

Topological properties of superconducting fermionic systems in the presence of spin-orbit coupling

Propiedades topológicas de sistemas fermiónicos superconductores en presencia del acople espín-órbita

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Abstract

The establishment of a topological phase is shown under the existence of Majorana fermions for a superconducting nanowire with spin-orbit coupling and Zeeman coupling, which is modeled as a one-dimensional Fermi gas in the continuum at zero temperature with a Hamiltonian in the second quantization framework, and then it is solved with the Bosonization technique. The signatures of the phase are observed via the density and pairing correlations and susceptibilities in the strong interaction limit for attractive interactions and strong coupling limit for the Zeeman coupling, and under the modulation of a chemical potential via the introduction of an external harmonic trap. As a result, phase diagrams of the nanowire are obtained, where a charge density phase and spin-singlet superconductivity of trivial topology can be established for a dominant superconducting gap, while spin-singlet and spin-triplet superconductivity of non-trivial topology can be established for a dominant magnetic gap.

Keywords: condensed matter, topological phase, bosonization, one-dimensional, correlations, superconductivity.

Resumen

Se demuestra el establecimiento de una fase topológica bajo la existencia de fermiones de Majorana en un nanohilo superconductor con acoplamientos de espín-órbita y Zeeman, el cual se modela con un Hamiltoniano en segunda cuantización de un gas de fermiones unidimensional en el continuo a temperatura cero, y se resuelve por medio del método de Bosonización. Las signatures de la fase se observan por medio de las correlaciones y susceptibilidades de densidad y emparejamiento del sistema en el límite de interacción atractiva fuerte y acoplamiento Zeeman fuerte, y bajo la modulación de un potencial químico de acuerdo con una trampa armónica externa. Como resultado, se obtienen diagramas de fase del nanohilo, en donde se pueden establecer fases de densidad de carga y superconductividad tipo singlete de topología trivial para un gap superconductor dominante, mientras que se pueden establecer fases superconductoras de tipo singlete y triplete de topología no trivial para un gap magnético dominante.

Palabras clave: materia condensada, fase topológica, bosonización, unidimensional, correlaciones, superconductividad.

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Chapter 1

Introduction

Since the arrival of quantum mechanics in the 20th century, and the vast evidence for all things to be made of fundamental entities which obey those mechanics [1–3], many problems have arisen from the fact that the quantum physics contains unimaginable phenomena which do not appear among the classic objects. Even so, many classical problems are so difficult to solve that the most powerful nowadays tools simply do not work (such as supercomputers or experimental setups). Known classical approaches seem to be exhausted, so throughout the late 20th century, and during the course of the 21st century, there has been an increasing interest in finding a way to use novel quantum approaches and built devices with some sort of quantum control to tune different quantum phenomena, as a TV control displays the show a person wants to watch. This work refers particularly to the approach of the *quantum computer* and the quantum control concept of a *quantum simulator*.

Quantum computers are a hot topic these days, not only for physics but for many different areas, even the ones not really close to science. For example, it has become usual that a quantum mechanics lecture is taught in computer science programs although it has a heavy content on physics [4]. That is because, it is so desirable to move from classical computers to quantum ones as they offer quite remarkable improvements, such as speed-ups to many tasks in computation and communication, and more secure information transmissions and authentications [4, 5]. Today, IBM has made great efforts along with other companies like Microsoft, using different forms of developing those computers based on their basic concept [6]. So, how does it work? A quantum computer exploits the superposition principle of quantum states, using every single one of them at the same time for doing a task whenever those states remain coherent in time, and then via a measurement, the final result of the task is acquired [5]. If the system has only two levels (two possible states), that system is a *qubit*. The biggest problem in this definition is the coherence because any perturbation from surroundings leads the system to decoherence, where the superposition collapses and the expected information is lost [5]. A measurement is a perturbation itself, so using the system implies damaging it as well. The qubits need to be protected somehow or be constantly repaired [5]. The challenge today is to get the most optimizations to develop at least a functional quantum computer that can surpass the best supercomputers available.

On the other hand, **quantum simulators** try to reproduce phenomena that cannot be easily realized or isolated even if there is a deep understanding of how it happens, making it possible to solve hard problems associated with that phenomena if produced naturally (*analog*

quantum simulators) or to have it in some device with effective couplings given that it is hard to find it in nature (*digital* quantum simulators) [7]. In last years, many researchers have proven quantum simulators to offer incredible utilities, like the production of pure unexpected quantum many-body phenomena, this is, not described by thermodynamics as these systems are affected by quantum fluctuations and not thermal fluctuations [7–9]. Because of it, the expectation of creating new materials with exotic behavior in the macro-world is growing a lot as time passes (for example with superconducting, topological, or transport efficiency) [7]. Researchers still working on improvements that lead to more possibilities for the future world, which could mold how we understand quantum physics and all the technology we use [7]. The fact is, quantum simulators can be used to emulate many phenomena for the desired building of the quantum computer and also quantum computers can be used to simulate required phenomena.

One of these improvements, by the usage of electronics, is that information and energy are not lost by heating, desiring materials with less and less electric resistance up to the point it is zero. The phenomenon where the electric resistance is zero is well known as **superconductivity** [5]. In 1911, Onnes [10] discovers that some metals suffer an electric resistance fall after cooling it below a critical temperature (which reduced the electric resistance up to fourteen orders of magnitude [11]). The theory that explains superconductivity in most metals emerges from the interaction between electrons and vibrations of the lattice where the electrons belong [12]. The interaction is given by means of virtual phonon exchange, which causes electron pairs to attract under an energy condition (BCS theory) [13]. In general, BCS theory (Barden, Cooper, and Schrieffer in 1957) is not able to explain superconductivity in all metals [12], and yet it is sufficiently simple for calculating thermal and electromagnetic properties of the system in which is applicable, with good experimental results [12, 13]. Hence, it is usually indispensable for the building of quantum computers and not so accessible due to temperature conditions, as it can be shown in highly coherent systems created with some quantum simulators such as superconducting circuits and trapped ions [7], showing the importance of dropping the resistance.

Alongside superconductivity, researchers have proposed to introduce the key feature of topology to overcome the decoherence mechanism always present in realistic setups, emerging in this way the introduction of **topological phases**. The system enters a topological phase when some of its observable properties are invariant under soft deformations of the space-time where the system exists, under low temperatures, low energies, and long wavelength conditions [14]. Recent theories and experiments show that a strong spin-orbit coupling (SOC) brings two and three-dimensional systems to the topological superconducting phase, in which high coherent modes are created at the edges [15]. In one dimension it is possible that, in addition to the strong SOC, an external magnetic field and the standard s-wave superconducting coupling, may bring a fermionic system to the superconducting topological phase [16, 17], where the edge modes are known as Majorana fermions (Majoranas). In this phase, the original fermions arrange in pairs of spatially-separated Majoranas, being highly coherent under local perturbations, useful to protect the information of the quantum computer, for example [16–18]. Although there have been many efforts to find the Majoranas, it has been quite elusive for current status experimental setups [17–19], or although measured there is no consensus of whether the observed excitations are indeed Majoranas. Thus, the proposal of systems that could support them, to have a final suitable setup, is a hot topic in physics these days.

To illustrate, one trending research from the past decade proposed one specific setup, showing

the existence of Majoranas theoretically and experimentally [20–24]. The system is based on a one-dimensional gas of atoms coupled by SOC, made from a heterostructure of Indium and Arsenide that is cut in a cylindrical form with a diameter in the scale of nm and a length-to-width ratio of 1000 or more (an InAs semiconductor nanowire). The superconductivity is induced by proximity putting the wire in contact with a superconducting material at low temperatures and both are exposed to a uniform magnetic field created by coils, as shown in figure 1.1.

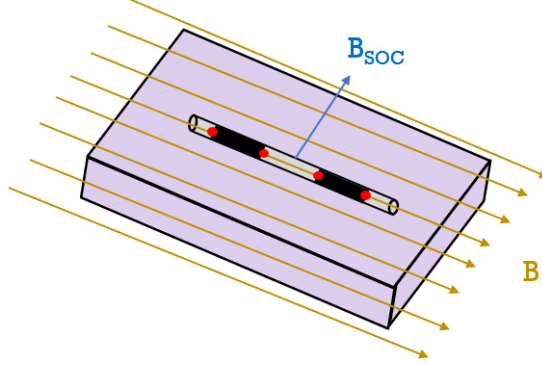


Figure 1.1: The drawing shows the setup used by Lutchyn, Oren, and Bakkers, where the nanowire is the grey cylinder, the superconductor is the purple block, and the arrows are the magnetic field lines. The uniform magnetic field is painted golden and goes lengthwise with the wire, while the SOC can be interpreted as another inner magnetic field that is perpendicular to the golden one (with an arrow painted blue), as explained further when describing the SOC model. The phase transition from a non-topological to a topological phase is achieved with a modulated chemical potential on the wire, where two black regions appear with Majoranas at the edges (red points). As of the usage of InAs, the SOC energy scale is around $50\mu eV$, the superconducting gap around $250\mu eV$ so the theoretical expectation of the zero temperature phase is at a magnetic field of $0.15T$ and below $1K$ temperature on the laboratory [22].

The objective of the present work is to solve a Hamiltonian model of the latter system and show that it could support the Majorana fermions, which fundamentally is: a superconducting fermionic system of spin $1/2$ coupled with SOC and a magnetic field. The physics of the system arises from the collective interactions of the particles since the system is made of many bodies in one dimension, where that dimensionality plays an important role [25]. If the system is a three-dimensional one, the perturbation theory can be applied, as short-range interactions could be neglected if particles do not encounter themselves: they have enough mean free path to escape the interaction range [25]. But strong interactions could be relevant in one dimension for the sought phases [16, 25], so the one-dimensional model may not accept the perturbation theory framework as only collective excitations dominate the physics (Fermi liquid theory does not hold). The *bosonization* technique constitutes an interesting choice for the treatment of such strongly correlated systems, allowing the writing of the Hamiltonian model into a quantum field theory [25]. Once the Hamiltonian is solved in terms of the fields, the information of the phases is acquired from the calculation of *correlations* via the mode decomposition of the fields by means of the *Path Integral* formalism [26–28], and are interpreted from the *Linear Response Theory* (LRT) [25, 26]. The information on the presented model and techniques is expanded in the following sections.

1.1 Second quantization

The most common and suitable description for a many-body system is given in the second quantization formalism [1–3]. It establishes the re-quantization of first quantized operators depending on whether the operator represents a one-particle or two-particle process [1–3]. One-particle processes refer to the scattering of one particle by means of an external coupling, like for example a chemical potential, a magnetic field, a potential trap, or simply the kinetic energy (dynamics information of one particle). The two-particle processes refer to the scattering processes involving two particles, like the possible interaction potential among them which produce collisions. If $\hat{t}$ is a one-particle operator and $\hat{u}$ a two-particle operator, the operators are represented in second quantization as [1–3]:

$$\hat{T} = \sum_{ij} \hat{\psi}_i^\dagger \langle i | \hat{t} | j \rangle \hat{\psi}_j, \quad (1.1)$$

$$\hat{U} = \frac{1}{2} \sum_{ijnm} \hat{\psi}_i^\dagger \hat{\psi}_j^\dagger \langle ij | \hat{u} | nm \rangle \hat{\psi}_m \hat{\psi}_n, \quad (1.2)$$

where $\hat{\psi}_j$ ($\hat{\psi}_j^\dagger$) is the annihilation (creation) operator for a single-particle state $|j\rangle$ given a quantum number j .

