_{kl}^\beta] &= \langle i| \left(\int [O^\top O] \begin{array}{c} -O^\top - \mathbb{1}_\beta - O - \\ -O^\top - \mathbb{1}_\beta - O - \end{array} \right) |j\rangle \\ &= \sum_{a,b=1}^N [n_\beta^{E_a} n_\beta^{E_b}] \langle i| \left(\int [O^\top O] \begin{array}{c} -O^\top - |a\rangle \langle a| - O - \\ -O^\top - |b\rangle \langle b| - O - \end{array} \right) |j\rangle \\ &= \sum_{a,b=1}^N [n_\beta^{E_a} n_\beta^{E_b}] \langle i| \boxed{\Phi_2^{\text{Haar}}(|a\rangle \langle a| \otimes |b\rangle \langle b|)} |j\rangle. \end{aligned}$$

Here, the average of the two-point function of the mean number of particles $[n_\beta^{E_a} n_\beta^{E_b}]$ is carried out over the eigenvalues of the different realizations of the ensemble of random matrices V taken from GOE.

The result of the two-fold quantum channel according to (70) is

$$\Phi_2^{\text{Haar}} \left(\begin{array}{c} -|a\rangle \langle a| - \\ -|b\rangle \langle b| - \end{array} \right) = \alpha \left(\begin{array}{c} -|a\rangle \langle a| - \\ -|b\rangle \langle b| - \end{array} \right) \text{---} + \beta \left(\begin{array}{c} -|a\rangle \langle a| - \\ -|b\rangle \langle b| - \end{array} \right) \text{X} + \gamma \left(\begin{array}{c} -|a\rangle \langle a| - \\ -|b\rangle \langle b| - \end{array} \right) \text{D} \text{C}.$$

Here, the coefficient α is given by

$$\begin{aligned} \alpha \left(\begin{array}{c} -|a\rangle \langle a| - \\ -|b\rangle \langle b| - \end{array} \right) &= \frac{1}{N^2 + N - 2} \left[\frac{1 + N}{N} \text{Tr} \left(\begin{array}{c} -|a\rangle \langle a| - \text{---} \\ -|b\rangle \langle b| - \text{---} \end{array} \right) \right. \\ &\quad \left. - \frac{1}{N} \text{Tr} \left(\begin{array}{c} -|a\rangle \langle a| - \text{X} \\ -|b\rangle \langle b| - \end{array} \right) - \frac{1}{N} \text{Tr} \left(\begin{array}{c} -|a\rangle \langle a| - \text{D} \text{C} \\ -|b\rangle \langle b| - \end{array} \right) \right]. \end{aligned}$$

The coefficient β is

$$\begin{aligned} \beta \left(\begin{array}{c} -|a\rangle \langle a| - \\ -|b\rangle \langle b| - \end{array} \right) &= \frac{1}{N^2 + N - 2} \left[-\frac{1}{N} \text{Tr} \left(\begin{array}{c} -|a\rangle \langle a| - \text{---} \\ -|b\rangle \langle b| - \text{---} \end{array} \right) \right. \\ &\quad \left. + \frac{1 + N}{N} \text{Tr} \left(\begin{array}{c} -|a\rangle \langle a| - \text{X} \\ -|b\rangle \langle b| - \end{array} \right) - \frac{1}{N} \text{Tr} \left(\begin{array}{c} -|a\rangle \langle a| - \text{D} \text{C} \\ -|b\rangle \langle b| - \end{array} \right) \right]. \end{aligned}$$

And γ is

$$\begin{aligned} \gamma \left(\begin{array}{c} -|a\rangle \langle a| - \\ -|b\rangle \langle b| - \end{array} \right) &= \frac{1}{N^2 + N - 2} \left[-\frac{1}{N} \text{Tr} \left(\begin{array}{c} -|a\rangle \langle a| - \text{---} \\ -|b\rangle \langle b| - \text{---} \end{array} \right) \right. \\ &\quad \left. - \frac{1}{N} \text{Tr} \left(\begin{array}{c} -|a\rangle \langle a| - \text{X} \\ -|b\rangle \langle b| - \end{array} \right) + \frac{1 + N}{N} \text{Tr} \left(\begin{array}{c} -|a\rangle \langle a| - \text{D} \text{C} \\ -|b\rangle \langle b| - \end{array} \right) \right]. \end{aligned}$$

Explicit evaluation of the traces in this case gives:

$$\begin{aligned}
 \text{Tr} \begin{pmatrix} -|a\rangle \langle a| - \\ -|b\rangle \langle b| - \end{pmatrix} &= \begin{array}{c} \text{---}|a\rangle \langle a| \text{---} \\ \text{---}|b\rangle \langle b| \text{---} \end{array} = 1, \\
 \text{Tr} \begin{pmatrix} -|a\rangle \langle a| - \\ -|b\rangle \langle b| - \end{pmatrix} &= \begin{array}{c} \text{---}|a\rangle \langle a| \text{---} \\ \text{---}|b\rangle \langle b| \text{---} \end{array} = \begin{array}{c} |a\rangle \langle a|b\rangle \langle b| \\ \text{---}|b\rangle \langle b| \text{---} \end{array} = \delta_{ab}, \\
 \text{Tr} \begin{pmatrix} -|a\rangle \langle a| - \\ -|b\rangle \langle b| - \end{pmatrix} &= \begin{array}{c} \text{---}|a\rangle \langle a| \text{---} \\ \text{---}|b\rangle \langle b| \text{---} \end{array} = \begin{array}{c} |a\rangle \langle a|b\rangle \langle b| \\ \text{---}|b\rangle \langle b| \text{---} \end{array} = \delta_{ab}.
 \end{aligned}$$

Then

$$\begin{aligned}
 \alpha \begin{pmatrix} -|a\rangle \langle a| - \\ -|b\rangle \langle b| - \end{pmatrix} &= \frac{1}{N^2 + N - 2} \left[\frac{1 + N}{N} - \frac{1}{N} \delta_{ab} - \frac{1}{N} \delta_{ab} \right], \\
 \beta \begin{pmatrix} -|a\rangle \langle a| - \\ -|b\rangle \langle b| - \end{pmatrix} &= \frac{1}{N^2 + N - 2} \left[-\frac{1}{N} + \frac{1 + N}{N} \delta_{ab} - \frac{1}{N} \delta_{ab} \right], \\
 \gamma \begin{pmatrix} -|a\rangle \langle a| - \\ -|b\rangle \langle b| - \end{pmatrix} &= \frac{1}{N^2 + N - 2} \left[-\frac{1}{N} - \frac{1}{N} \delta_{ab} + \frac{1 + N}{N} \delta_{ab} \right].
 \end{aligned}$$

The average of the two-point correlation functions is therefore

$$\begin{aligned}
 [C_{ij}^\beta C_{kl}^\beta] &= \sum_{a,b=1}^N [n_\beta^{E_a} n_\beta^{E_b}] \langle i | \text{---} \boxed{\Phi_2^{\text{Haar}}(|a\rangle \langle a| \otimes |b\rangle \langle b|)} \text{---} | j \rangle \\
 &= \frac{1}{N^2 + N - 2} \sum_{a,b=1}^N [n_\beta^{E_a} n_\beta^{E_b}] \left[\left(\frac{1 + N}{N} - 2 \frac{1}{N} \delta_{ab} \right) \langle i | \text{---} | j \rangle \right. \\
 &\quad \left. + \left(-\frac{1}{N} + \delta_{ab} \right) \langle i | \text{---} | j \rangle + \left(-\frac{1}{N} + \delta_{ab} \right) \langle i | \text{---} | j \rangle \right] \\
 &= \frac{1}{N^2 + N - 2} \sum_{a,b=1}^N [n_\beta^{E_a} n_\beta^{E_b}] \left(\frac{1 + N}{N} \delta_{ij} \delta_{kl} - \frac{1}{N} \delta_{il} \delta_{kj} - \frac{1}{N} \delta_{ik} \delta_{jl} \right) \\
 &\quad + \sum_{a=1}^N [n_\beta^{E_a} n_\beta^{E_a}] \left(-2 \frac{1}{N} \delta_{ij} \delta_{kl} + \delta_{il} \delta_{kj} + \delta_{ik} \delta_{jl} \right). \tag{2.24}
 \end{aligned}$$

Here, it is important to note that the two-point energy averages $[n_\beta^{E_a} n_\beta^{E_b}]$ are insensitive to the particular choices of a and b due to the lack of topology and structure in the system. Therefore, we are safe to do the replacements;

$$\sum_{a,b=1}^N [n_{\beta}^{E_a} n_{\beta}^{E_b}] \rightarrow N^2 [n_{\beta}^{\gamma+\lambda_1} n_{\beta}^{\gamma+\lambda_2}], \quad \sum_{a=1}^N [n_{\beta}^{E_a} n_{\beta}^{E_a}] \rightarrow N [(n_{\beta}^{\gamma+\lambda})^2].$$

Here, we have renamed the eigenvalues ϵ_a of the random matrix V as λ_i for simplicity, and then $E_i = \gamma + \lambda_i$ according to Equation (2.6). Now, in order to solve the two-point average (2.24), we need to calculate the term $[n_{\beta}^{\gamma+\lambda_1} n_{\beta}^{\gamma+\lambda_2}]$, where both λ_1 and λ_2 are a pair of eigenvalues of the matrix V taken from GOE, and as we know, they are random variables. Since $[\cdot]$ is an average over different realizations of V and therefore different realizations of the product $n_{\beta}^{\gamma+\lambda_1} n_{\beta}^{\gamma+\lambda_2}$, to compute such average we must take into account the joint probability density function of the eigenvalues of a matrix taken from GOE (Appendix A). If $P_{GOE}(\lambda)$ is the aforementioned probability distribution, we obtain the probability density function for the system to be in states with energies λ_1 or λ_2 as the two-point marginal probability distribution [15],

$$\rho^{(2)}(\lambda_1, \lambda_2) = \int d\lambda_3 \dots d\lambda_N P_{GOE}(\lambda_1, \dots, \lambda_N).$$

With this, it is clear that the two-point average of the Fermi-Dirac distribution is given by

$$[n_{\beta}^{E_1} n_{\beta}^{E_2}] = [n_{\beta}^{\gamma+\lambda_1} n_{\beta}^{\gamma+\lambda_2}] = \int d\lambda_1 d\lambda_2 \frac{1}{(e^{\beta(\gamma+\lambda_1)} + 1)} \frac{1}{(e^{\beta(\gamma+\lambda_2)} + 1)} \rho^{(2)}(\lambda_1, \lambda_2). \quad (2.25)$$

The explicit calculation of $\rho^{(2)}(\lambda_1, \lambda_2)$ is done for a general N -point correlation with the use of kernel matrices that take different forms for each Gaussian ensemble (see Appendix B). A complete treatment of this in the context of spectral form factors is presented in [37].

2.2.3 Energy average of the 2-point correlator for GOE

The 2-point correlation function for the eigenvalues λ of a random matrix of size $N \times N$ taken from GOE is given by (see Appendix B),

$$\begin{aligned} \rho^{(2)}(\lambda_1, \lambda_2) &= \int d\lambda_3 \dots d\lambda_N P_{GOE}(\lambda_1, \dots, \lambda_N) = \frac{N}{N-1} \left(\frac{1}{2\pi} \sqrt{4 - \lambda_1^2} \right) \left(\frac{1}{2\pi} \sqrt{4 - \lambda_2^2} \right) \\ &\quad - \frac{N}{(N-1)\pi^2} \hat{s}(N(\lambda_1 - \lambda_2))^2 + \frac{N}{(N-1)\pi^2} \mathbf{D} \hat{s}(N(\lambda_1 - \lambda_2)) \mathbf{I} \hat{s}(N(\lambda_1 - \lambda_2)) \\ &\quad - \frac{N}{(N-1)\pi^2} \mathbf{D} \hat{s}(N(\lambda_1 - \lambda_2)) \epsilon(N(\lambda_1 - \lambda_2)). \end{aligned} \quad (2.26)$$

The two-point correlation function in (2.26) consists of four terms involving derivatives and integrations of the S_i function; in order to simplify the calculations, we compute these correlators in the regimes of high- and low-temperature limits.

Low-temperature limit

In this regime, we may consider $\beta \rightarrow \infty$, so the Fermi-Dirac distribution might be approximated by

$$\frac{1}{e^{\beta E} + 1} \approx e^{-\beta E} . \quad (2.27)$$

Then the ensemble average is⁴

$$[n_{\beta}^{E_1} n_{\beta}^{E_2}] = \int d\lambda_1 d\lambda_2 e^{-\beta(\gamma+\lambda_1)} e^{-\beta(\gamma+\lambda_2)} \rho^{(2)}(\lambda_1, \lambda_2) .$$

The contribution of the first term in equation (2.26) is given by⁵

$$\begin{aligned} [n_{\beta}^{E_1} n_{\beta}^{E_2}]^{(1)} &= \int d\lambda_1 d\lambda_2 e^{-\beta(\gamma+\lambda_1)} e^{-\beta(\gamma+\lambda_2)} \frac{N}{N-1} \left(\frac{1}{2\pi} \sqrt{4-\lambda_1^2} \right) \left(\frac{1}{2\pi} \sqrt{4-\lambda_2^2} \right) \\ &= \frac{N}{N-1} \left(\int d\lambda e^{-\beta(\gamma+\lambda)} \frac{1}{2\pi} \sqrt{4-\lambda^2} \right)^2 = \frac{N}{N-1} [n_{\beta}^{\gamma+\lambda}]^2 . \end{aligned}$$

Performing the substitution $\lambda = 2 \cos(\theta)$ for $\pi \leq \theta \leq 2\pi$ we get

$$[n_{\beta}^{\lambda}] = \frac{2}{\pi} e^{-\beta\gamma} \int_0^{\pi} \sin^2(\theta) e^{-2\beta \cos(\theta)} d\theta .$$

We recognize this as one of the integral representations of the modified Bessel functions of the first kind, if z is a complex number and ν is the order, these are given by

$$I_{\nu}(z) = \frac{\left(\frac{1}{2}z\right)^{\nu}}{\pi^{1/2}\Gamma\left(\nu + \frac{1}{2}\right)} \int_0^{\pi} e^{\pm z \cos(\theta)} (\sin(\theta))^{2\nu} d\theta .$$

For $\nu = 1$:

$$I_1(2\beta) = \frac{2}{\pi} \beta \int_0^{\pi} e^{\pm 2\beta \cos(\theta)} \sin(\theta)^2 d\theta .$$

Then

$$[n_{\beta}^{\lambda}] = e^{-\beta\gamma} \frac{I_1(2\beta)}{\beta} .$$

Furthermore, it is known [38, 39] that in the limit $|z| \rightarrow \infty$ with $|Arg(z)| < \frac{\pi}{2}$, it holds

$$I_{\nu}(z) \rightarrow \frac{e^z}{\sqrt{2\pi z}} \left(1 + \mathcal{O}\left(\frac{1}{z}\right) \right) .$$

Concluding that the contribution of the disconnected part is

$$[n_{\beta}^{E_1} n_{\beta}^{E_2}]^{(1)} = \frac{N}{(N-1)} \frac{1}{4\pi\beta^3} e^{-2\beta(\gamma-2)} . \quad (2.28)$$

⁴Although all integrals over the λ values are taken from $-\infty$ to ∞ , along this section we sometimes omit the limits of integration in order to simplify the notation.

⁵From now on, the notation $[n_{\beta}^{E_1} n_{\beta}^{E_2}]^{(m)}$ refers to the contribution of the m -th term in Equation (2.26) to the two-point average $[n_{\beta}^{E_1} n_{\beta}^{E_2}]$ in (2.25).

The contribution of the second term involves the Si function squared

$$[n_\beta^{E_1} n_\beta^{E_2}]^{(2)} = \frac{N}{(N-1)\pi^2} \int d\lambda_1 d\lambda_2 \frac{1}{(e^{\beta(\gamma+\lambda_1)} + 1)} \frac{1}{(e^{\beta(\gamma+\lambda_2)} + 1)} \hat{s}(N(\lambda_1 - \lambda_2))^2.$$

It is convenient to introduce the change of variables $u_1 = \lambda_1 + \gamma - (\lambda_2 + \gamma) = \lambda_1 - \lambda_2$, and $u_2 = \lambda_2 + \gamma$. With this change and the low-temperature approximation in Equation (2.27), we get the following result:

$$[n_\beta^{E_1} n_\beta^{E_2}]^{(2)} = \frac{N}{(N-1)\pi^2} \int du_2 e^{-2\beta u_2} \int du_1 e^{-\beta u_1} \left(\frac{\sin(Nu_1)}{Nu_1} \right)^2.$$

The integral over u_2 is non-divergent for positive u_2 . Since we are free to choose the parameter γ , we can always choose this parameter such that u_2 is positive. The integral is then

$$\int_0^\infty du_2 e^{-2\beta u_2} = -\frac{e^{-2\beta u_2}}{2\beta} \Big|_0^\infty = \frac{1}{2\beta}.$$

For the integral with respect to u_1 we substitute $x = Nu_1$. Clearly x is different from zero since we assumed from the beginning that $\lambda_1 \neq \lambda_2$, obtaining

$$[n_\beta^{E_1} n_\beta^{E_2}]^{(2)} = \frac{1}{2\beta\pi^2(N-1)} \int_0^\infty dx e^{-\frac{\beta}{N}x} \left(\frac{\sin(x)}{x} \right)^2.$$

The last integral is solved numerically for positive values of x , the integral over x on the negative part is divergent unless we assume that $N \gg \beta$, in which case it contributes with a factor of $\pi/2$, and we ignore that contribution since we are more interested in the dependence on N . The second contribution in the low-temperature limit is therefore

$$[n_\beta^{E_1} n_\beta^{E_2}]^{(2)} = \frac{1}{2\beta\pi^2(N-1)} \left[\operatorname{arccot} \left(\frac{\beta}{2N} \right) - \frac{1}{4} \frac{\beta}{N} \ln \left(1 + \frac{4N^2}{\beta^2} \right) \right]. \quad (2.29)$$

The contribution of the third term in the low-temperature limit is

$$[n_\beta^{E_1} n_\beta^{E_2}]^{(3)} = \frac{N}{(N-1)\pi^2} \int du_2 e^{-2\beta u_2} \int du_1 e^{-\beta u_1} \frac{\partial}{\partial(Nu_1)} \left(\frac{\sin(Nu_1)}{Nu_1} \right) \int_0^{Nu_1} dt \frac{\sin(t)}{t}.$$

Again, doing the integration from 0 to ∞ and considering the new variable $x = Nu_1$, we get

$$[n_\beta^{E_1} n_\beta^{E_2}]^{(3)} = \frac{1}{2\beta\pi^2(N-1)} \int dx e^{-\frac{\beta}{N}x} \frac{d}{dx} \left(\frac{\sin(x)}{x} \right) \int_0^x dt \frac{\sin(t)}{t}.$$

In order to solve the integral

$$I = \int dx e^{-\frac{\beta}{N}x} \frac{d}{dx} \left(\frac{\sin(x)}{x} \right) \int_0^x dt \frac{\sin(t)}{t},$$

note that

$$\begin{aligned} \frac{d^2}{dx^2} \left(\int_0^x dt \frac{\sin(t)}{t} \right)^2 &= 2 \frac{d}{dx} \left[\left(\int_0^x dt \frac{\sin(t)}{t} \right) \frac{\sin(x)}{x} \right] \\ &= 2 \left[\left(\frac{\sin(x)}{x} \right)^2 + \left(\int_0^x dt \frac{\sin(t)}{t} \right) \frac{d}{dx} \left(\frac{\sin(x)}{x} \right) \right]. \end{aligned}$$

Then

$$\int dx e^{-\frac{\beta}{N}x} \frac{d^2}{dx^2} \left(\int_0^x dt \frac{\sin(t)}{t} \right) = 2 \int dx e^{-\frac{\beta}{N}x} \left(\frac{\sin(x)}{x} \right)^2 + 2I.$$

Now, all these integrals are non-negligible for $1 \gg x \frac{\beta}{N}$ since in the low-temperature limit $\beta \rightarrow \infty$, this implies that the integrals contribute for really small values of x . In this case, we may approximate $\sin(x) \approx x$, obtaining

$$\int dx e^{-\frac{\beta}{N}x} \frac{d^2}{dx^2} \left(\int_0^x dt \right) = 2 \int dx e^{-\frac{\beta}{N}x} + 2I.$$

So

$$I \approx - \int_0^\infty dx e^{-\frac{\beta}{N}x} = -\frac{N}{\beta}.$$

The third contribution is therefore

$$[n_\beta^{E_1} n_\beta^{E_2}]^{(3)} \approx -\frac{N}{2\beta^2 \pi^2 (N-1)}. \quad (2.30)$$

Now, the contribution of the fourth term is divided into two parts, depending on the sign of the integration variable:

$$\begin{aligned} [n_\beta^{E_1} n_\beta^{E_2}]^{(4)} &= \frac{N}{2\pi^2 (N-1)} \int_{-\infty}^\infty du_2 e^{-2\beta u_2} \left[- \int_{-\infty}^0 du_1 e^{-\beta u_1} \frac{d}{d(Nu_1)} \left(\frac{\sin(Nu_1)}{Nu_1} \right) \right. \\ &\quad \left. \int_0^\infty du_1 e^{-\beta u_1} \frac{d}{d(Nu_1)} \left(\frac{\sin(Nu_1)}{Nu_1} \right) \right]. \end{aligned}$$

Again, we consider only the non-divergent term and substitute $x = Nu_1$ obtaining

$$[n_\beta^{E_a} n_\beta^{E_b}]^{(4)} = \frac{1}{4\beta\pi^2 (N-1)} \left[\int_0^\infty dx e^{-\frac{\beta}{N}x} \frac{d}{dx} \left(\frac{\sin(x)}{x} \right) \right].$$

Now, the integral is solved with integration by parts, giving the final result

$$[n_\beta^{E_a} n_\beta^{E_b}]^{(4)} = \frac{1}{4\beta\pi^2 (N-1)} \left(-1 + \frac{\beta}{N} \operatorname{arccot} \left(\frac{\beta}{N} \right) \right). \quad (2.31)$$

Joining Equations (2.28), (2.29), (2.30), and (2.31), we conclude that the ensemble average in the low-temperature limit is given by;

$$[n_\beta^{E_1} n_\beta^{E_2}] = \frac{N}{(N-1)} \frac{1}{4\pi\beta^3} e^{-2\beta(\gamma-2)} - \frac{1}{2\beta\pi^2(N-1)} \left[\operatorname{arccot} \left(\frac{\beta}{2N} \right) - \frac{1}{4} \frac{\beta}{N} \ln \left(1 + \frac{4N^2}{\beta^2} \right) \right] - \frac{N}{2\beta^2\pi^2(N-1)} - \frac{1}{4\beta\pi^2(N-1)} \left(-1 + \frac{\beta}{N} \operatorname{arccot} \left(\frac{\beta}{N} \right) \right). \quad (2.32)$$

We recall that Equation (2.32) is valid for $u_1 = \lambda_1 - \lambda_2 \in (0, \infty)$ and $u_2 = \lambda_2 + \gamma \in (0, \infty)$.

In the low-temperature regime, the contribution of the first three terms in the last expression is negligible, and to order $1/N$ in the system's size we conclude that

$$[n_\beta^{E_1} n_\beta^{E_2}] \approx -\frac{1}{2\beta^2\pi^2} + \frac{1}{4\beta\pi^2 N}. \quad (2.33)$$

High-temperature limit

In this regime, we consider $\beta \rightarrow 0$, which allows a double Taylor series expansion of the exponential function

$$\frac{1}{e^{\beta(\gamma+\lambda)} + 1} \approx \frac{1}{\beta(\gamma+\lambda) + 2} \approx \frac{1}{2} - \frac{1}{4}\beta(\gamma+\lambda). \quad (2.34)$$

Substitution of (2.34) in (2.23) for the first contribution gives

$$[n_\beta^{E_1} n_\beta^{E_2}]^{(1)} \approx \int d\lambda_1 d\lambda_2 \left[\left(\frac{1}{2} - \frac{1}{4}\beta(\gamma+\lambda_1) \right) \left(\frac{1}{2} - \frac{1}{4}\beta(\gamma+\lambda_2) \right) \frac{N}{N-1} \left(\frac{1}{2\pi} \sqrt{4-\lambda_1^2} \right) \left(\frac{1}{2\pi} \sqrt{4-\lambda_2^2} \right) \right].$$

The simplification of this integral to the first order in β results in the first moment of λ that cancels out due to the symmetric interval of integration. Additionally, the integral over the whole probability distribution sums to one due to normalization. The contribution of the first term is then given by

$$[n_\beta^{E_1} n_\beta^{E_2}]^{(1)} \approx \frac{N}{N-1} \left(\frac{1}{4} - \frac{1}{4}\beta\gamma \right). \quad (2.35)$$

The contribution of the second term is the following:

$$[n_\beta^{E_1} n_\beta^{E_2}]^{(2)} \approx \frac{N}{(N-1)\pi^2} \int \int du_1 du_2 \left(\frac{1}{2} - \frac{1}{4}\beta(u_1+u_2) \right) \left(\frac{1}{2} - \frac{1}{4}\beta u_2 \right) \left(\frac{\sin(Nu_1)}{Nu_1} \right)^2.$$

Here, we have used the substitutions $u_1 = \lambda_1 + \gamma - (\lambda_2 + \gamma) = \lambda_1 - \lambda_2$ and $u_2 = \lambda_2 + \gamma$. Simplification of the integrand to the first order in β gives rise to integrals of odd functions over symmetric intervals and integrals that are easily solved numerically, yielding

$$[n_\beta^{E_1} n_\beta^{E_2}]^{(2)} \approx \frac{1}{4\pi(N-1)} \int du_2 .$$

To regularize the integral with respect to u_2 , we assume that the eigenvalues are small enough so that one can cut the integration interval around zero: In this regime, the Wigner semicircle distribution might be approximated by the constant $\frac{1}{\pi}$. Imposing the normalization condition implies that the integration interval must be $[-\frac{\pi}{2}, \frac{\pi}{2}]$. This method, is called the box approximation [37], is a brute-force approach used to obtain a finite value in divergent integrals. Using this approximation, we get

$$\int_{-\infty}^{\infty} du_2 \approx \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} du_2 = \pi .$$

Thus, the contribution of the second term is

$$[n_\beta^{E_1} n_\beta^{E_2}]^{(2)} \approx \frac{1}{4(N-1)} . \quad (2.36)$$

Now, the third term is given by

$$[n_\beta^{E_1} n_\beta^{E_2}]^{(3)} \approx \frac{N}{(N-1)\pi^2} \int \int du_1 du_2 \left(\frac{1}{2} - \frac{1}{4}\beta(u_1 + u_2) \right) \left(\frac{1}{2} - \frac{1}{4}\beta u_2 \right) \frac{\partial}{\partial(Nu_1)} \left(\frac{\sin(Nu_1)}{Nu_1} \right) \int_0^{Nu_1} dt \frac{\sin(t)}{t} .$$

Again, we simplify the integrand to first order in β and use the substitution $x = Nu_1$ to obtain

$$[n_\beta^{E_1} n_\beta^{E_2}]^{(3)} \approx \frac{1}{2(N-1)\pi} \left[\int_0^\infty dx \frac{d}{dx} \left(\frac{\sin(x)}{x} \right) \int_0^x dt \frac{\sin(t)}{t} \right] .$$