The particles are in a continuum for a Fermi gas, so the first quantum number is the position x . Also, the particles have spin $1/2$, and it introduces discrete quantum numbers: the two projections $\sigma = \uparrow$ ($\frac{1}{2}$) and $\sigma = \downarrow$ ($-\frac{1}{2}$) (from here $\hbar = 1$ is set). As position and spin reside in different Hilbert spaces, the state in space coordinates is $|j\rangle = |x\sigma\rangle$, where $\langle \sigma' x' | x\sigma \rangle = \delta(x - x')\delta_{\sigma\sigma'}$ and then $\hat{\psi}_j^\dagger = \hat{\psi}_{x\sigma}^\dagger = \hat{\psi}_\sigma^\dagger(x)$ (which represents a fermion of spin σ in position x) [1–3], where fermions satisfy $\{\hat{\psi}_\sigma(x), \hat{\psi}_{\sigma'}^\dagger(x')\} = \delta(x - x')\delta_{\sigma\sigma'}$ and $\{\hat{\psi}_\sigma^{(\dagger)}(x), \hat{\psi}_{\sigma'}^{(\dagger)}(x')\} = 0$. On the other hand, the space coordinates and momentum coordinates are connected via a Fourier transform as they are canonical variables, where the energy or dispersion relation is determined in the momentum space [1–3]. In second quantization, once the system is described by $|x\sigma\rangle$ states, the momentum representation is given by the inverse Fourier transform of the creation and annihilation operators, which is [1–3, 29]:

$$\hat{\psi}_\sigma(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk e^{ikx} \hat{\psi}_\sigma(k), \quad (1.3)$$

for any spin index. In the transform, the hermitian conjugate state $|k\sigma\rangle$ is defined, with k the momentum, represented in space coordinates as $k = -i\partial_x$.

In the following, any operator is considered in second quantization and its treatment (to solve the Hamiltonian) is based on it. In particular, the density number operator $\hat{n}_j = \hat{\psi}_j^\dagger \hat{\psi}_j$ gives the number of particles in the state j , so $\hat{N} = \sum_j \hat{n}_j$ is the number operator [1–3]. Here must be clarified that the many-body state $|j\rangle$ is in fact $|\{j\}\rangle$, so the index j refers to the occupation on that state among the many others in the set $\{j\}$ of states, say $|j\rangle = |\{j\}\rangle = |j_1, j_2, j_3, \dots, j_s, j_{s+1}, \dots\rangle$. Hence, the operator $\hat{n}_j$ counts the number of particles in one of the j_s states and $\hat{N}$ is the sum of all occupations in all possible states. The index s in j_s is suppressed for simplification [1–3].

If the model is to be solved, in second quantization the solving method resides in making the Hamiltonian proportional to the number operator (i.e. expressing it in quadratic forms), and

when that proportion is given the Hamiltonian is *diagonal* [30]. For a general Hamiltonian, the task of diagonalizing could be very difficult, and commonly it may need many approximations. In any way, the fundamental form of the Hamiltonian to diagonalize is a *quadratic* one in creation and annihilation operators, which can be reached using the bosonization technique, a fundamental tool to address one-dimensional systems.

1.2 Bosonization formalism

For a free one-dimensional Fermi gas without spin, the Hamiltonian is:

$$\hat{H} = \int dk \epsilon(k) \hat{\psi}^\dagger(k) \hat{\psi}(k), \quad \epsilon(k) = \frac{k^2}{2m}, \quad (1.4)$$

where the dispersion relation is a parabola. As quantum simulators have been idealized with ultracold atoms in the last years [7], a low energy approximation is considered, where the fermions arrange in all momenta up to the last particle, and the last momentum defines the Fermi energy (E_F , also called Fermi level or Fermi surface). In that state, all particles below the Fermi energy are frozen, so it is a good approximation to consider the processes only around this energy. Thus, the dispersion relation may be approximated as straight lines around the Fermi momenta $\pm k_F$ (Taylor expansion up to first order). The bosonization method consists in extend those lines from $(-\infty, \infty)$, defining chiral fermions for each linear dispersion relation at left (around $-k_F$) and right (around k_F), and solving the model for the chiral fermions, as it is simpler and captures the same original system at low energies, inside a cutoff, to keep the validity of the approximation [25]. The schematic of the procedure is shown in the following figure 1.2.

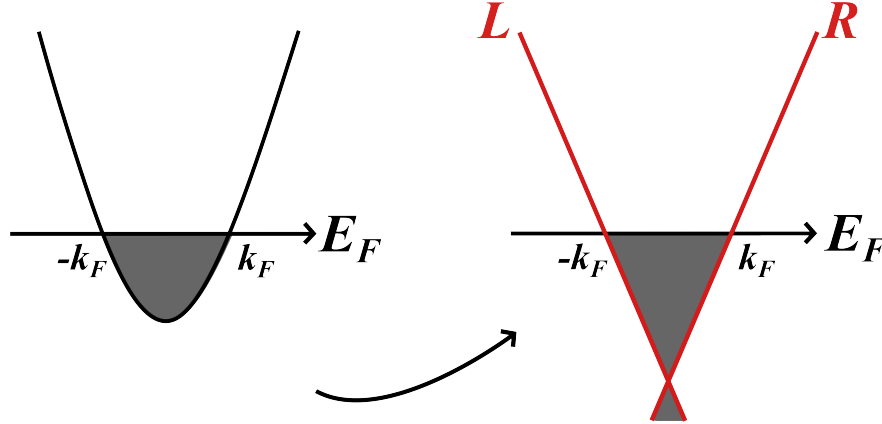


Figure 1.2: Original dispersion relation of the free one-dimensional fermions (at left) and the linearized dispersion relation (at right). The two lines define the dispersion relation for the chiral fermions, where the L line is for the left-moving fermions and the R is for the right-moving fermions. The figure also shows the Fermi energy and Fermi momenta, alongside the grey blur which is the filled part of the dispersion relation.

The left-moving and right-moving particles are defined as separating the Hamiltonian into two

parts that sum around the Fermi momenta:

$$\begin{aligned}\hat{H} &= \int dk \epsilon(k) \hat{\psi}^\dagger(k) \hat{\psi}(k) \\ &\approx \lim_{\Lambda \rightarrow 0} \int_{-k_F - \Lambda}^{-k_F + \Lambda} dk \epsilon_L(k) \hat{\psi}^\dagger(k) \hat{\psi}(k) + \int_{k_F - \Lambda}^{k_F + \Lambda} dk \epsilon_R(k) \hat{\psi}^\dagger(k) \hat{\psi}(k),\end{aligned}\quad (1.5)$$

where Λ is a short interval around each of the Fermi momenta (as the limit specifies it), $\epsilon_L(k)$ and $\epsilon_R(k)$ energies are the Taylor series expansion of $\epsilon(k)$ up to first order (linearization) around their respective Fermi momenta,

$$\epsilon_L(k) = \epsilon(-k_F) + (k - (-k_F)) \left(\frac{d\epsilon(k)}{dk} \right)_{-k_F} = E_F - \nu_F(k + k_F), \quad (1.6)$$

$$\epsilon_R(k) = \epsilon(k_F) + (k - k_F) \left(\frac{d\epsilon(k)}{dk} \right)_{k_F} = E_F + \nu_F(k - k_F), \quad (1.7)$$

defining the Fermi velocity as $\nu_F = \partial_k \epsilon(k)_{k_F} = k_F/m$, that is by definition the velocity of the particles that move in their respective branch (negative for the left-moving and positive for the right-moving). If the energies are replaced in the approximated Hamiltonian of equation 1.5, it yields:

$$\begin{aligned}\hat{H} &\approx \lim_{\Lambda \rightarrow 0} \int_{-k_F - \Lambda}^{-k_F + \Lambda} dk (E_F - \nu_F(k + k_F)) \hat{\psi}^\dagger(k) \hat{\psi}(k) \\ &\quad + \int_{k_F - \Lambda}^{k_F + \Lambda} dk (E_F + \nu_F(k - k_F)) \hat{\psi}^\dagger(k) \hat{\psi}(k) \\ &= \lim_{\Lambda \rightarrow 0} E_F \int_{-k_F - \Lambda}^{-k_F + \Lambda} dk \hat{\psi}^\dagger(k) \hat{\psi}(k) + E_F \int_{k_F - \Lambda}^{k_F + \Lambda} dk \hat{\psi}^\dagger(k) \hat{\psi}(k) \\ &\quad - \int_{-k_F - \Lambda}^{-k_F + \Lambda} dk \nu_F(k + k_F) \hat{\psi}^\dagger(k) \hat{\psi}(k) + \int_{k_F - \Lambda}^{k_F + \Lambda} dk \nu_F(k - k_F) \hat{\psi}^\dagger(k) \hat{\psi}(k) \\ &= \lim_{\Lambda \rightarrow 0} 2E_F - \int_{-\Lambda}^{\Lambda} dp \nu_F p \hat{\psi}^\dagger(p - k_F) \hat{\psi}(p - k_F) + \int_{-\Lambda}^{\Lambda} dp \nu_F p \hat{\psi}^\dagger(p + k_F) \hat{\psi}(p + k_F) \\ &= \lim_{\Lambda \rightarrow 0} 2E_F - \int_{-\Lambda}^{\Lambda} dp \nu_F p \hat{\psi}_L^\dagger(p) \hat{\psi}_L(p) + \int_{-\Lambda}^{\Lambda} dp \nu_F p \hat{\psi}_R^\dagger(p) \hat{\psi}_R(p),\end{aligned}\quad (1.8)$$

where in the third line, the integrals with E_F coefficient are considered the expected value of the Hamiltonian ($\langle \hat{H} \rangle$) because they express the density number operators in the region of integration, which in the limit of $\Lambda \rightarrow 0$ are the particle number at the Fermi energy, i.e. just one. Then, as those are the particles in $-k_F$ and k_F , there are two particles at the Fermi energy; hence the term reduces to $2E_F$, a constant for every equilibrium state. Also in the third line, a variable change is applied in both integrals to consider the variation $k \pm k_F$, and thus in the last line, the chiral fermion operators are defined according to $\hat{\psi}(p - k_F) = \hat{\psi}_L(p)$ and $\hat{\psi}(p + k_F) = \hat{\psi}_R(p)$, setting now $p = 0$ as the point where both linear dispersion relations meet each other.