The last integral is solved using integration by parts together with the Leibniz integral rule, obtaining

$$[n_\beta^{E_1} n_\beta^{E_2}]^{(3)} \approx -\frac{1}{4(N-1)} . \quad (2.37)$$

Finally, the contribution of the fourth term to first order in β is

$$[n_\beta^{E_1} n_\beta^{E_2}]^{(4)} = \frac{N}{2\pi^2(N-1)} \left[- \int_{-\infty}^{\infty} du_2 \int_{-\infty}^0 du_1 \left(\frac{1}{4} - \frac{1}{8}\beta u_1 - \frac{1}{4}\beta u_2 \right) \frac{d}{d(Nu_1)} \left(\frac{\sin(Nu_1)}{Nu_1} \right) \right. \\ \left. \int_{-\infty}^{\infty} du_2 \int_0^\infty du_1 \left(\frac{1}{4} - \frac{1}{8}\beta u_1 - \frac{1}{4}\beta u_2 \right) \frac{d}{d(Nu_1)} \left(\frac{\sin(Nu_1)}{Nu_1} \right) \right] .$$

Using the substitution $x = Nu_1$ and applying the box approximation to the integral with respect to u_2 , we get

$$[n_\beta^{E_1} n_\beta^{E_2}]^{(4)} \approx \frac{1}{2\pi(N-1)} \left[-\frac{1}{2} - \frac{\beta}{4N} \int_0^\infty dx \cos(x) + \frac{\beta}{4N} \int_0^\infty dx \frac{\sin(x)}{x} \right].$$

Applying the box approximation now for the integral with respect to u_1 , we get the following:

$$[n_\beta^{E_1} n_\beta^{E_2}]^{(4)} = -\frac{1}{4\pi(N-1)} - \frac{\beta}{8\pi N(N-1)} \sin(N\pi) + \frac{\beta}{8\pi N(N-1)} \text{Si}(N\pi). \quad (2.38)$$

Combining Equations (2.35), (2.36), (2.37), and (2.38), we get the result for the high-temperature limit

$$[n_\beta^{E_1} n_\beta^{E_2}] = \frac{N}{N-1} \left(\frac{1}{4} - \frac{1}{4}\beta\gamma \right) - \frac{1}{2(N-1)} + \frac{1}{4\pi(N-1)} + \frac{\beta}{8\pi N(N-1)} \sin(N\pi) - \frac{\beta}{8\pi N(N-1)} \text{Si}(N\pi). \quad (2.39)$$

Due to the box approximation used to evaluate the integrals, Equation (2.39) is only valid for $u_1 = \lambda_1 - \lambda_2 \in [-\pi, \pi]$ and $u_2 = \lambda_2 + \gamma \in [-\frac{\pi}{2} + \gamma, \frac{\pi}{2} + \gamma]$.

In the high-temperature regime, the dominant contribution comes from the terms independent of β , that is

$$[n_\beta^{E_1} n_\beta^{E_2}] \approx \frac{1}{4} - \frac{1}{2N} + \frac{1}{4\pi(N-1)}. \quad (2.40)$$

The calculation of the term $[n_\beta^{\gamma+\lambda_1} n_\beta^{\gamma+\lambda_2}]$ when $\lambda_1 = \lambda_2 = \lambda$ involves a marginal probability density function that corresponds to the first term in Equation (2.26), which in the high- and low-temperature limits may be written in terms of the disconnected average $[n_\beta^{\gamma+\lambda}]^2 = [n_\beta]^2$ as

$$[n_\beta^{\gamma+\lambda} n_\beta^{\gamma+\lambda}] = [(n_\beta^{\gamma+\lambda})^2] = [n_\beta^2] = \begin{cases} [n_\beta]^2 = \frac{1}{4} - \frac{1}{4}\beta\gamma, & \beta \rightarrow 0, \\ \sqrt{\frac{\pi}{2}}\beta^{3/2}[n_\beta]^2, & \beta \rightarrow \infty. \end{cases} \quad (2.41)$$

Now, plugging the energy averages for high- and low-temperatures given by Equations (2.33), (2.40), and (2.50) back into Equation (2.24), we obtain the expression for the thermal correlator $[C_{ij}^\beta C_{kl}^\beta]$. To order $1/N$, the result of the variance is, therefore,

$$[C_{ij}^\beta C_{kl}^\beta] - [C_{ij}^\beta][C_{kl}^\beta] \approx \begin{cases} ([n_\beta]^2 - \frac{1}{2} + \frac{1}{4\pi}) \frac{1}{N} \delta_{ij} \delta_{kl} + ([n_\beta^2] - [n_\beta]^2) \frac{1}{N} (\delta_{il} \delta_{kj} + \delta_{ik} \delta_{jl}), & \beta \rightarrow 0, \\ \left(\frac{[n_\beta]^2}{N} - \frac{1}{\beta^2 \pi^2} \right) \delta_{ij} \delta_{kl} + \frac{1}{N} \left(\frac{1}{\beta^2 \pi^2} + [n_\beta^2] - [n_\beta]^2 \right) (\delta_{il} \delta_{kj} + \delta_{ik} \delta_{jl}), & \beta \rightarrow \infty. \end{cases} \quad (2.42)$$

2.3 Gaussian unitary ensemble

2.3.1 Correlators in eigenvectors

The two- and four-point correlators over the GUE differ from the last calculations in the result of the k -fold quantum channel. The 1-fold quantum channel for the orthogonal group coincides with the 1-fold quantum channel for the unitary group. Then, the two-point correlation in the eigenvectors for GUE is given by the same Equation (2.8) as for GOE. According to Equation (64) in Appendix E.2, the two-fold quantum channel for GUE acting on the operator $\mathbb{1}_{\mathcal{A}} \otimes \mathbb{1}_{\mathcal{A}}$ with $\mathbb{1}_{\mathcal{A}} = \sum_{a=1}^N |a\rangle \langle a| n_{\beta}^{E_a}$ as defined before, is given by

$$\Phi_2^{\text{Haar}}(\mathbb{1}_{\mathcal{A}} \otimes \mathbb{1}_{\mathcal{A}}) = \frac{1}{N^2 - 1} \left[\text{Tr} \left(\text{---} \mathbb{1}_{\mathcal{A}} \otimes \mathbb{1}_{\mathcal{A}} \text{---} \right) \text{---} + \text{Tr} \left(\text{---} \mathbb{1}_{\mathcal{A}} \otimes \mathbb{1}_{\mathcal{A}} \text{---} \right) \text{---} \right. \\ \left. - \frac{1}{N} \text{Tr} \left(\text{---} \mathbb{1}_{\mathcal{A}} \otimes \mathbb{1}_{\mathcal{A}} \right) \text{---} - \frac{1}{N} \text{Tr} \left(\text{---} \mathbb{1}_{\mathcal{A}} \otimes \mathbb{1}_{\mathcal{A}} \right) \text{---} \right]. \quad (2.43)$$

Compared with the result for GOE, (2.11), we see in this case, that the only permutation operators that appear are the identity and the swap.

By Equations (2.15) and (2.16), we get

$$\Phi_2^{\text{Haar}}(\mathbb{1}_{\mathcal{A}} \otimes \mathbb{1}_{\mathcal{A}}) = \frac{1}{N^2 - 1} \left(N_p^2 \text{---} + N_p \text{---} - \frac{N_p^2}{N} \text{---} - \frac{N_p}{N} \text{---} \right). \quad (2.44)$$

Similarly to Equation (2.10), we have that the two-point correlator in the effective description is given by

$$[C_{ij}^{\Psi} C_{kl}^{\Psi}] = \langle i | \text{---} \left(\int [U^{\dagger} U] \begin{array}{c} -U^{\dagger} - \mathbb{1}_{\mathcal{A}} - U - \\ -U^{\dagger} - \mathbb{1}_{\mathcal{A}} - U - \end{array} \text{---} | j \rangle \right) \langle k | \text{---} \left(\int [U^{\dagger} U] \begin{array}{c} -U^{\dagger} - \mathbb{1}_{\mathcal{A}} - U - \\ -U^{\dagger} - \mathbb{1}_{\mathcal{A}} - U - \end{array} \text{---} | l \rangle \right) = \langle i | \text{---} \left[\Phi_2^{\text{Haar}}(\mathbb{1}_{\mathcal{A}} \otimes \mathbb{1}_{\mathcal{A}}) \right] \text{---} | j \rangle \langle k | \text{---} \left[\Phi_2^{\text{Haar}}(\mathbb{1}_{\mathcal{A}} \otimes \mathbb{1}_{\mathcal{A}}) \right] \text{---} | l \rangle. \quad (2.45)$$

Substituting the quantum channel (2.44), we obtain

$$[C_{ij}^{\Psi} C_{kl}^{\Psi}] = \frac{1}{N^2 - 1} \left(\left(N_p^2 - \frac{N_p}{N} \right) \delta_{ij} \delta_{kl} + N_p \delta_{il} \delta_{kj} - \frac{N_p^2}{N} \delta_{il} \delta_{kj} \right).$$

Thus, the variance in eigenvectors is given by,

$$[C_{ij}^{\Psi} C_{kl}^{\Psi}] - [C_{ij}^{\Psi}][C_{kl}^{\Psi}] = \frac{1}{N^2 - 1} \left(\frac{N_p^2}{N^2} - \frac{N_p}{N} \right) \delta_{ij} \delta_{kl} \pm \frac{1}{N^2 - 1} \left(N_p - \frac{N_p^2}{N} \right) \delta_{il} \delta_{kj}.$$

As in the case of $O(N)$, we take the limit of large N up to order $1/N$ of this expression while keeping the density $\zeta = N_p/N$ fixed, obtaining

$$[C_{ij}^{\Psi} C_{kl}^{\Psi}] - [C_{ij}^{\Psi}][C_{kl}^{\Psi}] = \frac{\zeta}{N} (1 - \zeta) \delta_{il} \delta_{kj}. \quad (2.46)$$

Clearly, in the limit in which N_p is fixed and N is big such that $\zeta \rightarrow 0$ goes to zero, the variance is of the order of $\frac{N_p}{N^2} \delta_{il} \delta_{kj}$.

We conclude that the correlator in this symmetry class might be written as follows,

$$C_{ij}^\Psi \approx [n_\beta] \delta_{ij} + \frac{\zeta}{N} (1 - \zeta) \delta_{il} \delta_{kj}. \quad (2.47)$$

2.3.2 Thermal correlators

As the one-fold quantum channel is the same for both GOE and GUE, the two-point correlation in the thermal ensemble coincides also. Then the two-point correlator is the same as in (2.23).

By the same procedure as in Equation (2.24), but now using the two-fold quantum channel for GUE, Equation (2.44), we obtain the variance

$$\begin{aligned} [C_{ij}^\beta C_{kl}^\beta] - [C_{ij}^\beta][C_{kl}^\beta] &= \sum_{a,b=1}^N [\eta_\beta^{\gamma+\lambda_1} \eta_\beta^{\gamma+\lambda_2}] \frac{1}{N^2 - 1} \left(\delta_{ij} \delta_{kl} - \frac{1}{N} \delta_{il} \delta_{kj} \right) \\ &+ \frac{N}{N^2 - 1} [\eta_\beta^2] \left(\delta_{il} \delta_{kj} - \frac{1}{N} \delta_{ij} \delta_{kl} \right) - [\eta_\beta]^2 \delta_{ij} \delta_{kl}. \end{aligned} \quad (2.48)$$

2.3.3 Energy average of the 2-point correlator for GUE

Now, using the kernel for the GUE, we find the two-point correlation function to be

$$\rho^{(2)}(\lambda_1, \lambda_2) = \frac{(N-2)!}{N!} \left[\frac{N^2}{4\pi^2} \sqrt{4 - \lambda_1^2} \sqrt{4 - \lambda_2^2} - \frac{N^2}{\pi^2} \frac{\sin^2(N(\lambda_1 - \lambda_2))}{N^2(\lambda_1 - \lambda_2)^2} \right]. \quad (2.49)$$

The first term corresponds to the disconnected part. The contribution of this was calculated for GOE in high- and low-temperature limits in Equations (2.35) and (2.28), respectively:

$$[\eta_\beta^{\gamma+\lambda_1} \eta_\beta^{\gamma+\lambda_2}]^{(1)} \approx \begin{cases} \frac{N}{N-1} \left(\frac{1}{4} - \frac{1}{4} \beta \gamma \right) & \text{for } \beta \rightarrow 0, \\ \frac{N}{4\pi\beta^3(N-1)} e^{-2\beta(\gamma-2)} & \text{for } \beta \rightarrow \infty. \end{cases}$$

The second term corresponds to the connected part, and the high- and low-temperature limits are given by Equations (2.36) and (2.29), respectively,

$$[\eta_\beta^{\gamma+\lambda_1} \eta_\beta^{\gamma+\lambda_2}]^{(2)} \approx \begin{cases} \frac{1}{4(N-1)} & \text{for } \beta \rightarrow 0, \\ \frac{1}{2\beta\pi^2(N-1)} \left[\operatorname{arccot} \left(\frac{\beta}{2N} \right) - \frac{1}{4} \frac{\beta}{N} \ln \left(1 + \frac{4N^2}{\beta^2} \right) \right] & \text{for } \beta \rightarrow \infty. \end{cases}$$

Substituting these results into Equation (2.48), we obtain the variance of the average over the GUE in the high- and low-temperature limits:

$$[C_{ij}^\beta C_{kl}^\beta] - [C_{ij}^\beta][C_{kl}^\beta] \approx \begin{cases} -\frac{1}{4N} \delta_{ij} \delta_{kl} + \frac{1}{N} ([n_\beta^2] - [n_\beta]^2) \delta_{il} \delta_{kj} & \text{for } \beta \rightarrow 0, \\ -\frac{1}{2\pi^2\beta^2} \delta_{ij} \delta_{kl} + \frac{1}{N} ([n_\beta^2] - [n_\beta]^2) \delta_{il} \delta_{kj} & \text{for } \beta \rightarrow \infty. \end{cases} \quad (2.50)$$

2.4 Analytical results

The results of the last sections are condensed in the following tables.

	$[C_{ij}^\beta]$	$[C_{ij}^\Psi]$	$[C_{ij}^\Psi C_{kl}^\Psi] - [C_{ij}^\Psi][C_{kl}^\Psi]$
GOE	$[n_\beta]\delta_{ij}$	$\zeta\delta_{ij}$	$\frac{\zeta}{N}(1-\zeta)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$
GUE	$[n_\beta]\delta_{ij}$	$\zeta\delta_{ij}$	$\frac{\zeta}{N}(1-\zeta)\delta_{il}\delta_{kj}$

Table 2.1: **Column 2:** two-point thermal correlator averaged over $O(N)$ and $U(N)$, valid for all temperatures. **Columns 3 and 4:** Asymptotic behavior, $N \rightarrow \infty$ with $\frac{N_p}{N} = \zeta$ fixed, of the two-point correlator and its variance in eigenstates averaged over $O(N)$ and $U(N)$, valid for all temperatures.

	$[C_{ij}^\beta C_{kl}^\beta] - [C_{ij}^\beta][C_{kl}^\beta]$ (bajas temperaturas)
GOE	$\left(\frac{[n_\beta]^2}{N} - \frac{1}{\beta^2\pi^2}\right)\delta_{ij}\delta_{kl} + \frac{1}{N}\left(\frac{1}{\beta^2\pi^2} + [n_\beta^2] - [n_\beta]^2\right)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$
GUE	$-\frac{1}{2\pi^2\beta^2}\delta_{ij}\delta_{kl} + \frac{1}{N}([n_\beta^2] - [n_\beta]^2)\delta_{il}\delta_{kj}$

Table 2.2: Asymptotic behavior, $N \rightarrow \infty$ with N_p fixed, of the thermal variance averaged over $O(N)$ and $U(N)$ at low temperatures, $\beta \rightarrow \infty$.

	$[C_{ij}^\beta C_{kl}^\beta] - [C_{ij}^\beta][C_{kl}^\beta]$ (altas temperaturas)
GOE	$\left([n_\beta]^2 - \frac{1}{2} + \frac{1}{4\pi}\right)\frac{1}{N}\delta_{ij}\delta_{kl} + ([n_\beta^2] - [n_\beta]^2)\frac{1}{N}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$
GUE	$-\frac{1}{4N}\delta_{ij}\delta_{kl} + \frac{1}{N}([n_\beta^2] - [n_\beta]^2)\delta_{il}\delta_{kj}$

Table 2.3: Asymptotic behavior, $N \rightarrow \infty$ with N_p fixed, of the thermal variance averaged over $O(N)$ and $U(N)$ at high temperatures, $\beta \rightarrow 0$.

	$[C_{ij}^\Psi]$	$[C_{ij}^\Psi C_{kl}^\Psi] - [C_{ij}^\Psi][C_{kl}^\Psi]$
GOE	$\frac{N_p}{N}\delta_{ij}$	$\frac{N_p}{N^2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$
GUE	$\frac{N_p}{N}\delta_{ij}$	$\frac{N_p}{N^2}\delta_{il}\delta_{kj}$

Table 2.4: Asymptotic behavior, $N \rightarrow \infty$ with N_p fixed, of the two-point correlator and its variance in eigenstates averaged over $O(N)$ and $U(N)$, valid for all temperatures. This corresponds to dropping the term ζ^2 in the results of Table 2.1.

Note that, since there is no special structure for each operator $\hat{d}_a$ and $\hat{d}_a^\dagger$, and all states $\binom{N}{N_p}$ are physically equivalent, the only relevant parameter is the number of particles N_p . This is reflected in the results of Table 2.1 for the first moment of the two-point correlator. Furthermore, from Table 2.1 we see that for both GOE and GUE, the first moment of the two-point correlator is diagonal in eigenstates and in a thermal state, and they are equal for any temperature β whenever β is selected by imposing the equality $[n_\beta] = \frac{N_p}{N}$.

According to Equation (1.16), the generalized ETH is the statement that the cactus-type diagrams in Figure 1.8 are the only ones that contribute to the thermal moments. Thus, for the thermal operator defined before,

$$C^\beta = O^\top \mathbb{1}_\beta O, \quad \mathbb{1}_\beta = \sum_{a=1}^N |a\rangle \langle a| \frac{1}{e^{\beta E_a} + 1}.$$

The thermal contribution to the average $[C_{ij}^\beta C_{kl}^\beta]$ comes only from the terms of the form $[C_{ij}^\beta C_{ji}^\beta]$, that is, the variance should be of the form,

$$[C_{ij}^\beta C_{kl}^\beta] - [C_{ij}^\beta][C_{kl}^\beta] \sim \delta_{il}\delta_{jk} + \delta_{ij}\delta_{kl}.$$

For both ensembles. However, the variance at low- and high-temperature presents different contractions for each ensemble, which is a consequence of the different symmetry properties in the system, these symmetries are not lost in the limit of high Hilbert space dimension. In Tables 2.2 and 2.3 we observe the presence of the additional contraction $\delta_{ik}\delta_{jl}$ in the variance term for GOE, in calculations of Section 2.1 we considered a large- N limit, while keeping the N -dependence on the obtained quantities. For this reason, we still conserve the structure of the symmetry of the system, and the presence of the additional contraction is precisely the subleading contribution not captured by ETH and free probability that was mentioned in Section 1.3.3. This shows that the generalized version of ETH must be used with caution when considering non-trivial thermodynamic limits.

The direct comparison with the averages of free probability goes as follows. Since $C_{ij}^\Psi = \langle i | O^\top \mathbb{1}_\mathcal{A} O | j \rangle$, we can write it in matrix form as $C^\Psi = O^\top \mathbb{1}_\mathcal{A} O$. Then, we might write⁶

$$[C^\Psi]^U = [C^\Psi]^O = \int [O^\top dO] O^\top \mathbb{1}_\mathcal{A} O = \Phi_1^{H_{\text{aar}}}(\mathbb{1}_\mathcal{A}) \mathbb{1} = \frac{\text{Tr}(\mathbb{1}_\mathcal{A})}{N} \mathbb{1} = \frac{N_p}{N} \mathbb{1}.$$

That is, a diagonal matrix with diagonal elements $\frac{N_p}{N}$ as found before for $[C_{ij}^\Psi]^O$. And since the one-fold quantum channel for the unitary group is the same, we have $[C^\Psi]^O = [C^\Psi]^U$.

The last result implies that $[C^\Psi]_{ij}^O = [C_{ij}^\Psi]^O$ and the same holds for $U(N)$. Now, notice that for the two-point correlator in the effective description, we have $[C^\Psi C^\Psi]_{ij} \neq [(C^\Psi C^\Psi)_{ij}]$, since

$$\begin{aligned} [C^\Psi C^\Psi]^U &= [C^\Psi C^\Psi]^O \\ &= \int [O^\top dO] (O^\top \mathbb{1}_\mathcal{A} O) (O^\top \mathbb{1}_\mathcal{A} O) = \int [O^\top dO] O^\top \mathbb{1}_\mathcal{A}^2 O = \int [O^\top dO] O^\top \mathbb{1}_\mathcal{A} O \\ &= \Phi_1^{H_{\text{aar}}}(\mathbb{1}_\mathcal{A}) \mathbb{1} = \frac{N_p}{N} \mathbb{1}. \end{aligned} \tag{2.51}$$

Again, a diagonal matrix with components $\frac{N_p}{N}$. Meanwhile, from Equation (2.18) for the average over $O(N)$, we have

⁶In this section $[.]^U$, $[.]^O$ and $[.]^{FP}$ denotes the ensemble average calculated over $O(N)$, $U(N)$ and using free probability respectively.

$$[C_{ik}^\Psi C_{kj}^\Psi] = \frac{1}{N^2 + N - 2} \left[\left(\frac{(1+N)}{N} N_p - 2 \frac{N_p}{N} \right) \delta_{ik} \delta_{kj} + \left(N_p - \frac{N_p^2}{N} \right) \delta_{ij} \delta_{kk} + \left(N_p - \frac{N_p^2}{N} \right) \delta_{ik} \delta_{kj} \right] = \frac{N_p}{N} \left(\frac{N_p + 2}{N + 2} \right) \delta_{ij} .$$

So,

$$(C^\Psi C^\Psi)_{ij} = \sum_{k=1}^N C_{ik}^\Psi C_{kj}^\Psi \rightarrow [(C^\Psi C^\Psi)_{ij}] = \sum_{k=1}^N [C_{ik}^\Psi C_{kj}^\Psi] = N \frac{N_p}{N} \left(\frac{N_p + 2}{N + 2} \right) \delta_{ij} .$$

Here, we can see that the RMT calculation allows us to find the ensemble average of the components, which are different from the averages obtained by free probability, which require for us to consider averages of the operator as a whole matrix. It is straightforward to check that

$$\frac{1}{N} \text{Tr}([C^\Psi C^\Psi]^U) = \frac{1}{N} \text{Tr}([C^\Psi C^\Psi]^O) = \frac{N_p}{N} .$$

Furthermore, plugging the operators $A = \mathbb{1}$ and $B = \mathbb{1}_A$ into Equations (1.22) and (1.21), or considering directly Equation (2.51), we obtain the free average

$$\begin{aligned} \langle \mathbb{1}_A U \mathbb{1} U^\dagger \mathbb{1}_A U \mathbb{1} U^\dagger \rangle &= \langle \mathbb{1}_A O \mathbb{1} O^\dagger \mathbb{1}_A O \mathbb{1} O^\dagger \rangle = \langle \mathbb{1}^2 \rangle \langle \mathbb{1}_A \rangle^2 + \langle \mathbb{1} \rangle^2 \langle \mathbb{1}_A^2 \rangle - \langle \mathbb{1} \rangle^2 \langle \mathbb{1}_A \rangle^2 \\ &= \frac{1}{N} \text{Tr}(\mathbb{1}) \frac{1}{N^2} \text{Tr}(\mathbb{1}_A)^2 + \frac{1}{N^2} \text{Tr}(\mathbb{1})^2 \frac{1}{N} \text{Tr}(\mathbb{1}_A) - \frac{1}{N^2} \text{Tr}(\mathbb{1})^2 \frac{1}{N^2} \text{Tr}(\mathbb{1}_A)^2 \\ &= \frac{1}{N} N \frac{1}{N^2} N_p^2 + \frac{1}{N^2} N^2 \frac{1}{N} N_p - \frac{1}{N^2} N^2 \frac{1}{N^2} N_p^2 = \frac{N_p}{N} = \frac{1}{N} \text{Tr}([C^\Psi C^\Psi]^{FP}) . \end{aligned}$$

And by the definition of the free average it is also clear that

$$\langle U^\dagger \mathbb{1}_A U \rangle = \langle O^\dagger \mathbb{1}_A O \rangle = \frac{1}{N} \text{Tr}(\Phi_1^{Haar}(\mathbb{1}_A)) = \frac{N_p}{N} .$$

Under this description, we then conclude that

$$[C^\Psi]^{FP} = [C^\Psi]^O = [C^\Psi]^U ,$$

and

$$\frac{1}{N} \text{Tr}([C^\Psi C^\Psi]^{FP}) = \frac{1}{N} \text{Tr}([C^\Psi C^\Psi]^O) = \frac{1}{N} \text{Tr}([C^\Psi C^\Psi]^U) .$$

This is as far as one can go in comparing the three averages in terms of the whole matrix. This is the reason why we instead computed the average of the components, since in this way the comparison with ETH is direct.

Another important remark, and in some way a sanity check for our results, is that in the limit of large N with fixed N_p , shown in Table 2.4, we recover the results that we had obtained if, instead of doing an exact calculation, we would have approximated the terms $[O_{ai} O_{aj}]$ by the known assumption of i.i.d. Gaussian entries:

$$[O_{ai}] = 0 \quad [O_{ai}O_{bj}] = \frac{1}{N}\delta_{ab}\delta_{ij}, \quad (2.52)$$

where the square brackets $[\cdot]$ denote the average over the orthogonal group. This is equivalent to say that we consider each eigenvector Haar distributed itself [40], ignoring the correlations imposed by the orthogonality condition. With this assumption, the mean value of the 2-point correlation function is given by

$$[C_{ij}^\Psi] = \sum_{a \in \mathcal{A}} [O_{ai}O_{aj}] = \sum_{a \in \mathcal{A}} \frac{1}{N}\delta_{ij} = \frac{N_p}{N}\delta_{ij},$$

with standard deviation given by

$$\begin{aligned} [C_{ij}^\Psi C_{kl}^\Psi] - [C_{ij}^\Psi] [C_{kl}^\Psi] &= \left[\left(\sum_{a \in \mathcal{A}} O_{ai}O_{aj} \right) \left(\sum_{b \in \mathcal{A}} O_{bk}O_{bl} \right) \right] - [C_{ij}^\Psi] [C_{kl}^\Psi] \\ &= \left[\sum_{a, b \in \mathcal{A}} O_{ai}O_{aj}O_{bk}O_{bl} \right] - [C_{ij}^\Psi] [C_{kl}^\Psi] = \sum_{a, b \in \mathcal{A}} [O_{ai}O_{aj}O_{bk}O_{bl}] - [C_{ij}^\Psi] [C_{kl}^\Psi]. \end{aligned} \quad (2.53)$$