Now, since the lines are extended along the $\mathbb{R}$ (so the limit $\Lambda \rightarrow 0$ is no longer considered), and the Hamiltonian is replaced for:

$$\tilde{\hat{H}} = \int dp \tilde{\epsilon}(p) \left(\hat{\psi}_R^\dagger(p) \hat{\psi}_R(p) - \hat{\psi}_L^\dagger(p) \hat{\psi}_L(p) \right), \quad \tilde{\epsilon}(p) = \nu_F p, \quad (1.9)$$

which is the Tomonaga-Luttinger model [31–33]. Note that the constant factor $2E_F$ has been removed since a constant will not affect the further analysis of the Hamiltonian by means of the Path Integral approach (see the definition in 1.14). According to the Hamiltonian of equation 1.9, the system space separates into two chiral spaces, so the chiral fermions commute, $[\hat{\psi}_R^{(\dagger)}(k), \hat{\psi}_L^{(\dagger)}(k')] = 0$ (the separability of the space and the commutator are shown in the introduction of the appendix A) [25, 34, 35]. But, in contrast with the figure 1.2, the dispersion relations touch at $k = 0$, so there is a point where the spaces converge. Then, the commutator is not strictly zero, instead, it is considered that they commute by leaving aside that specific point, and as it is just a point, the integral in equation 1.9 is not sensitive to it. This particular choice means this work uses *abelian bosonization*. One important feature of this extension is that if the Fermi energy is still there, then the system will be infinitely filled (the linearized dispersion relation is not bounded from below, as shown in figure 1.2), so for example, the constant term $2E_F$ must be infinite now! In fact, there are many infinities arising from this extension, like the expected number of particles on the ground state ($\langle \hat{\psi}^\dagger(k) \hat{\psi}(k) \rangle \rightarrow \infty$). Hence, as in quantum field theory, the normal ordering is introduced to deal with those infinities, where it takes the form [25–28]:

$$: \hat{\psi}^\dagger(k) \hat{\psi}(k) := \hat{\psi}^\dagger(k) \hat{\psi}(k) - \langle \hat{\psi}^\dagger(k) \hat{\psi}(k) \rangle, \quad (1.10)$$

so only the normal ordered operators contribute (like $\hat{\psi}^\dagger(k) \hat{\psi}(k)$) and others do not (like $\hat{\psi}(k) \hat{\psi}^\dagger(k)$) [25–28]. The normal ordering of the operators are assumed from now on and it will not be denoted to avoid extra notation (for example, the Hamiltonian is in fact $:\hat{H}:$, but in the document is simply denoted as $\hat{H}$). This consideration clearly has implications for calculations that eventually are mentioned.

Now, one of the most useful relations in bosonization is the writing of the annihilation operator in terms of the chiral fields in space coordinates. This is done using the Fourier transform of equation 1.3 and only taking the contributions of the Fermi momenta, which follows:

$$\begin{aligned} \hat{\psi}(x) &= \frac{1}{\sqrt{2\pi}} \int dk e^{ikx} \hat{\psi}(k) \approx \lim_{\Lambda \rightarrow \infty} \frac{1}{\sqrt{2\pi}} \int_{-k_F - \Lambda}^{-k_F + \Lambda} dk e^{ikx} \hat{\psi}(k) + \frac{1}{\sqrt{2\pi}} \int_{k_F - \Lambda}^{k_F + \Lambda} dk e^{ikx} \hat{\psi}(k) \\ &= \lim_{\Lambda \rightarrow \infty} \frac{1}{\sqrt{2\pi}} \int_{-\Lambda}^{\Lambda} dp e^{i(p - k_F)x} \hat{\psi}(p - k_F) + \frac{1}{\sqrt{2\pi}} \int_{-\Lambda}^{\Lambda} dp e^{i(p + k_F)x} \hat{\psi}(p + k_F) \\ &= e^{-ik_F x} \frac{1}{\sqrt{2\pi}} \int dp e^{ipx} \hat{\psi}_L(p) + e^{ik_F x} \frac{1}{\sqrt{2\pi}} \int dp e^{ipx} \hat{\psi}_R(p) \\ &= e^{-ik_F x} \hat{\psi}_L(x) + e^{ik_F x} \hat{\psi}_R(x). \end{aligned} \quad (1.11)$$

The extension is introduced in the third line of equation 1.11. The latter is the corresponding mapping of the Fermi gas to the Tomonaga-Luttinger liquid, where that equation is vastly used to bosonize a Hamiltonian, clearly depending on the dispersion relation. In the low energy approximation, the rapid fluctuations are given by the phases $e^{\pm ik_F x}$ while the chiral fields are slow-varying ones. Then, the most relevant physics comes from fluctuations of the density ($\hat{\rho}$) and current ($\hat{J}$) operators, interesting physics arises when the Hamiltonian of equation 1.9 is written in terms of these operators (see appendix A, equations A.18 and A.19). The revealing part which makes the technique be called bosonization, is that the defined densities commute as bosons, so a theory written in these densities is a theory in a bosonic space: the original fermionic system is mapped into a bosonic one, where the canonical bosonic fields

$\hat{\phi}(x, t)$ and $\hat{\pi}(x, t)$ ($[\hat{\phi}(x), \hat{\pi}(y)]_{t=0} = i\delta(x - y)$) diagonalize the system via the fundamental identity of bosonization:

$$\hat{\psi}_\alpha(x) = \frac{1}{\sqrt{2\pi\delta}} \hat{\eta}_\alpha e^{i\alpha 2\sqrt{\pi}\hat{\phi}_\alpha(x)}, \quad (1.12)$$

where α stands for chirality in ± 1 values ($\alpha = 1$ means R and $\alpha = -1$ means L), δ is a cutoff parameter introduced by hand in calculations to make quantities converge, $\hat{\eta}_\alpha$ are chiral Klein factors and $\hat{\phi}_\alpha(x)$ are the chiral canonical fields (for the derivation of the theory in densities and currents, the definition of the cutoff, Klein factors, canonical fields, and their chiral parts see appendix A). With the bosonization identity, the free Fermi gas of equation 1.4 is mapped as follows:

$$\hat{H} = \int dk \epsilon(k) \hat{\psi}^\dagger(k) \hat{\psi}(k) \approx \frac{\nu_F}{2} \int dx \left[\left(\partial_x \hat{\phi}(x) \right)^2 + \hat{\pi}(x)^2 \right]. \quad (1.13)$$

The mapping corresponds to the massless Klein-Gordon equation (the wave equation), in which a mode decomposition of the fields is based, and it shows the final writing of the system in terms of bosonic fields (see appendix A, equation A.58). Such decomposition is fundamental for the approach of this work on the analysis of the Hamiltonian, and so the efforts are directed to achieve the last diagonal form of the Hamiltonian in equation 1.13. This form, given by a quantum field theory, is of widely use as it is universal [36], useful when reviewing lattice models as when working in the continuum limit, since in both cases the same quantum field theory analysis is valid [36]. Unfortunately, as in almost every theory, this form is not trivial to get and there is much physics that arises in non-diagonal terms, like cosine functions of the fields that appear in the interaction terms such as the operator in equation 1.2, which moves the model from the Klein-Gordon to the sine-Gordon [37–39]. An analysis of those cosines and possible treatments are given in the section A.4 of appendix A and section B.8 of appendix B. The procedure allows us to calculate correlations if the Hamiltonian is quadratic. In order to do so, a combination of the Path Integral formalism together with the Linear Response Theory opens this possibility, hence in the following section the methodology is reviewed.

1.3 Correlations, Path Integral, and LRT

The information revisited for the sought phases resides on correlations, which are expected values of products of creation and annihilation operators following the structure $\langle \hat{a}^\dagger \hat{a} \dots \rangle$. According to the bosonization technique, these correlations should be given in terms of the canonical fields of the theory, defining the n -point correlation function $\langle \hat{\phi}(x_1) \hat{\phi}(x_2) \hat{\phi}(x_3) \dots \hat{\phi}(x_n) \rangle$, and if time evolution is considered, time order is introduced, so $\langle \mathcal{T} \hat{\phi}(x_1, t_1) \hat{\phi}(x_2, t_2) \hat{\phi}(x_3, t_3) \dots \hat{\phi}(x_n, t_n) \rangle$ is a standard form of a correlation for the bosonized system. From now on, if the correlations have time dependence, a time order operator is set in the correlations although it is not written explicitly.

In this case, due to the mode decomposition of the fields in bosonic operators, and in general, the 1-point correlation vanishes ($\langle \hat{\phi}(x) \rangle$) as it measures the expectation value of the field amplitude. The first non-vanishing correlation is the 2-point one ($\langle \hat{\phi}(x) \hat{\phi}(y) \rangle$), where it is conceived as a propagator in the sense that if the field represents some property at some point, thus in the ground state, for example, the expectation value $\langle \hat{\phi}(x) \hat{\phi}(y) \rangle$ tells us how that property in point x is somehow related with the property in another point y . Here is when the name of those expected values as correlations start to have sense, as it measures how a property value in one point is correlated with the same in another point as if the system

starts to have a directory in which that property is given and it is not just random. It is said the system begins to *order*.

The property by itself does not correlate, it needs an external coupling, say $f(x, t)$ mediated by a parameter A , so the Hamiltonian changes via the strength of that coupling [36, 40, 41]. In the low energy approximation, the coupling changes drastically the ground state if the parameter approaches some critical value A_c (which maps into a macroscopic change of a system property), where is said such parameter tunes a different behavior of the system, it goes through a transition. If only the ground state is considered, the transition occurs at zero temperature, which is called a *quantum phase transition*, because it does not involve thermal fluctuations. Relevant transitions of this kind discovered in past years are for example the paramagnetic-ferromagnetic, superconductor-insulator, and superfluid-Mott insulator transitions.

Fundamentally, tuning an external coupling provokes a breaking of some symmetry of the system such that a local order starts to appear on a small region around the point of affection, which can be extended via a high response of the system according to the tuning parameters that break the symmetries. In the case of topological order, the coupling changes some non-local parameters of long-range correlations on the ground state [40]. Thus, for the remaining work, this is the procedure to establish if a phase appears: the correlation is associated with some property, then it is calculated on the ground state via the mode decomposition of the fields, and the condition for the parameters to cause a high response are found according to the LRT [26, 36]. In case of a topological phase, the non-local order will be captured by Majoranas [16]. The conditions of the parameters are specific values that constitute a phase diagram [25, 34, 35, 42].

In this work, the correlations represent three properties: charge density ($\hat{\rho}$), spin density ($\hat{\sigma}$), and pairing ($\hat{O}$). The first two are associated with ordering in charge and spin precisely, while the pairings are associated with an ordering where particles arrange in pairs [12, 13, 25]. For example, the spinless charge density is given as $\hat{\rho}(x) = \hat{\psi}^\dagger(x)\hat{\psi}(x)$, while the spin 1/2 charge density is $\hat{\rho}(x) = \hat{\psi}_\uparrow^\dagger(x)\hat{\psi}_\uparrow(x) + \hat{\psi}_\downarrow^\dagger(x)\hat{\psi}_\downarrow(x)$. According to [25], the interesting correlations to study in this work are eight:

- Charge density
- Axial spin density (z axis).
- Planar spin densities (x and y axes).
- Spin-singlet pairing.
- Axial spin-triplet pairing.
- Planar spin-triplet pairings.

All of them are presented and calculated generically in appendix B.