And now since the entries O_{ai} are Gaussian distributed, we can use Wick's theorem to compute the four-point correlation function in Equation (2.53). This theorem states that the average of the four-point correlation function $[O_{ai}O_{aj}O_{bk}O_{bl}]$ can be expanded as a sum of two-point averages:

$$\begin{aligned} [O_{ai}O_{aj}O_{bk}O_{bl}] &= [O_{ai}O_{aj}][O_{bk}O_{bl}] + [O_{ai}O_{bk}][O_{aj}O_{bl}] + [O_{ai}O_{bl}][O_{aj}O_{bk}] \\ &= \left(\frac{1}{N}\delta_{ij} \right) \left(\frac{1}{N}\delta_{kl} \right) + \left(\frac{1}{N}\delta_{ab}\delta_{ik} \right) \left(\frac{1}{N}\delta_{ab}\delta_{jl} \right) + \left(\frac{1}{N}\delta_{ab}\delta_{il} \right) \left(\frac{1}{N}\delta_{ab}\delta_{jk} \right), \end{aligned}$$

where we have used condition (2.52). The variance is then given by

$$\begin{aligned} [C_{ij}^\Psi C_{kl}^\Psi] - [C_{ij}^\Psi] [C_{kl}^\Psi] &= \sum_{a, b \in \mathcal{A}} \left(\frac{1}{N^2}\delta_{ij}\delta_{kl} + \frac{1}{N^2}\delta_{ab}\delta_{ik}\delta_{jl} + \frac{1}{N^2}\delta_{ab}\delta_{il}\delta_{jk} \right) - \frac{N_p^2}{N^2}\delta_{ij}\delta_{kl} \\ &= \frac{N_p}{N^2} (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}). \end{aligned}$$

Then, the two-point correlation function in eigenstates may be written as a sum of an average diagonal part plus a noise part that is subleading as N increases,

$$C_{ij}^\Psi = \frac{N_p}{N}\delta_{ij} \pm \frac{N_p}{N^2} (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}). \quad (2.54)$$

The same result as in Table 2.4. The last equivalence can be represented by the following commutative diagram

$$\begin{array}{ccc}
 & C_{ij}^\Psi = [C_{ij}^\Psi] \pm [C_{ij}^\Psi C_{kl}^\Psi] - [C_{ij}^\Psi][C_{kl}^\Psi] & \\
 \nearrow \text{Exact calculation} & & \searrow \text{Thermodynamic limit } N \rightarrow \infty \\
 C_{ij}^\Psi = \sum_{a \in \mathcal{A}} \psi_i^a \psi_j^a & \xrightarrow{[\psi_i^a] = 0, [\psi_i^a \psi_j^b] = \frac{1}{N} \delta_{ab} \delta_{ij} \text{ and Wick's theorem}} & C_{ij}^\Psi = \frac{N_p}{N} \delta_{ij} \pm \frac{N_p}{N^2} (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})
 \end{array}$$

2.5 Numerical solution

As in the case of the SYK model in Section 1.2.1, we performed several experiments for the expectation value of the number operator at site s in the states $|\Psi^{N_p}\rangle$.

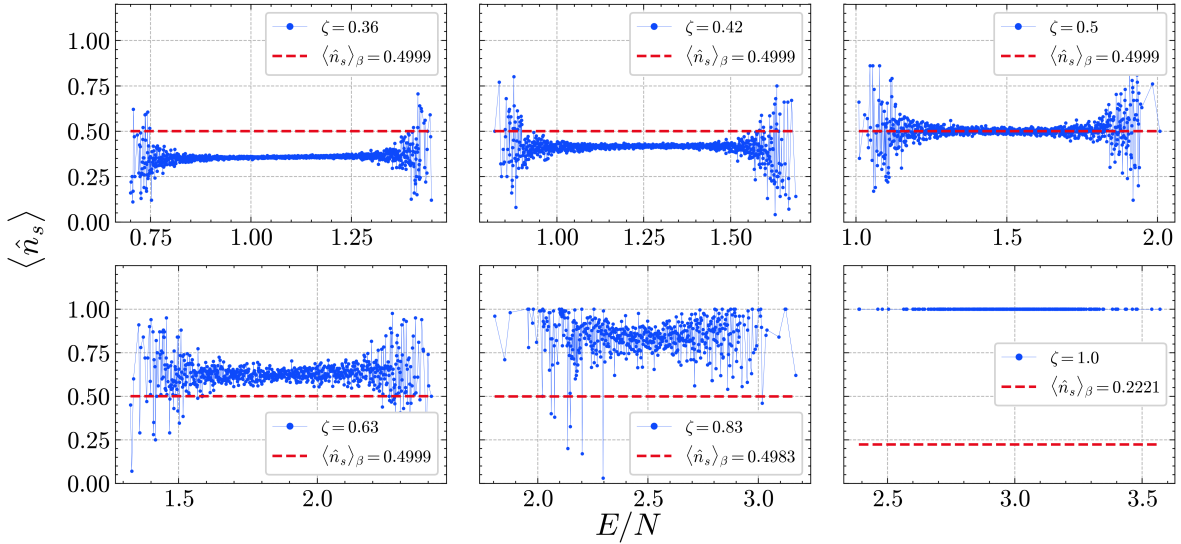


Figure 2.2: The GOE ensemble average over 10^3 realizations of the expected value in eigenvectors of the mean number operator at site s , $\langle \hat{n}_s \rangle = C_{ss}^\Psi = \langle \Psi^{N_p} | c_s^\dagger c_s | \Psi^{N_p} \rangle$ (blue dots), and the average thermal expectation $\langle \hat{n}_s \rangle_\beta = C_{ss}^\beta = \frac{1}{Z} \text{Tr} (e^{-\beta H} c_s^\dagger c_s)$ (dashed red line) for different densities $\zeta = \frac{N_p}{N}$ corresponding to $N = 5, 6, 8, 10, 12, 14$ (from bottom right to upper left) with $N_p = 5$ and $\alpha = 3$ fixed. Note that in all cases the mean energy is around $N_p \alpha$.

In Figures 2.2 and 2.3, we observe a match between the expectation values in thermal states and in eigenstates for density $\zeta = 0.5$, corresponding to half-filling in the system, in which case $N_p = \frac{N}{2}$. This is consistent with the analytical results. Since the number operator at site s can only take a value of zero or one for fermions, in the thermal state it is reasonable that the expected value takes the average value between those numbers, which is $\frac{1}{2}$. From the equality $\frac{N_p}{N} = [n_\beta]$, it is clear that this would happen for $N_p = \frac{N}{2}$. On the other hand, Figures 2.4 and 2.5 show the convergence of the off-diagonal matrix elements to a smooth function of the energy difference ω , this function becomes smoother and closer to zero as N increases just as ETH (1.4) states.

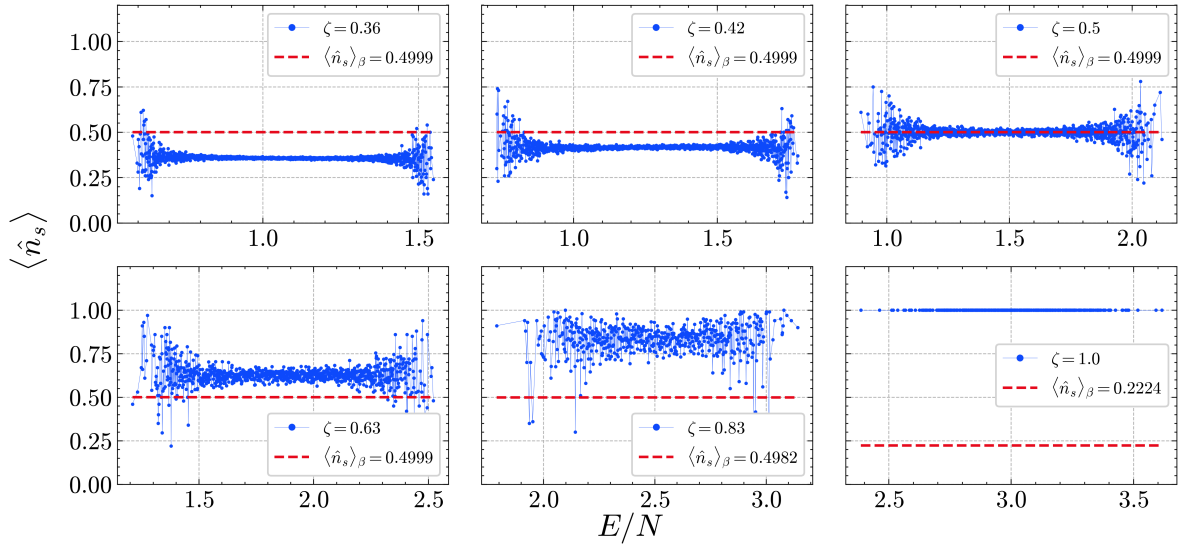


Figure 2.3: The GUE ensemble average over 10^3 realizations of the expected value in eigenvectors of the mean number operator at site s , $\langle \hat{n}_s \rangle = C_{ss}^\Psi = \langle \Psi^{N_p} | c_s^\dagger c_s | \Psi^{N_p} \rangle$ (blue dots), and the average thermal expectation $\langle \hat{n}_s \rangle_\beta = C_{ss}^\beta = \frac{1}{Z} \text{Tr} (e^{-\beta H} c_s^\dagger c_s)$ (dashed red line) for different densities $\zeta = \frac{N_p}{N}$ corresponding to $N = 5, 6, 8, 10, 12, 14$ (from bottom right to upper left) with $N_p = 5$ and $\alpha = 3$ fixed. Note that in all cases the mean energy is around $N_p \alpha$.

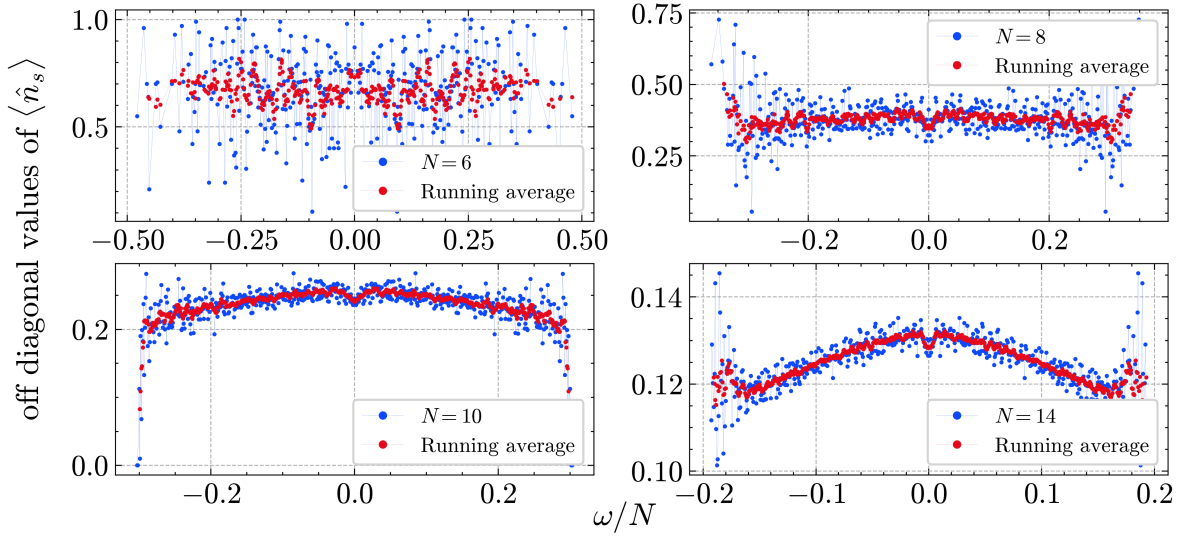


Figure 2.4: The GOE ensemble average over 50 realizations of the off-diagonal expectation values in eigenvectors of the mean number operator at site s , $\langle \hat{n}_s \rangle = C_{ss}^\Psi = \langle \Psi^{N_p} | c_s^\dagger c_s | \Psi^{N_p} \rangle$ for different densities $\zeta = \frac{N_p}{N}$ corresponding to $N = 6, 8, 10, 14$ with $N_p = 5$ and $\alpha = 3$ fixed. In all cases we show the running average.

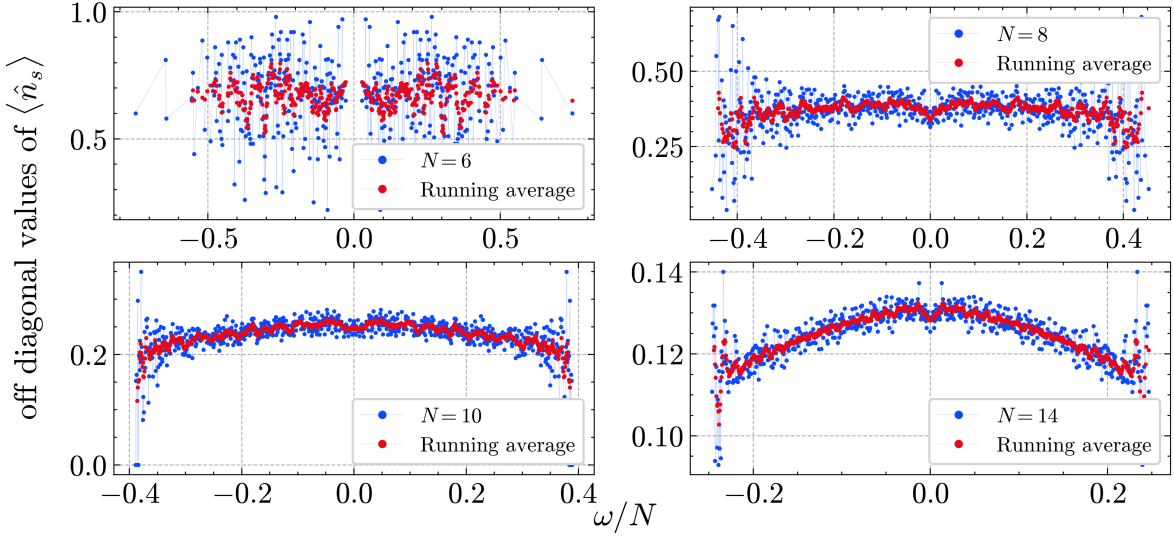


Figure 2.5: The GUE ensemble average over 50 realizations of the off-diagonal expectation values in eigenvectors of the mean number operator at site s , $\langle \hat{n}_s \rangle = C_{ss}^\Psi = \langle \Psi^{N_p} | c_s^\dagger c_s | \Psi^{N_p} \rangle$ for different densities $\zeta = \frac{N_p}{N}$ corresponding to $N = 6, 8, 10, 14$ with $N_p = 5$ and $\alpha = 3$ fixed. In all cases we show the running average.

2.6 Conclusions and perspectives

In this work, we introduced the phenomenon of thermalization in isolated many-body quantum systems. We explored existing classical hypotheses, such as ergodicity and typicality, and the challenges we face when extending these concepts to the quantum regime. We focused on the Eigenstate Thermalization Hypothesis (ETH) as a leading explanation for how quantum systems reach thermal equilibrium. Through the study of different systems, we identified key characteristics of thermalization.

In Section 1.2.1, we examined a disordered, highly chaotic fermionic system that thermalizes, contrasting it with the system of random free fermions discussed in Chapter 2.1, which also thermalizes but without chaotic behavior. We studied a bosonic system in Section 1.2.2, where it was proven that thermalization, in the sense of ETH, could also occur in a gapped phase. It also demonstrated a direct link between chaos and thermalization when next-nearest neighbor interactions are included. This showed that chaos can arise from different sources, not just randomness in the Hamiltonian. Notably, the system of random free fermions, despite having randomness in its Hamiltonian, does not thermalize due to chaos; rather, it does so through increasing quantum correlations like entanglement.

We analytically computed the subleading terms for the two-point correlation functions in the system of random free fermions. Our results condensed in Section 2.4 confirmed our initial hypothesis; the subleading contributions can be interpreted in terms of the presence or absence of time-reversal symmetry. This crucial difference is not present in the generalized version of ETH, suggesting the need for caution in the applicability of the generalized ETH to systems with time-reversal symmetries in non-trivial thermodynamic limits.

This work provides a strong foundation for further exploration into the role of subleading contribu-

tions in thermalization. The main conclusion is that, the generalized ETH does not explain the thermalization of non-chaotic quantum systems with dynamics governed by periodic trajectories. Furthermore, the subdominant terms not captured by the generalized ETH may contain relevant information about the symmetries of the system.

A natural next step is to investigate the effect of time-reversal symmetry in the thermalization of the system of random free fermions with spin. This could be done by taking the Hamiltonian from the Gaussian Symplectic Ensemble (GSE) and averaging with the Haar measure of the symplectic group, then completing the study of the system over the 3 classical Gaussian ensembles. Also, it would be interesting to study the robustness of the ETH against perturbations that could break the symmetries, and study in depth the exact relationship between the non-crossing partitions and the hidden conserved quantities that are not captured by the generalized ETH. Ultimately, refining our understanding of thermalization in quantum systems could lead to new developments in quantum computing, quantum thermodynamics, insights into the behavior of chaotic gravitational systems such as black holes, and enrich our perspective of the fundamental laws of physics.

Appendices

A Gaussian ensembles

The universality class of statistical properties of random matrices depends on the ensemble considered. In classical random matrix theory, there are 3 universality classes: GOE, GUE, and GSE, each of them taking their names from the probability distribution of the matrix elements, and the group of matrices that constitute the canonical transformations: orthogonal matrices for GOE, unitary for GUE, and symplectic for GSE. In this Appendix, we present known results about these ensembles as presented in [35]. For the sake of simplicity, consider the smallest non-trivial dimension, $N = 2$. In this case, the matrices that are invariant under orthogonal transformations are the real and symmetric 2×2 matrices. We are interested in finding the probability distribution $P(H)$ of the matrix elements that is normalized

$$\int_{-\infty}^{\infty} dH_{11} dH_{22} dH_{12} P(H) = \int dH P(H) = 1,$$

and meets the following two conditions:

1) Invariance under orthogonal transformations:

$$dH P(H) = d\tilde{H}(\tilde{P}), \quad \tilde{H} = OHO^T.$$

2) Uncorrelated matrix elements:

$$P(H) = P_{11}(H_{11})P_{22}(H_{22})P_{12}(H_{12}).$$

Next, following [35], we consider a change of basis $H' = OHO^T$, with the particular orthogonal matrix O given by

$$O = \begin{pmatrix} 1 & -\Theta \\ \Theta & 1 \end{pmatrix} \quad \text{With } \Theta \ll 1,$$

obtaining

$$H' = \frac{1}{\Theta^2 + 1} \begin{pmatrix} H_{11} - 2\Theta H_{12} + \Theta^2 H_{22} & H_{11}\Theta - \Theta H_{22} + H_{12}(1 - \Theta^2) \\ \Theta H_{11} - \Theta H_{22} + H_{12}(1 - \Theta^2) & H_{11}\Theta^2 + H_{22} + 2H_{12}\Theta \end{pmatrix}.$$

Ignoring second-order terms in Θ we get

$$\begin{aligned} H'_{11} &= H_{11} - 2\Theta H_{12}, \\ H'_{22} &= H_{22} + 2\Theta H_{12}, \\ H'_{12} &= H_{12} + \Theta(H_{11} - H_{22}) . \end{aligned}$$

The last two conditions yield the equation

$$P(H) = P(H) \left[1 - \Theta \left(2H_{12} \frac{d \ln P_{11}}{dH_{11}} - 2H_{12} \frac{d \ln P_{22}}{dH_{22}} - (H_{11} - H_{22}) \frac{d \ln P_{12}}{dH_{12}} \right) \right] .$$

Since the infinitesimal parameter might take any value, for equality to hold, the coefficient must vanish, so

$$\frac{1}{H_{12}} \frac{d \ln P_{12}}{dH_{12}} - \frac{2}{H_{11} - H_{22}} \left(\frac{d \ln P_{11}}{dH_{11}} - \frac{d \ln P_{22}}{dH_{22}} \right) = 0 .$$

This equation gives rise to three differential equations for each P_{ij} , whose solutions are Gaussian functions of the form

$$P(H) = C e^{-A \text{Tr } H^2} .$$

Similar calculations give the same results for *GUE* and *GSE*, see [35]. This explains the ‘‘Gaussian’’ part of the name. Furthermore, although this was shown for $N = 2$ this is true for any finite value of N .

A.1 Level density

We can perform a change of coordinates from the matrix elements to the eigenstates of the matrix, and from this, obtain the density of levels for each ensemble. See [29] for a complete deduction. The result is given by

$$P(\lambda_1, \dots, \lambda_N) \propto \prod_{i < j} |\lambda_i - \lambda_j|^\beta e^{-A \sum_{i=1}^N \lambda_i^2}, \quad (55)$$

where

$$\beta = \begin{cases} 1 & \text{GOE}, \\ 2 & \text{GUE}, \\ 4 & \text{GSE}. \end{cases}$$

We can obtain the average level density as the marginal probability density function of the joint level density (see Appendix 55),

$$\varrho(\lambda) = \int d\lambda_2 \dots d\lambda_N P(\lambda, \lambda_2, \dots, \lambda_N) .$$

By setting an appropriate energy scale (see [35]), we obtain the Wigner semicircle distribution,

$$\varrho(\lambda) = \begin{cases} \frac{2}{\pi} \sqrt{1 - \lambda^2} & |\lambda| < 1, \\ 0 & |\lambda| > 1. \end{cases}$$

A.2 Level spacing distributions

The level spacing distributions for each of the Gaussian ensembles are also well-known results

$$P(S) = \begin{cases} \frac{\pi}{2} S e^{-\frac{\pi}{4} S^2} & GOE, \\ \frac{32}{\pi^2} S^2 e^{-\frac{4}{\pi} S^2} & GUE, \\ \frac{2^{18}}{3^6 \pi^3} S^4 e^{-\frac{64}{9\pi} S^2} & GSE. \end{cases}$$

B N -point correlation functions

For an $N \times N$ matrix taken from GOE and GSE the probability distribution of eigenvalues is given by;

$$P(\lambda_1, \dots, \lambda_N) \propto e^{-\beta \frac{N}{4} \sum_i \lambda_i^2} \prod_{i < j} |\lambda_i - \lambda_j|^\beta,$$

where $\beta = 1, 4$ for GOE and GSE, respectively. Following [37] the n -point correlation function is defined as

$$\rho^{(n)}(\lambda_1, \dots, \lambda_n) = \int d\lambda_{n+1} \dots d\lambda_N P(\lambda_1, \dots, \lambda_N).$$

That is, as a marginal probability distribution. Physically, this represents the probability density function for the system to be in states with energy $\lambda_1, \dots, \lambda_n$.

The two-point correlator for GOE and GSE is given by

$$\rho^{(2)}(\lambda_1, \lambda_2) = \frac{1}{N(N-1)} \left(\frac{1}{4} \text{Tr}(K(\lambda_1, \lambda_2)) \text{Tr}(K(\lambda_1, \lambda_2)) - \frac{1}{2} \text{Tr}(K(\lambda_1, \lambda_2) K(\lambda_2, \lambda_1)) \right), \quad (56)$$

where $K(\lambda_i, \lambda_j)$ is the quaternion kernel. If we define the functions

$$\begin{aligned} \hat{s}(r) &= \frac{\sin(r)}{r} & \epsilon(r) &= \frac{1}{2} \text{sign}(r), \\ \mathbf{D}\hat{s}(r) &= \partial_r \hat{s}(r) & \mathbf{I}\hat{s}(r) &= \int_0^r \hat{s}(t) dt, \end{aligned}$$

the quaternion kernel for GOE may be written as

$$K(\lambda_i, \lambda_j) \equiv \begin{cases} \frac{N}{\pi} \begin{pmatrix} \hat{s}(N(\lambda_i - \lambda_j)) & \mathbf{D}\hat{s}(N(\lambda_i - \lambda_j)) \\ \mathbf{I}\hat{s}(N(\lambda_i - \lambda_j)) - \epsilon(N(\lambda_i - \lambda_j)) & \hat{s}(N(\lambda_i - \lambda_j)) \end{pmatrix} & \text{for } i \neq j, \\ \frac{N}{2\pi} \sqrt{4 - \lambda_i^2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{for } i = j, \end{cases}$$

while for GSE it is

$$K(\lambda_i, \lambda_j) \equiv \begin{cases} \frac{N}{\pi} \begin{pmatrix} \hat{s}(2N(\lambda_i - \lambda_j)) & \mathbf{D}\hat{s}(2N(\lambda_i - \lambda_j)) \\ \mathbf{I}\hat{s}(2N(\lambda_i - \lambda_j)) & \hat{s}(2N(\lambda_i - \lambda_j)) \end{pmatrix} & \text{for } i \neq j, \\ \frac{N}{2\pi} \sqrt{4 - \lambda_i^2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{for } i = j. \end{cases}$$

On the other hand, the large- N limit of the n -point correlation function for the GUE is given by

$$\rho^{(2)}(\lambda_1, \lambda_2) = \frac{(N-n)!}{N!} \det(K(\lambda_i, \lambda_j))_{i,j=1}^n, \quad (57)$$

and the kernel K , in the large- N limit is given by

$$K(\lambda_i, \lambda_j) \equiv \begin{cases} \frac{N}{\pi} \frac{\sin(N(\lambda_i - \lambda_j))}{N\lambda_i - \lambda_j} & \text{for } i \neq j, \\ \frac{N}{2\pi} \sqrt{4 - \lambda_i^2} & \text{for } i = j. \end{cases}$$

C Unfolding of the spectra

For discrete systems, the result of a measurement is a sequence of energies $E_1, \dots, E_N$, which can be described as a function of several Dirac delta functions,

$$S(E) = \sum_{n=1}^N \delta(E - E_n). \quad (58)$$

Since the level density of a standard random matrix ensemble is not directly related to the physical level density of the investigated systems, in order to describe the fluctuations of the spectrum in the framework of RMT, we need to remove the influence of the particular level density of the system. This procedure is called unfolding [35]. The objective of unfolding is to rescale the energies, so that the local density of states of the re normalized eigenvalues is equal to 1. This ensures automatically that also the mean level spacing is equal to 1. A simple way to achieve this is to divide the whole spectrum into different sets and divide each energy by the mean level spacing of the set to which it belongs [41].

The last method is good enough for many purposes. However, when the density of levels is known, the unfolding is performed by calculating the cumulative spectral function, also known as the staircase function $\eta(E)$, which counts the number of levels with energy less than or equal to E ,

$$\eta(E) = \int_{-\infty}^E S(E') dE' = \sum_{n=1}^N \Theta(E - E_n) .$$

This function is composed of an average part and a fluctuating part. The smooth part $\varepsilon(E)$ is given by the cumulative spectral function of the mean level density $\varrho(E)$,

$$\varepsilon(E) = \int_{-\infty}^E \varrho(E') dE' . \quad (59)$$

In order to compute the mean level density $\varrho(E)$, which is the smooth version of (58), several procedures may be used, for instance, the delta functions $\delta(E - E_n)$ in (58) can be fit by Gaussian functions $g(E - E_n)$ [35], where

$$g(E) = \frac{1}{\Delta\sqrt{\pi}} e^{-\frac{E^2}{\Delta^2}} .$$

So

$$\varrho(E) = \sum_{n=1}^N g(E - E_n) .$$

By finding the parameter Δ that minimizes the mean square deviation between the level staircase $\eta(E)$ and the smooth part $\varepsilon(E)$ the unfolded spectrum is obtained by the transformation

$$\varepsilon_i = \varepsilon(E_i) .$$

When the exact form of the mean level density is unknown, another approach consists of fitting the staircase function to some continuous function directly. The election of this function is arbitrary, but is usually done with polynomials of degree n . However, it has been shown that statistics that measure long-range level correlations are very sensitive to the degree of the polynomial used, while statistics that measure short-range correlations are independent of the unfolding procedure used [42].

D The Haar measure

In this appendix, we present a brief review of the fundamental definitions of measure and group theory required to state Theorem 3, which postulates the existence and uniqueness of the Haar measure. We comment on the practical application in the calculation of integrals of the form 60 and review the simple example of the Haar measure in the complex circle S^1 . The theorems and definitions in this Appendix are taken from [43].

Measure theory

In order to perform a measurement, we need to define a domain. In mathematical terms this is called a σ -algebra, and its definition contains the intuitive ideas of how domains like areas or volumes behave in everyday life.