If the correlation is of the general sort of $\langle f(\hat{\phi}(x_1))g(\hat{\phi}(x_2)) \rangle$, when assuming a thermal reservoir it is calculated as [1–3, 26]:

$$\begin{aligned} \langle f(\hat{\phi}(x_1))g(\hat{\phi}(x_2)) \rangle &= \frac{1}{\mathcal{Z}} \text{Tr} \left[f(\hat{\phi}(x_1))g(\hat{\phi}(x_2))e^{-\beta\hat{H}} \right] = \frac{1}{\mathcal{Z}} \int Dl \langle l | f(\hat{\phi}(x_1))g(\hat{\phi}(x_2))e^{-\beta\hat{H}} | l \rangle, \\ \mathcal{Z} &= \text{Tr} \left[e^{-\beta\hat{H}} \right] = \int Dl \langle l | e^{-\beta\hat{H}} | l \rangle, \end{aligned} \quad (1.14)$$

where the trace is calculated for some base $|l\rangle$, $\mathcal{Z}$ is the partition function, $\beta = 1/k_B T$ the inverse temperature, and the Hamiltonian $\hat{H}$ is of the form of equation 1.13 in the fields. Clearly, the base $|l\rangle$ must be the one that diagonalizes the Hamiltonian, then those have to be the corresponding states $|\phi\rangle$ and $|\pi\rangle$ of the fields. According to the mode decomposition of the fields in appendix A (see equations A.59, A.60 and A.70), the states of the fields must be given by coherent states, and using this specific basis the correlation takes the form:

$$\langle f(\hat{\phi}(x_1))g(\hat{\phi}(x_2)) \rangle = \frac{1}{\mathcal{Z}} \int D\phi D\pi f(\phi(x_1))g(\phi(x_2)) \langle \phi, \pi | e^{-\beta \hat{H}} | \phi, \pi \rangle, \quad (1.15)$$

which is pretty similar to the amplitude probability of evolution of a state from one position y to x in time t , with $\langle x | e^{-it\hat{H}} | y \rangle$ the propagator.

Feynman solves with the brilliant idea of the many paths which constitute the Path Integral approach. This approach is used to calculate the propagator of the correlations, that in this case is $\langle \phi, \pi | e^{-\beta \hat{H}} | \phi, \pi \rangle$. The form is not exactly the same as the first propagator presented, there are two things to analyze: the first is that the evolution is between $\langle \phi, \pi |$ and $|\phi, \pi\rangle$, so it is an evolution with equal initial and final points (closed paths), and the second is that the parameter β does the rol of a time in the evolution. This is solved by mapping $\beta = i\tau$ where τ is an *imaginary time* [26]. Hence, the correlations are calculated in imaginary time, where the fields are evaluated in $\tau = it$, with t the real time (Wick's rotation). In order to analyze solely the ground state properties, the zero temperature limit must be considered at the end of the calculation ($\beta \rightarrow \infty$).

The latter presents how the Path Integral formalism is used in this calculation, but for in this work it is useful to argue the apparition of imaginary time and separability, while the calculations are done with the mode expansion presented in A.59, A.60 and A.70 of appendix A for the ground state defined in the bosonization, which means the calculation is at zero temperature.

Although complex elements are used in the calculations, such as imaginary time and zero temperature, they can have real physical meaning. In fact, one has to recall that the problem of thermal fluctuations is not that they vanish, is that the scale of energy for such fluctuations is not relevant compared with quantum fluctuations. Thus, despite zero temperature being impossible according to the third law of thermodynamics, quantum phase transitions can still be observed in a laboratory at finite temperature by identifying the behavior of the phase under the correct energy scale [36, 41]. On the other hand, imaginary time correlations are not the ones directly measured, but they are related to the real functions that enable experiments to show the phases [41]. In this work, the approach is given by the LRT, where the correlations can be used to calculate how the system responds to the external coupling $f(x, t)$, which are the *susceptibilities*.

To understand a bit more about the susceptibility, let us review the magnetic susceptibility χ_m , which is defined in linear coupling with the equation $\mathbf{M} = \chi_m \mathbf{H}$ [43]. $\mathbf{M}$ is the magnetization of some sample when an external magnetic field $\mathbf{H}$ is applied. The tinier χ_m the bigger $\mathbf{H}$ for a given $\mathbf{M}$, so it means there is a low response (as big magnetic fields are needed to magnetize the sample). Instead, if χ_m is big it means tiny $\mathbf{H}$ will cause big $\mathbf{M}$, so the system responds highly. In the case where an infinite susceptibility is triggered, according to the tuning parameter, the system responds immensely to any $\mathbf{H}$, no matter how tiny it is. When the susceptibility diverges, the system “wants” to order [25], which means the corresponding property would

take a constant value and not just a random one that averages to zero if there is a mechanism that allows the ordering to happen (like some specific coupling which breaks a symmetry), where in this magnetic example is the case for ferromagnets, where the magnetic field breaks the axial symmetry of orientation of the spins.

In the most general case, LRT allows us to calculate the susceptibility in linear order coupling corresponding to a correlation in equation 1.16 for a tiny external force (enough for a system with high response) [25, 26],

$$\langle f(\hat{\phi}(x', t'))g(\hat{\phi}(x, t)) \rangle = \int_{x', t'} dy dt'' \chi(x, t; y, t'') f(y, t'') + \mathcal{O}(f^2). \quad (1.16)$$

To calculate the susceptibility from the latter equation, as the Hamiltonian in equation 1.13 is translational-invariant and time-invariant, the correlation will depend only on space and time separation, no matter what the points are ($\langle f(\hat{\phi}(x', t'))g(\hat{\phi}(x, t)) \rangle = F(x, t; x', t') = F(|x - x'|, |t - t'|) = F(\mathbf{x}, \mathbf{t})$). In this case, the Fourier transform of equation 1.16 gives a simpler LRT equation [25, 26]:

$$F(\mathbf{q}, \omega) = \chi(\mathbf{q}, \omega) f(\mathbf{q}, \omega), \quad (1.17)$$

where it is in complete analogy with the relation for the magnetic susceptibility, and the functions evaluated in momentum $\mathbf{q}$ and frequency ω are the Fourier transform of the same functions in equation 1.16. An illustration of the susceptibility meaning is shown in figure 1.3.

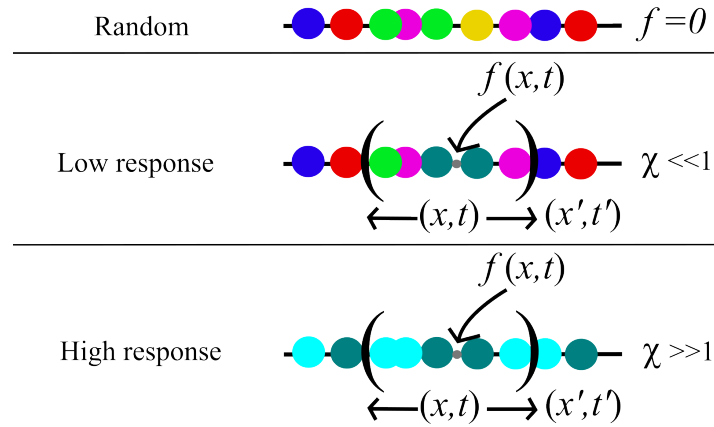


Figure 1.3: Illustration of the susceptibility role on LRT, where it shows a one-dimensional arrangement of particles in which a color is their respective property. At the top, no external force acts so the displayed property is random, and it averages to zero locally (normalized to zero by normal order in equation 1.10). At the center, the external force is introduced and, according to equation 1.16, it acts on a region around the points of the evaluated correlation. If the susceptibilities are tiny, the property tries to remain at zero average and does respond little to the external force, hence partial random patterns appear. At the bottom of the figure, as the susceptibilities are immense, the system responds highly to the external force such that eventually, all particles fix their property according to the others in their vicinity, establishing a long or quasi-long range order displaying a non-vanished local average.

One important feature realized from figure 1.3 is the symmetry breaking, where before the external force acts the property averages to zero anywhere on the system, afterwards there is a color symmetry all around it. When there is a high response, this symmetry is broken and there are different values for the color in different places [36, 40, 41]. If such symmetry is not somehow broken by a real external coupling, then the order is not true although the susceptibility diverges (that is why it was said the system wanted to order, not to order indeed) [25].

One crucial relation between the calculated correlation and the real susceptibility is given in equation 1.17, and from this equation the conditions for $\chi(\mathbf{q}, \omega)$ to diverge are derived, happening at certain $\mathbf{q}$ and ω (a maximum peak). The emerged peaks are not usually delta functions (possible poles of the relation), as those peaks belong to well-defined quasiparticle-like excitations [25, 36], which do not exist purely because of the strongly correlated nature of one-dimensional systems [25]. Instead, there is a dissipative continuum of critical excitations outside the characteristic momentum and frequency of the low-energy process which enables the ordering to happen, in which the process is associated with an external coupling or interaction term [36]. The conditions for the latter behavior are of great importance in the present work, as it is a generic property of a strongly coupled quantum critical point [36]. Finally, the conditions are usually compiled in a phase diagram, summarizing the purpose of the work.

The present document is divided as follows: chapter 2 establishes the model with the corresponding terms discussed in this introduction, and afterwards the bosonization is applied to each of them in chapter 3, according to the background of the theory given in appendix A. In this chapter 3, the operators for the mentioned correlations are also bosonized, and are calculated following the general results of the appendix B. In chapter 4, the analysis of the susceptibilities and final results are presented, showing the behavior of the properties of the system and all possible phase diagrams. At last, in chapter 5 some conclusions and perspectives of the work are discussed.

Chapter 2

The model

The system to model is a Fermi gas in one dimension, with internal degrees of freedom of spin 1/2 that are coupled via an effective SOC and an external linear Zeeman field. Additionally, the particles are subjected to an effective attractive interaction between opposite spins. Considered at zero temperature, the Fermi level can be fixed by a chemical potential μ , and the ensemble could be in the presence of an external trapping potential $V(x)$. Hence, when modeling each part of the system, one gets a Hamiltonian $\hat{H}$ made of three fundamental parts:

$$\hat{H} = \hat{H}_F + \hat{H}_C + \hat{H}_I, \quad (2.1)$$

$$\hat{H}_F = \int dx \hat{\Psi}_{\uparrow\downarrow}^\dagger(x) \left[-\frac{\partial_x^2}{2m} + V(x) - \mu \right] \hat{\Psi}_{\uparrow\downarrow}(x), \quad (2.2)$$

$$\hat{H}_C = \int dk \hat{\Psi}_{\uparrow\downarrow}^\dagger(k) \left[\sum_{j=x,y,z} (\alpha_j k_j + h_j) \sigma^j \right] \hat{\Psi}_{\uparrow\downarrow}(k), \quad (2.3)$$

$$\hat{H}_I = -U \int dx \hat{\psi}_\uparrow^\dagger(x) \hat{\psi}_\downarrow^\dagger(x) \hat{\psi}_\downarrow(x) \hat{\psi}_\uparrow(x), \quad (2.4)$$

where $\hat{\Psi}_{\uparrow\downarrow}^\dagger(x) = (\hat{\psi}_\uparrow^\dagger(x), \hat{\psi}_\downarrow^\dagger(x))$ are the field operators in space representation, whereas $\hat{\Psi}_{\uparrow\downarrow}^\dagger(k)$ in momentum representation, conveniently used for retrieving the second quantized form of the one-and two-particle operators in one dimension, note the integral depending on one solely component. It is understood the operators in the Hamiltonians of equations 2.2 and 2.3 are multiplied by the 2x2 identity matrix, which in the following the matrix elements are associated as $(1,1) \rightarrow (\uparrow, \uparrow)$, $(1,2) \rightarrow (\uparrow, \downarrow)$, $(2,1) \rightarrow (\downarrow, \uparrow)$ and $(2,2) \rightarrow (\downarrow, \downarrow)$ [1–3]. $\hat{H}_F$ contains the kinetic energy, the trap, and the chemical potential. $\hat{H}_C$ are the couplings, including the SOC with strengths $\boldsymbol{\alpha} = (\alpha_x, \alpha_y, \alpha_z)$, and momentum $\mathbf{k} = (k_x, k_y, k_z)$ respectively, while the Zeeman coupling strengths are denoted as $\mathbf{h} = (h_x, h_y, h_z)$ with Pauli matrices $\boldsymbol{\sigma} = (\sigma^x, \sigma^y, \sigma^z)$. This term is a general one related to its experimental realization [44], however, in the present work the SOC is considered in the $\hat{e}_x$ direction (planar), and the Zeeman coupling will have planar and a component in the $\hat{e}_z$ direction (axial). $\hat{H}_I$ is the model of the attractive interaction, commonly seen as the BCS superconducting term with strength $U > 0$. In the following, every term is analyzed.

2.1 The Hamiltonian terms

2.1.1 Space-varying chemical potential

The first thing to discuss is the role of the chemical potential and the external trap in $\hat{H}_F$. The term can be seen in two ways:

1.

$$\hat{H}_F = \sum_{\sigma=\uparrow,\downarrow} \int dx \left[\hat{\psi}_{\sigma}^{\dagger}(x) \left(-\frac{\partial_x^2}{2m} + V(x) \right) \hat{\psi}_{\sigma}(x) - \mu \hat{\psi}_{\sigma}^{\dagger}(x) \hat{\psi}_{\sigma}(x) \right]. \quad (2.5)$$

2.

$$\hat{H}_F = \sum_{\sigma=\uparrow,\downarrow} \int dx \left[\hat{\psi}_{\sigma}^{\dagger}(x) \left(-\frac{\partial_x^2}{2m} \right) \hat{\psi}_{\sigma}(x) - (\mu - V(x)) \hat{\psi}_{\sigma}^{\dagger}(x) \hat{\psi}_{\sigma}(x) \right]. \quad (2.6)$$

Ergo, one can interpret the system of free fermions with a chemical potential μ under the presence of a trap, or it can be viewed as a system of free fermions with a chemical potential that varies in space as $\mu(x) = \mu - V(x)$ [45, 46]. Both are equivalent of course, but the second offers a way of tuning the system lengthwise. If we have N fermions, the trap introduces a distribution of them in space that varies as the trap does, constituting regions of higher or lower density, and decreasing at the edges due to the trap. The latter constitutes one way of modeling the modulation of the chemical potential required in the experimental setup of figure 1.1. Hereinafter $\hat{H}_F$ is understood as,

$$\hat{H}_F = \int dx \left[\hat{\Psi}_{\uparrow\downarrow}^{\dagger}(x) \left(-\frac{\partial_x^2}{2m} - \mu(x) \right) \hat{\Psi}_{\uparrow\downarrow}(x) \right]. \quad (2.7)$$

2.1.2 Diagonalization of the couplings

The SOC refers to the coupling between the internal atomic pseudo-degrees of freedom of spin and the external atomic degrees of freedom of movement (momentum) [47]. This leads to a fine structure in the momentum of the atomic spectra, developing a certain dispersion relation per spin species. In the case of the Zeeman coupling, it is a coupling between the same degrees of freedom of spin and an external magnetic field, causing the fine structure of the spectra but in energy. Thus, both are quite similar, in the sense that moving charges on an electric field can be seen as perceiving an additional magnetic field in their rest frame, as it is shown in classical electrodynamics [43].

The Zeeman coupling is modeled due to the coupling energy of a magnetic moment $\boldsymbol{\mu}$ and a uniform magnetic field $\mathbf{B}$, so the first quantized operator $\hat{H}_Z$ is given as:

$$\hat{H}_Z \propto \boldsymbol{\mu} \cdot \mathbf{B} \propto \boldsymbol{\sigma} \cdot \mathbf{B} = B\sigma^z, \quad (2.8)$$

where the magnetic moment is said proportional to the spin $\boldsymbol{\sigma}$ and at the end $\mathbf{B} = B\hat{e}_z$ is assumed, so the effective energy has strength B (which is not the magnetic field strength but more constants that resume the full coupling energy). This term appears in $\hat{H}_C$ with the coupling strengths h_j in general. Now, in the case of particles with that magnetic moment but

moving, if the magnetic field is internal to the structure where the particles exist, then it can be related to an electric field $\mathbf{E}$ via $\mathbf{B} \propto \mathbf{k} \times \mathbf{E}$ in the rest frame of the laboratory, so the inner Hamiltonian $\hat{H}_{inner}$ would be:

$$\hat{H}_{inner} \propto \boldsymbol{\sigma} \cdot (\mathbf{k} \times \mathbf{E}) = \mathbf{E} \cdot (\boldsymbol{\sigma} \times \mathbf{k}) = \alpha(\mathbf{k} \times \boldsymbol{\sigma}) \cdot \hat{z}, \quad (2.9)$$

which is a form of the SOC operator [48], given as a one-particle coupling between the spin $\boldsymbol{\sigma}$ and momentum $\mathbf{k}$, replacing again $\mathbf{E} = E\hat{e}_z$ and summerazing the effective coupling strength in the parameter α , so in $\hat{H}_C$ these kind of terms are given with the strengths α_j . Notice that in the system we are working charge is not considered, so the best way to represent SOC is with the inner magnetic field denoted B_{SOC} , as represented in figure 1.1. With the latter form, fundamentally as a coupling between the spin and a magnetic field, there is one specific feature that generalizes its effect on the fermions, and it is that **breaks inversion symmetries** [48]. For example, in a one-dimensional free Fermi gas of spin 1/2, the dispersion relations for spin $\uparrow$ and spin $\downarrow$ are equal parabolas centered at $k = 0$, so they satisfy $\epsilon_{k\uparrow} = \epsilon_{k\downarrow}$, and also

$$\epsilon_{k\uparrow} = \epsilon_{-k\uparrow}, \quad \epsilon_{k\downarrow} = \epsilon_{-k\downarrow}, \quad (2.10)$$

which implies

$$\epsilon_{k\uparrow} = \epsilon_{-k\downarrow}, \quad (2.11)$$

where the relations in equation 2.10 imply a spacial inversion symmetry for spin $\uparrow$ and $\downarrow$, and the equation 2.11 imply the time inversion symmetry of the system. It is certain that if an axial effective Zeeman field is applied, then both dispersion relations are not equal anymore, they separate vertically by a constant amount proportional to the coupling strength, breaking the time inversion symmetry in consequence, and also the SOC couples the spin with some specific momenta so the dispersion relations are displaced horizontally, breaking spacial and time inversion symmetries.

Now, following the full couplings at the equation 2.3, only the planar momentum contribute. In one dimension, it means we could write the equations for k_x or k_y , but the result would not be different since both directions are equivalent with respect to the z axis. The Zeeman coupling is taken in the way that one component is in the direction of the SOC coupling given by the chosen momentum, and the other is axial, obeying the equal-weight Rashba-Dresselhaus SOC as follows [44, 47]:

$$\hat{H}_C = \int dk \hat{\Psi}_{\uparrow\downarrow}^\dagger(k) \left[\left(\alpha k + \frac{h}{2} \right) \sigma^x - \frac{B}{2} \sigma^z \right] \hat{\Psi}_{\uparrow\downarrow}(k), \quad (2.12)$$

where $\boldsymbol{\lambda} = (\alpha, 0, 0)$, $\mathbf{k} = (k, 0, 0)$ and $\mathbf{h} = (h/2, 0, -B/2)$. In the case of k_y :

$$\hat{H}_C = \int dk \hat{\Psi}_{\uparrow\downarrow}^\dagger(k) \left[\left(\alpha k + \frac{h}{2} \right) \sigma^y - \frac{B}{2} \sigma^z \right] \hat{\Psi}_{\uparrow\downarrow}(k), \quad (2.13)$$

with $\boldsymbol{\lambda} = (0, \alpha, 0)$, $\mathbf{k} = (0, k, 0)$ and $\mathbf{h} = (0, h/2, -B/2)$. In any of the previous ways, the physics will behave exactly the same, the only difference is the diagonalization of the couplings. The planar SOC diagonalization would be different depending on whether the momentum is

in the $\hat{e}_x$ or $\hat{e}_y$ direction (which is the diagonalization of either σ^x or σ^y Pauli matrices). σ^x is diagonalized by what is denoted in this work as a *CS* basis, given as:

$$\begin{aligned}\hat{\psi}_c &= \frac{1}{\sqrt{2}} (\hat{\psi}_\uparrow + \hat{\psi}_\downarrow), & \hat{\psi}_s &= \frac{1}{\sqrt{2}} (\hat{\psi}_\uparrow - \hat{\psi}_\downarrow), \\ \hat{\psi}_\uparrow &= \frac{1}{\sqrt{2}} (\hat{\psi}_c + \hat{\psi}_s), & \hat{\psi}_\downarrow &= \frac{1}{\sqrt{2}} (\hat{\psi}_c - \hat{\psi}_s),\end{aligned}\tag{2.14}$$

either in space or momentum. And if it is σ^y , the diagonalization comes in what it is denoted in this work as the $\circ\bullet$ basis:

$$\begin{aligned}\hat{\psi}_\circ &= \frac{1}{\sqrt{2}} (\hat{\psi}_\uparrow + i\hat{\psi}_\downarrow), & \hat{\psi}_\bullet &= \frac{1}{\sqrt{2}} (\hat{\psi}_\uparrow - i\hat{\psi}_\downarrow), \\ \hat{\psi}_\uparrow &= \frac{1}{\sqrt{2}} (\hat{\psi}_\circ + i\hat{\psi}_\bullet), & \hat{\psi}_\downarrow &= \frac{1}{\sqrt{2}} (\hat{\psi}_\circ - i\hat{\psi}_\bullet).\end{aligned}\tag{2.15}$$

In any of those ways the matrices are transformed into σ^z , so it is nothing else than a coordinate system rotation, as illustrated in the figure 2.1. The attractive interaction term at equation 2.4 is not altered by such rotation, which is why it is not considered for this analysis. The latter bases that diagonalize these matrices are often called **helical states**, which are the center of many works on topological systems in recent years [49–53].

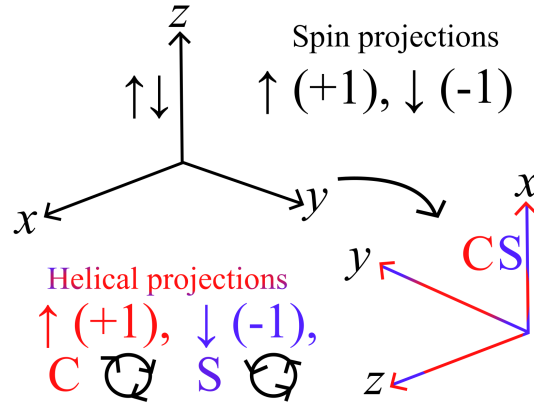


Figure 2.1: Rotation from a right-handed x, y, z frame to a left-handed z, y, x frame. This rotation exchanges the x axis with the z axis but the y axis is inverted ($y \rightarrow -y$). In the right-handed frame, the Hamiltonian is diagonalized by the spin 1/2 basis (black). In the left-handed frame, the Hamiltonian is diagonalized by the helical basis (red and blue). The helical modes are just two quantum numbers with fermionic statistics, as the spin, so they often are represented as arrows with red and blue colors, denoted in this work as C and S , or $\circ$ and $\bullet$, or in other works are represented as a clockwise rotating helix ($\uparrow$) and a counterclockwise rotating helix ($\downarrow$) [49–53].