Definition 2 (σ -algebra). *Let X be a set. Then $\mathcal{A}$ is a σ -algebra if for any countable index set I of some sequence $A_i \in \mathcal{A}$, we have that*

- $\bigcap_{i \in I} A_i \in \mathcal{A}$
- $\bigcup_{i \in I} A_i \in \mathcal{A}$
- $A_i^c \in \mathcal{A}$

Following intuition, a measure of the whole set must be somehow the sum of the measures of its components. In fact, this is the only defining property of a measure.

Definition 3 (Measure). *A measure is a function $\mu : \mathcal{A} \rightarrow [0, \infty]$ defined on a σ -algebra $\mathcal{A}$, such that the measure of the empty set is zero, $\mu(\Phi) = 0$, and for some countable index set I , and a disjoint sequence of sets $A_i \in \mathcal{A}$ we have that;*

$$\mu\left(\bigcup A_i\right) = \sum \mu(A_i).$$

Now, in order to generalize the notion of closeness and connectedness, we define the concept of Topological space, in terms of open sets. It is defined as follows.

Definition 4 (Topological space). *A topology τ on a set X , is a collection of subsets of X , called open sets, that satisfy the following axioms;*

- Φ and X are in τ .
- Any arbitrary union of open sets is an open set.
- The intersection of any finite number of open sets is an open set.

Again, these are natural conditions that a well-behaved space should have.

Joining the last definitions, we define a Borel σ -algebra, which are the kind of measure and spaces with which we want to work.

Definition 5 (Borel σ -algebra). *The Borel σ -algebra of a topological space X with topology τ is the σ -algebra generated by the topology $\sigma(\tau)$, that is, the smallest σ -algebra that contains all elements of τ . Similarly, a Borel measure on a space X is a measure on the Borel σ -algebra of X , $\mu : \mathcal{B}(X) \rightarrow \mathbb{R} \cup \infty$ such that if $S \subset X$ is compact, then $\mu(S) < \infty$.*

Definition 6 (Regular measure). *Let $\mathcal{F}$ be a σ -algebra on a topological space X . Then, a measure μ is deemed regular if the following conditions are met:*

- 1) (Outer regularity) *If $U \in \mathcal{F}$, then*

$$\mu(U) = \inf\{\mu(O) | U \in \mathcal{F}, U \subset O \text{ is open}\}.$$

2) (Inner regularity) If $U \subset X$ is open and $U \in \mathcal{F}$, then

$$\mu(U) = \sup\{\mu(K) | K \in \mathcal{F}, K \subset U \text{ is compact}\}.$$

A regular Borel measure is called a Radon measure.

Group Theory

Definition 7 (Topological group). Let G be a group with a Hausdorff topology, with the group operation denoted by $g \circ h = gh$, inverse function g^{-1} and identity e . G is a topological group if the functions,

$$m : G \times G \rightarrow G; \quad m(g, h) = gh \quad \text{and} \quad i : G \rightarrow G; \quad i(g) = g^{-1}$$

are each continuous functions.

A clear example of a topological group is the general linear group, that is, the set of matrices defined as

$$GL(E) = \{S \in \mathcal{L}(E), \text{ such that } S^{-1} \text{ exists} \},$$

where $\mathcal{L}(E)$ is the space of linear functions over a Banach space E .

Now we define what we mean by left and right translation.

Definition 8 (Left translation in the group). Let G be a topological group, for any continuous function defined on G , $f \in C(G)$, we define the left translation by g and the right translation by g of f , denoted by ${}_g f$ and f_g respectively, as

$${}_g f(x) = f(gx) \quad \text{and} \quad f_g(x) = f(xg)$$

Definition 9 (Left invariance of the measure). Let G be a compact topological group. A Borel measure $\mu : \mathcal{B}(G) \rightarrow [0, \infty)$ is left invariant if ${}_g \mu(\mathcal{A}) = \mu(\mathcal{A})$ for all $g \in G$ and $\mathcal{A} \in \mathcal{B}(G)$, where ${}_g \mu(\mathcal{A}) = \mu(g\mathcal{A})$ is the left translation of the measure μ ⁷.

The following is a practical result when the measure μ has an additional normalization property

Theorem 2 (Integration property of a Radon measure). Let G be a compact group and μ a Borel measure on $\mathcal{B}(G)$. For any $g \in G$ we have the set function, which is the left translation of the measure ${}_g \mu : \mathcal{B}(G) \rightarrow [0, \infty)$ given by

$${}_g \mu(g\mathcal{A}) \quad \forall \mathcal{A} \in \mathcal{B}(G).$$

Then ${}_g \mu$ is also a Borel measure, and if μ is also a Radon probability measure, ($\mu(G) = 1$), then ${}_g(\mu) = 1$ and $\forall f \in C(G)$;

$$\int_G {}_g f d\mu = \int_G f d_g \mu.$$

⁷The definition for right invariance is analogous.

The last property is important in practical terms, furthermore, for any compact group G it is proved that there is always a Radon measure which is left invariant and a Radon measure that is right invariant. All these definitions and results converge to the main result of this section, the Von Neumann theorem.

Theorem 3 (Existence and uniqueness of the Haar measure). *Let G be a compact topological group. Then there is a unique left-invariant probability measure μ in the Borel σ -algebra $\mathcal{B}(G)$, and μ is also right-invariant. This implies that μ is the unique left- and right-invariant probability measure. We call μ the Haar measure of G .*

The last two theorems imply that, if G is a compact topological group, μ is its Haar measure and f is any measurable function, for any $g \in G$ we have

$$\int_G f(gx) d\mu(x) = \int_G f d\mu. \quad (60)$$

This result is usually taken as the definition of the Haar measure itself, since it is the only measure for which it is valid.

Complex circle

To illustrate all these definitions, we consider a simple example in which explicit functions can be written. As is well known, the complex circle S^1 is the set of complex numbers with unit norm;

$$S^1 = \{c \in \mathbb{C} : \|c\| = 1\},$$

with the typical norm $\|c\| = \text{Re}(c)^2 + \text{Im}(c)^2$.

In order to define a Haar measure in S^1 we need to prove the following theorem:

Theorem 4. *S^1 is a compact topological group under the d_2 metric.*

Proof: Since the nonzero complex numbers are a group under complex multiplication, we only need to prove that S^1 is a subgroup. Hence, let $a, b \in S^1$. Then $\|a\| = \|b\| = 1$, which implies that $(\|a\|)(\|b\|) = 1$, then since S^1 is not empty and is closed under multiplication it is a subgroup of $\mathbb{C}$ and therefore a group.

For compactness, consider the function $f : [0, 1] \rightarrow S^1$ defined as $f(x) = e^{2\pi i x}$, this function is continuous and its image is all S^1 . Furthermore, $[0, 1]$ is a compact set. Since the continuous image of a compact set is compact, we conclude that S^1 is compact. Finally, since complex multiplication and inversion are continuous functions for non-zero complex numbers, we conclude that S^1 is a compact topological group. $\square$

Now, Theorem 3 implies the existence and uniqueness of a Haar measure on S^1 . Let's denote an arc on S^1 by $U_0 \subset S^1$, which is the image of the function $e^{2\pi i U}$ for some $U \subset [0, 1]$. Since that function is bijective on the domain $[0, 1]$, it is clear that $U_0 \cup V_0 = (U \cup V)_0$ and $U_0 \cap V_0 = (U \cap V)_0$ for any $U, V \subset [0, 1]$. With this, we can define the topology on S^1 as

$$\mathcal{A} = \left\{ U_0 \mid U = \bigcup_{i=1}^n (a_i, b_i) \cap [0, 1] \text{ Such that } a_i, b_i \in \mathbb{R} \text{ and } I \text{ is an index set of any order} \right\}.$$

That is, the topology on S^1 is the induced topology of the open sets in $[0, 1)$ and the Haar measure over S^1 is the measure induced by $\mu_0 : \tau \rightarrow \mathbb{R}_{\geq 0}$, with $\mu_0(a, b) = |b - a|$, that is;

$$\mu(\mathcal{A}) = \mu_0 \left(\bigcup_{i=1}^n (a_i, b_i)_0 \right) = \sum_{i=1}^n |b_i - a_i|,$$

where the (a_i, b_i) are disjoint.

E Quantum channels

E.1 Completely positive trace preserving operators

In this section, we consider superoperators of the form $\Phi : \mathcal{L}(\mathcal{X}) \rightarrow \mathcal{L}(\mathcal{Y})$, from the space of linear operators defined on some field $\mathcal{X}$, to the space of linear operators defined on the field $\mathcal{Y}$. In physics, superoperators are meant to transform a quantum state $\rho \in \mathcal{L}(\mathcal{X})$ into another $\Phi(\rho) \in \mathcal{L}(\mathcal{Y})$. In order to obtain a valid quantum state $\Phi(\rho)$, the following conditions must hold:

- Φ must preserve the trace: $\text{Tr}(\Phi(\rho)) = \text{Tr}(\rho)$, $\forall \rho \in \mathcal{L}(\mathcal{X})$.
- Φ must be completely positive: $\Phi \otimes \mathbb{1}_{\mathcal{L}(\mathcal{Z})}$ is positive for all Euclidean spaces $\mathcal{Z}$.

Superoperators for which the last conditions are met are called Quantum Channels, commonly known in the physics community as CPTP from “Completely positive trace preserving” operators. A quantum channel may be represented in different ways, each of which is useful in specific contexts. In this work, we use the Choi-Jamiolkowski representation in the calculation of the k -fold quantum channel over the orthogonal group in Appendix E.3, and the Kraus representation in its continuous form for the calculation of the four-point correlators in Chapter 2.

The Choi-Jamiolkowski representation

The Choi-Jamiolkowski representation of a quantum channel $\Phi : \mathcal{L}(\mathcal{H}_\infty) \rightarrow \mathcal{L}(\mathcal{H}_\epsilon)$ is the correspondence $J : T(\mathcal{H}_\infty, \mathcal{H}_\epsilon) \rightarrow \mathcal{L}(\mathcal{H}_\epsilon \otimes \mathcal{H}_\infty)$

$$J(\Phi) = (\Phi \otimes \mathbb{1}_{\mathcal{L}(\mathcal{H}_\infty)})(\text{vec}(\mathbb{1}_{\mathcal{H}_\infty}) \text{vec}(\mathbb{1}_{\mathcal{H}_\infty})^\dagger) \in \mathcal{L}(\mathcal{H}_\epsilon \otimes \mathcal{H}_\infty) \quad (61)$$

The Kraus representation

The Kraus representations describes the action of Φ as

$$\Phi(\rho) = \sum_{a \in \Gamma} A_a \rho B_a^\dagger,$$

where $A_a, B_a \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, $a \in \Gamma$, where Γ is the set of possible states of the system.

E.2 Integration over the unitary group $U(d)$

Let A be some operator acting on the Hilbert space $\mathcal{H}^{\otimes k}$, with $\mathcal{H} = \mathbb{C}^d$. The k -fold quantum channel of A with respect to the unitary group $U(d)$ is defined as [44],

$$\Phi_k^{\text{Haar}}(A) = \int [U^\dagger dU] (U^{\otimes k})^\dagger A U^{\otimes k} . \quad (62)$$

Note that this is the continuous version of the Kraus representation E.1. Here, $[U^\dagger dU]$ denotes the Haar measure of the unitary group. As stated in Appendix D, this is the only left and right invariant measure on the group. Thus, for some function f we have that

$$\int [U^\dagger dU] = 1, \quad \int [U^\dagger dU] f(VU) = \int [U^\dagger dU] f(UV) = \int [U^\dagger dU] f(U), \quad \forall V \in U(\mathcal{H}) .$$

In particular, if we take $f(U) = (U^{\otimes k})^\dagger A U^{\otimes k}$, the right-invariance property implies that

$$(V^{\otimes k})^\dagger (\Phi_k^{\text{Haar}}(A)) V^{\otimes k} = \Phi_k^{\text{Haar}}(A) .$$

That is, $\Phi_k^{\text{Haar}}(A)$ commutes with $V^{\otimes k}$. With this condition, the Schur-Weyl duality ensures that the quantum channel may be written as a linear combination of permutation operators W_π ,

$$\Phi_k^{\text{Haar}}(A) = \sum_{\pi \in S_k} W_\pi u_\pi(A) ,$$

where S_k is the permutation group of k elements and $u_\pi(A)$ are linear functions of A . Now, since $u_\pi(A)$ is linear, we may write it in terms of new operators C_π : $u_\pi(A) = \text{Tr}(C_\pi A)$. In a similar way, right invariance and Schur-Weyl duality imply that the operators C_π can be written in terms of linear operators W_σ . Putting this together, we obtain a general expression for the k -fold quantum channel on the unitary group

$$\Phi_k^{\text{Haar}}(A) = \sum_{\pi, \sigma} c_{\pi, \sigma} \text{Tr} W_\pi A ,$$

where the coefficients $c_{\pi, \sigma}$ can be written in a matrix, called the Weingarten matrix. This matrix is invertible, and if Q is the inverse matrix the quantum channel can be written as

$$\Phi_k^{\text{Haar}}(A) = \sum_{\pi, \sigma} (Q^{-1})_{\pi, \sigma} \text{Tr}\{W_\pi A\} , \quad (63)$$

where $Q_{\sigma, \lambda} = d^{\#\text{cycles}(\sigma\lambda)}$ is the inverse of the Weingarten matrix and $\#\text{cycles}(\sigma\lambda)$ refers to the number of cyclic permutations in the cyclic notation of the permutation $\sigma\lambda$.

Using Equation (63) we can compute the one- and two-fold quantum channels over the unitary group.