The real problem comes with the simultaneous diagonalization of the axial Zeeman coupling (of σ^z), and either $\hat{e}_x$ or $\hat{e}_y$ planar SOC. None of the previous transformations at 2.14 and 2.15 would diagonalize both. Although it is desirable to find a transformation to diagonalize

both couplings, it is a difficult task, on top of that, dealing these couplings and the interaction term with such transformation in space representation increases the complications, turning the theory in a complex formalism, while the objective is to solve it via a simple and known formalism as it is the further bosonization and quantum field theory of the model. Thus, as said previously, the $\hat{e}_x$ planar SOC is chosen, and the $\hat{H}_C$ term in equation 2.12 is transformed to space representation by the Fourier transform of equation 1.3:

$$\begin{aligned}\hat{H}_C &= \int dk \hat{\Psi}_{\uparrow\downarrow}^\dagger(k) \left[\left(\alpha k + \frac{h}{2} \right) \sigma^x - \frac{B}{2} \sigma^z \right] \hat{\Psi}_{\uparrow\downarrow}(k) \\ &= \int dx \hat{\Psi}_{\uparrow\downarrow}^\dagger(x) \left[\left(-i\alpha \partial_x + \frac{h}{2} \right) \sigma^x - \frac{B}{2} \sigma^z \right] \hat{\Psi}_{\uparrow\downarrow}(x).\end{aligned}\quad (2.16)$$

2.1.3 The interacting potential

The last Hamiltonian term is the attractive interaction one $\hat{H}_I$, which is constructed from the first quantization image of a two-particle interaction. This is, there is an energy potential between pairs of particles that make them interact when they approach, so from the second quantization picture, the operator must be modeled as a two-particle operator just as shown in equation 1.2. If a space and spin basis is used, the interaction term is given as:

$$\begin{aligned}\hat{H}_I &= \frac{1}{2} \sum_{\sigma\sigma'\sigma''\sigma'''} \int dx dx' dx'' dx''' \hat{\psi}_\sigma^\dagger(x) \hat{\psi}_{\sigma'}^\dagger(x') \langle x\sigma | x'\sigma' | \hat{u} | x''\sigma'' | x'''\sigma''' \rangle \hat{\psi}_{\sigma'''}(x''') \hat{\psi}_{\sigma''}(x'') \\ &= \frac{1}{2} \sum_{\sigma\sigma'\sigma''\sigma'''} \int dx dx' dx'' dx''' \hat{\psi}_\sigma^\dagger(x) \hat{\psi}_{\sigma'}^\dagger(x') U(x, x', x'', x''') \delta_{\sigma'\sigma''} \delta_{\sigma\sigma'''} \hat{\psi}_{\sigma'''}(x''') \hat{\psi}_{\sigma''}(x'') \\ &= \frac{1}{2} \sum_{\sigma\sigma'} \int dx dx' dx'' dx''' \hat{\psi}_\sigma^\dagger(x) \hat{\psi}_{\sigma'}^\dagger(x') U(x, x', x'', x''') \hat{\psi}_\sigma(x''') \hat{\psi}_{\sigma'}(x''),\end{aligned}\quad (2.17)$$

where it is assumed that the interaction does not depend on the spin of the particles but just in their position, as usual. In general, the matrix element $U(x, x', x'', x''')$ is unknown, but according to the set up in figure 1.1, the induced attractive interaction between pairs of particles happen at an approximately equal and constant rate U along the longitude of the wire. On the other hand, at low energies and that short scale the pairs are formed when particles approach each other, not at long distances, and when the pair is formed it exist as whole in one position, which reduces the matrix element as $U(x, x', x'', x''') \approx U \delta(x - x''') \delta(x' - x'') \delta(x - x')$, and:

$$\begin{aligned}\hat{H}_I &\approx \frac{1}{2} \sum_{\sigma\sigma'} \int dx dx' dx'' dx''' \hat{\psi}_\sigma^\dagger(x) \hat{\psi}_{\sigma'}^\dagger(x') U \delta(x - x''') \delta(x' - x'') \delta(x - x') \hat{\psi}_\sigma(x''') \hat{\psi}_{\sigma'}(x'') \\ &= \frac{1}{2} U \sum_{\sigma\sigma'} \int dx \hat{\psi}_\sigma^\dagger(x) \hat{\psi}_{\sigma'}^\dagger(x) \hat{\psi}_\sigma(x) \hat{\psi}_{\sigma'}(x) \\ &= \frac{1}{2} U \int dx \left[\hat{\psi}_\uparrow^\dagger(x) \hat{\psi}_\downarrow^\dagger(x) \hat{\psi}_\uparrow(x) \hat{\psi}_\downarrow(x) + \hat{\psi}_\downarrow^\dagger(x) \hat{\psi}_\uparrow^\dagger(x) \hat{\psi}_\downarrow(x) \hat{\psi}_\uparrow(x) \right] \\ &= -U \int dx \hat{\psi}_\uparrow^\dagger(x) \hat{\psi}_\downarrow^\dagger(x) \hat{\psi}_\downarrow(x) \hat{\psi}_\uparrow(x),\end{aligned}\quad (2.18)$$

where the latter and final form is achieved by using the anticommutation relations of the operators by means of the spin. This term leads to the same consequences of the regular

superconductivity theory, where at higher dimensions the term is approximated again using a mean field theory, for example, so the pairing gap appears as a Δ , or the superconductivity is induced as a coupling between the wire and a superconductor just as shown in figure 1.1 [24]. In this work there is not such notation initially, the pair attraction strength is given by U , but eventually when the bosonization is done a pairing gap proportional to U is defined, presenting a one-dimensional version of the superconductivity at low energies.

Finally, wrapping all discussed terms, equations 2.7, 2.16 and 2.18, the full Hamiltonian in the $\uparrow\downarrow$ basis is given by:

$$\hat{H} = \int dx \left[\hat{\Psi}_{\uparrow\downarrow}^\dagger \left(-\frac{\partial_x^2}{2m} - \mu(x) + \left(-i\alpha\partial_x + \frac{h}{2} \right) \sigma^x - \frac{B}{2} \sigma^z \right) \hat{\Psi}_{\uparrow\downarrow} - U \hat{\psi}_\uparrow^\dagger \hat{\psi}_\downarrow^\dagger \hat{\psi}_\downarrow \hat{\psi}_\uparrow \right], \quad (2.19)$$

whereas transforming to the CS basis the model is

$$\hat{H} = \int dx \left[\hat{\Psi}_{cs}^\dagger \left(-\frac{\partial_x^2}{2m} - \mu(x) + \left(-i\alpha\partial_x + \frac{h}{2} \right) \sigma^z - \frac{B}{2} \sigma^x \right) \hat{\Psi}_{cs} - U \hat{\psi}_c^\dagger \hat{\psi}_s^\dagger \hat{\psi}_s \hat{\psi}_c \right], \quad (2.20)$$

where $\hat{\Psi}_{cs}^\dagger = \hat{\Psi}_{cs}^\dagger(x) = \left(\hat{\psi}_c^\dagger(x), \hat{\psi}_s^\dagger(x) \right)$, as all the fields are evaluated in position x . The following procedure is for the Hamiltonian in the CS basis, translating the further results to the $\uparrow\downarrow$ basis to understand the original physics of the spin 1/2 system.

Chapter 3

Bosonization of the model and correlations

One must start with the Hamiltonian terms that get diagonalized in the CS basis, then $U = 0$ and $B = 0$ for the moment. On the other hand, the modulation of the chemical potential is recovered for further analysis, so $\mu(x) = \mu$ to simplify notation. The Fourier transform at equation 1.3 is applied to the Hamiltonian in equation 2.20, where the detailed transformation to momentum representation is given in the section C.1 of the appendix C, and the retrieved expression is given by:

$$\begin{aligned}\hat{H} &= \int dk \left[\frac{(k + m\alpha)^2}{2m} - \frac{1}{2}m\alpha^2 - \mu + \frac{h}{2} \right] \hat{\psi}_c^\dagger(k) \hat{\psi}_c(k) \\ &\quad + \int dk \left[\frac{(k - m\alpha)^2}{2m} - \frac{1}{2}m\alpha^2 - \mu - \frac{h}{2} \right] \hat{\psi}_s^\dagger(k) \hat{\psi}_s(k) \\ &= \int dk (\epsilon_c(k) - \mu_c) \hat{\psi}_c^\dagger(k) \hat{\psi}_c(k) + \int dk (\epsilon_s(k) - \mu_s) \hat{\psi}_s^\dagger(k) \hat{\psi}_s(k).\end{aligned}\quad (3.1)$$

In the last line, we realize that dispersion relations for the CS degrees of freedom and their chemical potentials are:

$$\epsilon_c(k) = \frac{(k + m\alpha)^2}{2m}, \quad \epsilon_s(k) = \frac{(k - m\alpha)^2}{2m}, \quad (3.2)$$

$$\mu_c = \mu + \frac{1}{2}m\alpha^2 - \frac{h}{2}, \quad \mu_s = \mu + \frac{1}{2}m\alpha^2 + \frac{h}{2}. \quad (3.3)$$

This result shows that the SOC separates the degenerated $\uparrow\downarrow$ system into a CS system with dispersion relations horizontally displaced by an amount of $k_0 = m\alpha$. There is also one last term with a constant coefficient of $\mu_\alpha = m\alpha^2/2$ that comes from completing squares and acts like an intrinsic chemical potential for the system, and in that sense the modulated chemical potential μ acts like a separation of the full chemical potential from that intrinsic chemical potential, and h separates both μ_c and μ_s (as an effective axial Zeeman field does for $\uparrow$ and $\downarrow$ spin), which can be seen as a vertical displacement of the curves in addition to the horizontal displacement. In this case the Fermi momenta are (see section C.1 of appendix C):

$$k_{Lc} = -k_{Fc} - k_0, \quad k_{Rc} = k_{Fc} - k_0, \quad k_{Fc} = \sqrt{2m\mu + m^2\alpha^2 - mh}, \quad (3.4)$$

$$k_{L_s} = -k_{F_s} + k_0, \quad k_{R_s} = k_{F_s} + k_0, \quad k_{F_s} = \sqrt{2m\mu + m^2\alpha^2 + mh}. \quad (3.5)$$

With the latter defined Fermi momenta, the dispersion relation for the coupled Fermi gas is shown in the next figure.

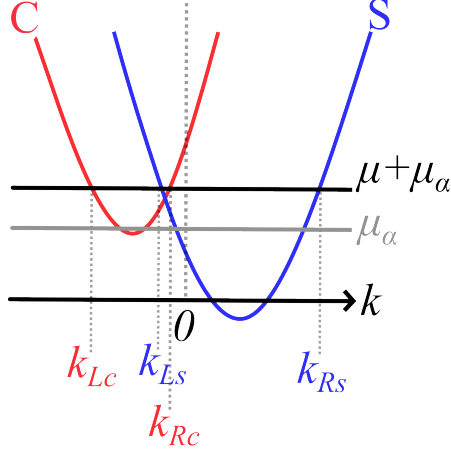


Figure 3.1: Dispersion relation of the $\uparrow\downarrow$ system with SOC. The curves split horizontally, breaking the degeneracy except for a critical crossing point. Again, the Fermi surface cut the curves on points but because of the raised degeneracy there are four in this case, that are the same $\pm k_F$ but displaced up and down by the amount $h/2$, defining k_{F_c} and k_{F_s} , and the two latter displaced to left and right by the amount k_0 , defining k_{L_c} , k_{R_c} , k_{L_s} and k_{R_s} . This figure considers $\mu > 0$ and $h > 0$, so the C dispersion relation is moved upwards and the L dispersion is moved downwards. If $h = 0$, then $k_{F_c} = k_{F_s}$.

When $B \neq 0$, the coupling introduces a gap between two bands as illustrated in the figure 3.2.

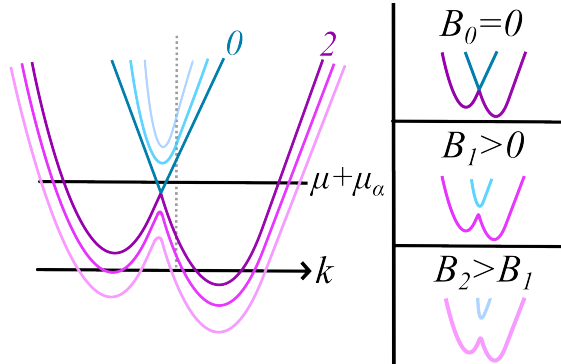


Figure 3.2: Dispersion relation of the $\uparrow\downarrow$ system with SOC and the axial Zeeman coupling. The horizontally split curves mix to form the 0-band (blue colored) and 2-band (purple colored) according to the notation in [45]. At the right, the curves show how the gap increases as the magnetic field changes from $B_0 = 0$ to $B_2 > 0$, and at the left, the curves are shown together

When $U \neq 0$, the proximity coupling introduces the superconductive gap Δ precisely, which results into another separation of the bands and thus another full diagonalization which has

been avoided so far [45, 54, 55]. Instead, the procedure consists in bosonizing the diagonal dispersion relation and map the non-diagonal terms with the bosonization mapping, which means that the non-diagonal Hamiltonian terms appear as low energy deviations of the quantum field theory from the quantum harmonic oscillator, as will be explained further.

The fermionic operators are expanded in chiral fields according to their kind and corresponding momentum following the linearized dispersion relations display in figure 3.3, in accordance with the expansion in equation 1.11 but twice, as now there are two quantum numbers leading to two copies of the bosonized fields, the C and S fields:

$$\hat{\psi}_c(x) = e^{i(-k_{Fc}-k_0)x}\hat{\psi}_{L_c}(x) + e^{i(k_{Fc}-k_0)x}\hat{\psi}_{R_c}(x) = e^{-i(k_{Fc}+k_0)x}\hat{\psi}_{L_c}(x) + e^{i(k_{Fc}-k_0)x}\hat{\psi}_{R_c}(x), \quad (3.6)$$

$$\hat{\psi}_s(x) = e^{i(-k_{Fs}+k_0)x}\hat{\psi}_{L_s}(x) + e^{i(k_{Fs}+k_0)x}\hat{\psi}_{R_s}(x) = e^{-i(k_{Fs}-k_0)x}\hat{\psi}_{L_s}(x) + e^{i(k_{Fs}+k_0)x}\hat{\psi}_{R_s}(x). \quad (3.7)$$

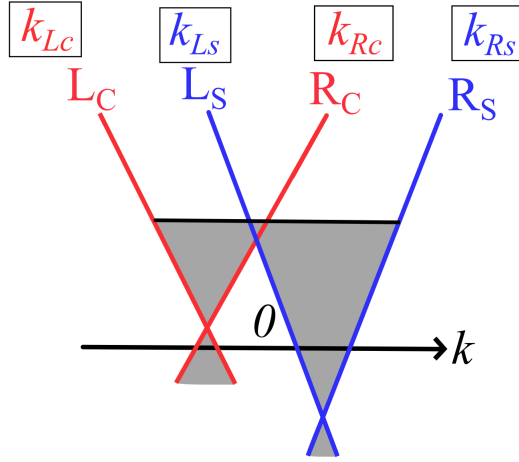


Figure 3.3: Linearization of the dispersion relations in figure 3.1. The chirality is associated with the momenta defined in equations 3.4 and 3.5.

We make use of the bosonization identities from equations A.73, A.74, A.75, and 1.12, for the two copies as it is shown for the spin 1/2 Fermi gas in appendix B, so the chiral fields are:

$$\begin{aligned} \hat{\psi}_{L_c}(x) &= \frac{1}{\sqrt{2\pi\delta}} e^{-2\sqrt{\pi}\hat{\phi}_{L_c}(x)}, & \hat{\psi}_{R_c}(x) &= \frac{1}{\sqrt{2\pi\delta}} e^{2\sqrt{\pi}\hat{\phi}_{R_c}(x)}, \\ \hat{\psi}_{L_s}(x) &= \frac{1}{\sqrt{2\pi\delta}} e^{-2\sqrt{\pi}\hat{\phi}_{L_s}(x)}, & \hat{\psi}_{R_s}(x) &= \frac{1}{\sqrt{2\pi\delta}} e^{2\sqrt{\pi}\hat{\phi}_{R_s}(x)}, \end{aligned} \quad (3.8)$$

where the bosonic fields are given by:

$$\begin{aligned} \hat{\phi}_{c,s}(x) &= \hat{\phi}_{L_{c,s}}(x) + \hat{\phi}_{R_{c,s}}(x), & \hat{\theta}_{c,s}(x) &= \hat{\phi}_{L_{c,s}}(x) - \hat{\phi}_{R_{c,s}}(x), \\ \hat{\phi}_{L_c}(x) &= \frac{1}{2} \left(\hat{\phi}_c(x) + \hat{\theta}_c(x) \right), & \hat{\phi}_{R_c}(x) &= \frac{1}{2} \left(\hat{\phi}_c(x) - \hat{\theta}_c(x) \right), \\ \hat{\phi}_{L_s}(x) &= \frac{1}{2} \left(\hat{\phi}_s(x) + \hat{\theta}_s(x) \right), & \hat{\phi}_{R_s}(x) &= \frac{1}{2} \left(\hat{\phi}_s(x) - \hat{\theta}_s(x) \right). \end{aligned} \quad (3.9)$$

All the usual commutation relations are satisfied in every space, and the operators commute between different spaces (mainly, see equation B.67). Having the identities, the rest is to map

every Hamiltonian term. For this purpose, it is better to divide the Hamiltonian into three specific parts, that are the digonal term, and the two other non-diagonal terms:

$$\hat{H} = \hat{H}_0 + \hat{H}_B + \hat{H}_U, \quad (3.10)$$

$$\hat{H}_0 = \int dk \frac{(k + k_0)^2}{2m} \hat{\psi}_c^\dagger(k) \hat{\psi}_c(k) + \int dk \frac{(k - k_0)^2}{2m} \hat{\psi}_s^\dagger(k) \hat{\psi}_s(k), \quad (3.11)$$

$$\hat{H}_B = -\frac{B}{2} \int dx \left[\hat{\psi}_c^\dagger(x) \hat{\psi}_s(x) + \hat{\psi}_s^\dagger(x) \hat{\psi}_c(x) \right], \quad (3.12)$$

$$\hat{H}_U = -U \int dx \hat{\psi}_c^\dagger(x) \hat{\psi}_s^\dagger(x) \hat{\psi}_s(x) \hat{\psi}_c(x). \quad (3.13)$$

The fermionic densities are mapped and are added or multiplied according to the latter parts of the Hamiltonian which is done in the appendix A and B (in equations A.95 and A.96, for example). $\hat{H}_0$ is already mapped, as the low energy approximation comes from it (see the equation 1.13), thus:

$$\begin{aligned} \hat{H}_0 &= \int dk \frac{(k + k_0)^2}{2m} \hat{\psi}_c^\dagger(k) \hat{\psi}_c(k) + \int dk \frac{(k - k_0)^2}{2m} \hat{\psi}_s^\dagger(k) \hat{\psi}_s(k) \\ &= \frac{\nu_c}{2} \int dx \left[\left(\partial_x \hat{\phi}_c(x) \right)^2 + \left(\partial_x \hat{\theta}_c(x) \right)^2 \right] + \frac{\nu_s}{2} \int dx \left[\left(\partial_x \hat{\phi}_s(x) \right)^2 + \left(\partial_x \hat{\theta}_s(x) \right)^2 \right], \end{aligned} \quad (3.14)$$

where

$$\frac{k + k_0}{m} \Big|_{k_{Lc}} = \frac{-k_{F_c} - k_0 + k_0}{m} = -\frac{k_{F_c}}{m} = -\nu_c, \quad \frac{k + k_0}{m} \Big|_{k_{Rc}} = \nu_c, \quad (3.15)$$

$$\frac{k - k_0}{m} \Big|_{k_{Ls}} = \frac{-k_{F_s} + k_0 - k_0}{m} = -\frac{k_{F_s}}{m} = -\nu_s, \quad \frac{k - k_0}{m} \Big|_{k_{Rs}} = \nu_s, \quad (3.16)$$

according with the analysis at the introduction (see equations 1.6 and 1.7), and $\hat{\pi}(x) = \partial_x \hat{\theta}(x)$ (see section A.2 in appendix A). For the remaining terms, it is only needed to map $\hat{\psi}_c^\dagger(x) \hat{\psi}_c(x)$ (as $\hat{\psi}_s^\dagger(x) \hat{\psi}_s(x)$ immediately would be mapped since an change of index $c \rightarrow s$ is enough), and $\hat{\psi}_c^\dagger(x) \hat{\psi}_s(x)$ (as $\hat{\psi}_s^\dagger(x) \hat{\psi}_c(x)$ would be mapped too for being the h.c of the latter). For a complete expansion of the terms see section C.2 of appendix C, which results in:

$$\hat{\psi}_c^\dagger(x) \hat{\psi}_c(x) = \frac{1}{\sqrt{\pi}} \partial_x \hat{\phi}_c(x) + \frac{1}{\pi \delta} \cos \left[2\sqrt{\pi} \hat{\phi}_c(x) + 2k_{F_c} x \right], \quad (3.17)$$

$$\hat{\psi}_s^\dagger(x) \hat{\psi}_s(x) = \frac{1}{\sqrt{\pi}} \partial_x \hat{\phi}_s(x) + \frac{1}{\pi \delta} \cos \left[2\sqrt{\pi} \hat{\phi}_s(x) + 2k_{F_s} x \right], \quad (3.18)$$

$$\begin{aligned} \hat{\psi}_c^\dagger(x) \hat{\psi}_s(x) &= \frac{1}{\pi \delta} e^{i2k_0 x} e^{i\sqrt{\pi}(\hat{\theta}_c(x) - \hat{\theta}_s(x))} \cos \left[\sqrt{\pi} \left(\hat{\phi}_c(x) - \hat{\phi}_s(x) \right) + (k_{F_c} - k_{F_s})x \right] \\ &+ \frac{1}{\pi \delta} e^{i2k_0 x} e^{i\sqrt{\pi}(\hat{\theta}_c(x) - \hat{\theta}_s(x))} \cos \left[\sqrt{\pi} \left(\hat{\phi}_c(x) + \hat{\phi}_s(x) \right) + (k_{F_c} + k_{F_s})x \right], \end{aligned} \quad (3.19)$$

$$\begin{aligned} \hat{\psi}_s^\dagger(x) \hat{\psi}_c(x) &= \frac{1}{\pi \delta} e^{-i2k_0 x} e^{-i\sqrt{\pi}(\hat{\theta}_c(x) - \hat{\theta}_s(x))} \cos \left[\sqrt{\pi} \left(\hat{\phi}_c(x) - \hat{\phi}_s(x) \right) + (k_{F_c} - k_{F_s})x \right] \\ &+ \frac{1}{\pi \delta} e^{-i2k_0 x} e^{-i\sqrt{\pi}(\hat{\theta}_c(x) - \hat{\theta}_s(x))} \cos \left[\sqrt{\pi} \left(\hat{\phi}_c(x) + \hat{\phi}_s(x) \right) + (k_{F_c} + k_{F_s})x \right]. \end{aligned} \quad (3.20)$$