1-fold quantum channel

When $k = 1$, the only permutation is the identity $\mathbb{1}$, which consists on the cycle (1). Thus, the matrix Q consists on the number $d^1 = d$. The Weingarten matrix is therefore $1/d$, and the 1-fold quantum channel is

$$\Phi_k^{\text{Haar}}(A) = \frac{1}{d} \text{Tr}(A) .$$

2-fold quantum channel

For $k = 2$ elements, we have two possible permutations, the identity $\mathbb{1}$ operator, which leaves the elements in the same position (1)(2), and the swap S operator that exchanges them (12). According to Equation (63), the matrix Q is

$$Q = \begin{pmatrix} d^{\# \text{cycles}(\mathbb{1}\mathbb{1})} & d^{\# \text{cycles}(\mathbb{1}S)} \\ d^{\# \text{cycles}(S\mathbb{1})} & d^{\# \text{cycles}(SS)} \end{pmatrix} = \begin{pmatrix} d^2 & d \\ d & d^2 \end{pmatrix},$$

and then the Weingarten matrix is

$$C = Q^{-1} = \begin{pmatrix} \frac{1}{d^2-1} & -\frac{1}{d(d^2-1)} \\ -\frac{1}{d(d^2-1)} & \frac{1}{d^2-1} \end{pmatrix}.$$

Then the 2-fold quantum channel for the unitary group is given by

$$\Phi_2^{\text{Haar}}(A) = \frac{1}{d^2-1} \left(\mathbb{1} \text{Tr}(A) + S \text{Tr}(SA) - \frac{1}{d} S \text{Tr}(A) - \frac{1}{d} \mathbb{1} \text{Tr}(SA) \right). \quad (64)$$

E.3 Integration over the orthogonal group $O(d)$

Definition (62) is equally valid for a k -fold quantum channel over the orthogonal group. For a 2-fold quantum channel it reads

$$\Phi_2^{\text{Haar}}(A) = \int [O^\top dO] (O^{\otimes 2})^\dagger A O^{\otimes 2} \quad \forall A \in \mathcal{H}_1 \otimes \mathcal{H}_2. \quad (65)$$

However, the process of solving this integral is somewhat more cumbersome. Although the set of orthogonal matrices is contained in the set of unitary matrices, the group structure in each case implies very different properties.

To compute this channel, special interest is paid to the set of operators A that project themselves, i.e., $\Phi_2^{\text{Haar}}(A) = A$. From Equation (65) it is clear that these operators must commute with $O^{\otimes 2}$, that is, $[A, O^{\otimes 2}] = 0$ for all O acting on $\mathcal{H}_1 \otimes \mathcal{H}_2$. The set of operators A for which that condition is true is called the commutant of the group and is denoted by $(O^{\otimes 2})'$.

In the rest of this section we will present the proof of the result (70) that gives the exact solution of (65). The idea is to obtain the commutant of the 2-fold quantum channel over $O \otimes O$ as the intersection of the commutants of $U \otimes U$ and $U \otimes \bar{U}$, where $\bar{U}$ is the complex conjugate transposed of U .

In the last section, we found the commutant for $U \otimes U$, (actually for $U^{\otimes k}$ for any k), so we only need to find a way to compute the commutant of $U \otimes \bar{U}$. To achieve this, following [45], we defined the partial transpose of an operator acting on $\mathcal{H}_1 \otimes \mathcal{H}_2$ in a product basis, as the transposition operation the only acts in the indices of the basis associated to $\mathcal{H}_2$, that is

$$T_2(A \otimes B) = A \otimes T(B),$$

where $T(B)$ is the ordinary matrix transposition of B .

Let's check how the partial transposition acts on a product of unitaries. Here, we assume that U_i, A_i are operators acting on $\mathcal{H}_i$, we have

$$\begin{aligned} T_2((U_1 \otimes U_2)(A_1 \otimes A_2)(U_1^* \otimes U_2^*)) &= T_2((U_1 A_1 U_1^*) \otimes (U_2 A_2 U_2^*)) \\ &= (U_1 A_1 U_1^*) \otimes (T(U_2^*) T(A_2) T(U_2)) \\ &= (U_1 A_1 U_1^*) \otimes (\bar{U}_2 T(A_2) \bar{U}_2^*) \\ &= (U_1 \otimes \bar{U}_2) T_2(A_1 \otimes A_2) (U_1 \otimes \bar{U}_2)^*, \end{aligned}$$

Where U^* denotes complex conjugation of U .

This result motivates the following definition: For any group G of product unitaries, we denote by $\tilde{G}$ the group of unitaries $U_1 \otimes \bar{U}_2$ where $U_1 \otimes U_2 \in G$. Now, integration of the partial transposition gives

$$\begin{aligned} T_2 \left(\int [U^\dagger dU] (U_1 \otimes U_2)(A_1 \otimes A_2)(U_1^* \otimes U_2^*) \right) &= \int [U^\dagger dU] T_2((U_1 \otimes U_2)(A_1 \otimes A_2)(U_1^* \otimes U_2^*)) \\ &= \int [U^\dagger dU] (U_1 \otimes \bar{U}_2) T_2(A_1 \otimes A_2) (U_1 \otimes \bar{U}_2)^*. \end{aligned} \quad (66)$$

From Equation (66) we define an analogue of the 2-fold quantum channel (62) but now for the product $U \otimes \bar{U}$ as

$$\tilde{\Phi}_2^{\text{Haar}}(A) = \int [U^\dagger dU] (U_1 \otimes \bar{U}_2) A (U_1 \otimes \bar{U}_2)^*.$$

With this definition, Equation (66) precisely states that

$$T_2(\tilde{\Phi}_2^{\text{Haar}}(A_1 \otimes A_2)) = \tilde{\Phi}_2^{\text{Haar}} T_2(A_1 \otimes A_2), \quad (67)$$

Which implies a direct relation between the commutants of the groups:

$$T_2(G') = \tilde{G}'. \quad (68)$$

Thus, the operators that are invariant under $\tilde{G}$ are the partial transposes of those invariant under G . This gives us the recipe to find the commutant of $U \otimes \bar{U}$ knowing the commutant of $U \otimes U$.

Commutant for $U \otimes U$

In this case, consider $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ and $G = U \otimes U$ with $U \in \mathcal{H}_1$. Thanks to the Schur-Weyl duality; we know that the commutant of G is spanned by the identity and the swap operators, which we can write in terms of the basis $|i\rangle$ of $\mathcal{H}_1$:

$$\mathbb{1} = \sum_{i,j} |ij\rangle \langle ij|, \quad \mathbb{S} = \sum_{ij} |ij\rangle \langle ji|. \quad (69)$$

Commutant for $U \otimes \bar{U}$

In this case $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ and $G = U \otimes \bar{U}$ with $U \in \mathcal{H}_1$. By Equation (68), we know that the commutant of G is spanned by the partial transpositions of the identity and swap operators:

$$T_2(\mathbb{1}) = T_2 \left(\sum_{ij} |ij\rangle \langle ij| \right) = \sum_{ij} (|i\rangle \langle i| \otimes T(|j\rangle \langle j|)) = \sum_{ij} |i\rangle \langle i| \otimes |j\rangle \langle j| = \sum_{ij} |ij\rangle \langle ij| = \mathbb{1},$$

$$T_2(\mathbb{S}) = T_2 \left(\sum_{ij} |ij\rangle \langle ji| \right) = \sum_{ij} (|i\rangle \langle j| \otimes T(|j\rangle \langle i|)) = \sum_{ij} |i\rangle \langle j| \otimes |i\rangle \langle j| = \sum_{ij} |ii\rangle \langle jj| = \hat{\mathbb{S}}.$$

The partial transposition of the swap gives a new operator that we call the *scalar* and denote by $\hat{\mathbb{S}}$.

Commutant for $O \otimes O$

The intersection $(U \otimes U) \cap (U \otimes \bar{U})$ implies $\bar{U} = U$, so U is a real orthogonal matrix. From the last calculations, we therefore have that $(O^{\otimes 2})'$ is generated by the identity, the swap, and the scalar operators: $\mathbb{1}$, $\mathbb{S}$, and $\hat{\mathbb{S}}$ [45]. For practical applications, it is useful to redefine the basis as

$$P_0 = \frac{1}{d} \hat{\mathbb{S}}, \quad P_1 = \frac{1}{2}(\mathbb{1} - \mathbb{S}), \quad P_2 = \frac{1}{2}(\mathbb{1} + \mathbb{S}) - \frac{1}{d} \hat{\mathbb{S}},$$

where P_1 and P_2 are then projections onto the symmetric and antisymmetric subspaces of $\mathcal{H}$. It is easy to show that if we denote by d_i the dimension of the subspace projected by the operator P_i , $d_i = \text{Tr}\{P_i\}$, then $d_0 + d_1 + d_2 = d^2 = \dim(\mathbb{C}^d) \dim(\mathbb{C}^d) = \dim(\mathbb{C}^d \otimes \mathbb{C}^d)$, as expected, since $\mathcal{H}_1 = \mathcal{H}_2 = \mathbb{C}^d$.

The operator A over which the quantum channel acts can be a state represented by a density matrix ρ , so since any operator A in $(O^{\otimes 2})'$ is spanned by the projectors P_0, P_1 and P_2 , we can also say that all states ρ belong to the convex hull generated by the normalized operators $\{P_0/d_0, P_1/d_1, P_2/d_2\}$. Furthermore, the Choi-Jamiolkowski representation (61) of a quantum channel described in Section E.1 shows an explicit one-to-one correspondence between quantum states and quantum channels (this is called the *channel-state* duality). This duality implies that we can obtain a basis for the quantum channel from the basis of quantum states in the commutant. Following [46], a straightforward calculation shows that

$$J(\mathbb{1}) = \rho_o, \quad J \left(\mathbb{1} \otimes \frac{\text{Tr}(\cdot)\mathbb{1} - T}{d-1} \right) = \rho_1, \quad J \left(\text{Tr}(\cdot) \frac{\mathbb{1}}{d} \right) = \frac{1}{d^2} (P_0 + P_1 + P_2),$$

where $J(\Phi)$ is the Choi-Jamiolkowski representation (61). Then, any 2-fold quantum channel over $O \otimes O$ is generated by the basis channels $\mathbb{1}$, $\frac{1}{d} \text{Tr}(\cdot)$, $\frac{\text{Tr}(\cdot)\mathbb{1} - T}{d-1}$, that is,

$$\Phi_2^{\text{Haar}}(A) = \alpha \mathbb{1} + \beta \frac{1}{d} \text{Tr}(A) + \gamma \frac{\text{Tr}(A)\mathbb{1} - T}{d-1}. \quad (70)$$

We can now plug in simple cases, such as $\Phi(\mathbb{1})$ and $\Phi(\hat{\mathbb{S}})$, for which the integral (65) is easy to solve, and impose the conservation of the trace in order to find the coefficients α, β and γ .

As a result we obtain

$$\alpha(A) = \frac{1}{d^2 + d - 2} \left[\frac{1+d}{d} \text{Tr}(A\mathbb{1}) - \frac{1}{d} \text{Tr}(A\mathbb{S}) - \frac{1}{d} \text{Tr}(A\hat{\mathbb{S}}) \right], \quad (71)$$

$$\beta(A) = \frac{1}{d^2 + d - 2} \left[-\frac{1}{d} \text{Tr}(A\mathbb{1}) + \frac{1+d}{d} \text{Tr}(A\mathbb{S}) - \frac{1}{d} \text{Tr}(A\hat{\mathbb{S}}) \right], \quad (72)$$

$$\gamma(A) = \frac{1}{d^2 + d - 2} \left[-\frac{1}{d} \text{Tr}(A\mathbb{1}) - \frac{1}{d} \text{Tr}(A\mathbb{S}) + \frac{1+d}{d} \text{Tr}(A\hat{\mathbb{S}}) \right]. \quad (73)$$

In conclusion, the calculation of the quantum channel (65) exploits the constraint that all operators in the commutant of $O^{\otimes k}$ must be projected on themselves together with the channel-state duality, to find a basis for the quantum channel from the basis of the commutant. It may be calculated from the basis of $U^{\otimes k}$ previously known from the Schur-Weyl duality.

F Graph interpretation of free probability

In this Appendix, we introduce the interpretation of free probability in terms of simple connected graphs as presented in the lectures of Novak [47].

The number m_n of simple undirected graphs that can be formed from a set of vertices $[n] = 1, \dots, n$ is given by $m_n = 2^{\binom{n}{2}}$. These graphs do not contain edges that connect a vertex to itself, and two vertices are not connected by more than one edge. Simple graphs can be either connected or disconnected. A graph is connected if there is a path of edges that connects any pair of points. The relationship between connected and disconnected structures is well known in probability. We can construct a graph by choosing a partition of the set $[n]$ formed by disjoint subsets $B_1, \dots, B_k$, and constructing a connected graph in each of these (which can be done in $c_{|B_1|} \dots c_{|B_k|}$ ways, where $c_{|B_i|}$ is the number of ways we can create a connected graph in the subset B_i that has cardinality $|B_i|$). The number of simple graphs m_n is then obtained as the sum over all partitions π of $[n]$,

$$m_n = \sum_{\pi \in P(n)} \prod_{B \in \pi} c_{|B|}.$$

In stochastic applications, the sequence of moments of a random variable X is related to simple graphs as $m_n = m_n(X) = \mathbb{E}[X^n]$. Thus, the last formula can be written for the sequence of quantities $c_n(X)$,

$$m_n(X) = \sum_{\pi \in P(n)} \prod_{B \in \pi} c_{|B|}(X).$$

The sequence $c_n(X)$ is called the sequence of cumulants. Cumulants are widely encountered in probability and statistics, where the first two cumulants have special names: mean and variance.

A variation of the previous procedure considers a representation where each node is located on the unit circle, and the edges are denoted by straight lines connecting them. The number of geometrically connected graphs κ_n takes into account not only the connectivity described earlier but also the geometric connectivity of the nodes. Analogous to the cumulants $c_n(X)$, we define the non-crossing cumulants of a random variable X based on the relationship between all structures and the geometrically connected ones,

$$m_n(X) = \sum_{\pi \in NC(n)} \prod_{B \in \pi} \kappa_{|B|}(X). \quad (74)$$

The first three values of the sequences of cumulants and non-crossing cumulants are identical. However, for $n \geq 4$, the non-crossing cumulants define a new type of probability theory called free probability, where the non-crossing cumulants are known as free cumulants.

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