Given the latter bosonized operators, $\hat{H}_B$ and $\hat{H}_U$ are bosonized as follows:

$$\begin{aligned}
\hat{H}_B &= -\frac{B}{2} \int dx \left[\frac{1}{\pi\delta} e^{i2k_0x} e^{i\sqrt{\pi}(\hat{\theta}_c(x) - \hat{\theta}_s(x))} \cos \left[\sqrt{\pi} \left(\hat{\phi}_c(x) - \hat{\phi}_s(x) \right) + (k_{F_c} - k_{F_s})x \right] \right. \\
&\quad \frac{1}{\pi\delta} e^{i2k_0x} e^{i\sqrt{\pi}(\hat{\theta}_c(x) - \hat{\theta}_s(x))} \cos \left[\sqrt{\pi} \left(\hat{\phi}_c(x) + \hat{\phi}_s(x) \right) + (k_{F_c} + k_{F_s})x \right] \\
&\quad \frac{1}{\pi\delta} e^{-i2k_0x} e^{-i\sqrt{\pi}(\hat{\theta}_c(x) - \hat{\theta}_s(x))} \cos \left[\sqrt{\pi} \left(\hat{\phi}_c(x) - \hat{\phi}_s(x) \right) + (k_{F_c} - k_{F_s})x \right] \\
&\quad \left. \frac{1}{\pi\delta} e^{-i2k_0x} e^{-i\sqrt{\pi}(\hat{\theta}_c(x) - \hat{\theta}_s(x))} \cos \left[\sqrt{\pi} \left(\hat{\phi}_c(x) + \hat{\phi}_s(x) \right) + (k_{F_c} + k_{F_s})x \right] \right] \\
&= -\frac{B}{2} \int dx \left[\right. \\
&\quad \frac{2}{\pi\delta} \cos \left[\sqrt{\pi} \left(\hat{\theta}_c(x) - \hat{\theta}_s(x) \right) + 2k_0x \right] \cos \left[\sqrt{\pi} \left(\hat{\phi}_c(x) - \hat{\phi}_s(x) \right) + (k_{F_c} - k_{F_s})x \right] \\
&\quad + \frac{2}{\pi\delta} \cos \left[\sqrt{\pi} \left(\hat{\theta}_c(x) - \hat{\theta}_s(x) \right) + 2k_0x \right] \cos \left[\sqrt{\pi} \left(\hat{\phi}_c(x) + \hat{\phi}_s(x) \right) + (k_{F_c} + k_{F_s})x \right] \\
&\quad \left. \right], \tag{3.21}
\end{aligned}$$

$$\begin{aligned}
\hat{H}_U &= -U \int dx \hat{\psi}_c^\dagger(x) \hat{\psi}_s^\dagger(x) \hat{\psi}_s(x) \hat{\psi}_c(x) = -U \int dx \hat{\psi}_c^\dagger(x) \hat{\psi}_c(x) \hat{\psi}_s^\dagger(x) \hat{\psi}_s(x) \\
&= -U \int dx \left[\frac{1}{\sqrt{\pi}} \partial_x \hat{\phi}_c(x) + \frac{1}{\pi\delta} \cos \left[2\sqrt{\pi} \hat{\phi}_c(x) + 2k_{F_c}x \right] \right] \\
&\quad \left[\frac{1}{\sqrt{\pi}} \partial_x \hat{\phi}_s(x) + \frac{1}{\pi\delta} \cos \left[2\sqrt{\pi} \hat{\phi}_s(x) + 2k_{F_s}x \right] \right] \\
&= -U \int dx \left[\frac{1}{\pi} \partial_x \hat{\phi}_c(x) \partial_x \hat{\phi}_s(x) \right. \\
&\quad \left. + \frac{1}{(\pi\delta)^2} \cos \left[2\sqrt{\pi} \hat{\phi}_c(x) + 2k_{F_c}x \right] \cos \left[2\sqrt{\pi} \hat{\phi}_s(x) + 2k_{F_s}x \right] \right] \\
&= -U \int dx \left[\frac{1}{\pi} \partial_x \hat{\phi}_c(x) \partial_x \hat{\phi}_s(x) \right. \\
&\quad \left. + \frac{2}{(2\pi\delta)^2} \cos \left[2\sqrt{\pi} \left(\hat{\phi}_c(x) - \hat{\phi}_s(x) \right) + (k_{F_c} - k_{F_s})x \right] \right], \tag{3.22}
\end{aligned}$$

where in the last equation, the products between densities and cosines vanish because they have odd powers of the bosonic operators (see a full explanation in section A.4 of appendix A). Both bosonizations show the distinct canonical transformation of the fields in the cosines (that is equivalent to the charge and spin separation in equations B.3 and B.4 for the model in appendix B). In this case the separations are denoted as + and - according to:

$$\hat{\phi}_\pm(x) = \frac{1}{\sqrt{2}} \left(\hat{\phi}_c(x) \pm \hat{\phi}_s(x) \right), \quad \hat{\theta}_\pm(x) = \frac{1}{\sqrt{2}} \left(\hat{\theta}_c(x) \pm \hat{\theta}_s(x) \right). \tag{3.23}$$

When applying the transformation to the bosonized terms in equations 3.14, 3.21 and 3.22,

the final result is:

$$\begin{aligned}\hat{H}_0 &= \frac{\bar{\nu}}{2} \int dx \left[\left(\partial_x \hat{\phi}_+(x) \right)^2 + \left(\partial_x \hat{\theta}_+(x) \right)^2 \right] + \frac{\bar{\nu}}{2} \int dx \left[\left(\partial_x \hat{\phi}_-(x) \right)^2 + \left(\partial_x \hat{\theta}_-(x) \right)^2 \right] \\ &\quad + \frac{\nu_c - \nu_s}{2} \int dx \left[\partial_x \hat{\phi}_+(x) \partial_x \hat{\phi}_-(x) + \partial_x \hat{\theta}_+(x) \partial_x \hat{\theta}_-(x) \right],\end{aligned}\quad (3.24)$$

$$\begin{aligned}\hat{H}_B &= -\frac{2B}{2\pi\delta} \int dx \left[\cos \left[\sqrt{2\pi} \hat{\theta}_-(x) + 2k_0 x \right] \cos \left[\sqrt{2\pi} \hat{\phi}_-(x) + (k_{F_c} - k_{F_s}) x \right] \right. \\ &\quad \left. + \cos \left[\sqrt{2\pi} \hat{\theta}_-(x) + 2k_0 x \right] \cos \left[\sqrt{2\pi} \hat{\phi}_+(x) + (k_{F_c} + k_{F_s}) x \right] \right],\end{aligned}\quad (3.25)$$

$$\begin{aligned}\hat{H}_U &= -\frac{U}{2\pi} \int dx \left(\partial_x \hat{\phi}_+(x) \right)^2 + \frac{U}{2\pi} \int dx \left(\partial_x \hat{\phi}_-(x) \right)^2 \\ &\quad - \frac{2U}{(2\pi\delta)^2} \int dx \cos \left[\sqrt{8\pi} \hat{\phi}_-(x) + (k_{F_c} - k_{F_s}) x \right],\end{aligned}\quad (3.26)$$

with $\bar{\nu} = \nu_c + \nu_s/2$. Reuniting all the terms, the final bosonized Hamiltonian is:

$$\begin{aligned}\hat{H} &= \int dx \left[\left(\frac{\nu_c + \nu_s}{4} - \frac{U}{2\pi} \right) \left(\partial_x \hat{\phi}_+(x) \right)^2 + \left(\frac{\nu_c + \nu_s}{4} \right) \hat{\pi}_+^2(x) \right] \\ &\quad + \int dx \left[\left(\frac{\nu_c + \nu_s}{4} + \frac{U}{2\pi} \right) \left(\partial_x \hat{\phi}_-(x) \right)^2 + \left(\frac{\nu_c + \nu_s}{4} \right) \hat{\pi}_-^2(x) \right] \\ &\quad + \frac{\nu_c - \nu_s}{2} \int dx \left[\partial_x \hat{\phi}_+(x) \partial_x \hat{\phi}_-(x) + \partial_x \hat{\theta}_+(x) \partial_x \hat{\theta}_-(x) \right] \\ &\quad - \frac{2B}{2\pi\delta} \int dx \left[\cos \left[\sqrt{2\pi} \hat{\theta}_-(x) + 2k_0 x \right] \cos \left[\sqrt{2\pi} \hat{\phi}_-(x) + (k_{F_c} - k_{F_s}) x \right] \right. \\ &\quad \left. + \cos \left[\sqrt{2\pi} \hat{\theta}_-(x) + 2k_0 x \right] \cos \left[\sqrt{2\pi} \hat{\phi}_+(x) + (k_{F_c} + k_{F_s}) x \right] \right] \\ &\quad - \frac{2U}{(2\pi\delta)^2} \int dx \cos \left[\sqrt{8\pi} \hat{\phi}_-(x) + (k_{F_c} - k_{F_s}) x \right] \rightarrow \\ \hat{H} &= \frac{\nu_+}{2} \int dx \left[\frac{1}{K_+} \left(\partial_x \hat{\phi}_+(x) \right)^2 + K_+ \hat{\pi}_+^2(x) \right] + \frac{\nu_-}{2} \int dx \left[\frac{1}{K_-} \left(\partial_x \hat{\phi}_-(x) \right)^2 + K_- \hat{\pi}_-^2(x) \right] \\ &\quad + \Delta\nu \int dx \left[\partial_x \hat{\phi}_+(x) \partial_x \hat{\phi}_-(x) + \partial_x \hat{\theta}_+(x) \partial_x \hat{\theta}_-(x) \right] \\ &\quad - 2g_B \int dx \left[\cos \left[\sqrt{2\pi} \hat{\theta}_-(x) + 2k_0 x \right] \cos \left[\sqrt{2\pi} \hat{\phi}_-(x) + (k_{F_c} - k_{F_s}) x \right] \right. \\ &\quad \left. + \cos \left[\sqrt{2\pi} \hat{\theta}_-(x) + 2k_0 x \right] \cos \left[\sqrt{2\pi} \hat{\phi}_+(x) + (k_{F_c} + k_{F_s}) x \right] \right] \\ &\quad - g_U \int dx \cos \left[\sqrt{8\pi} \hat{\phi}_-(x) + (k_{F_c} - k_{F_s}) x \right].\end{aligned}\quad (3.27)$$

The Luttinger parameters $K_{\pm}$ and velocities $\nu_{\pm}$ in the Hamiltonian of equation 3.27 are given in complete analogy with the ones derived for the model in the appendix B (see equations B.6

and B.7). In this case:

$$\frac{\nu_{\pm}}{2K_{\pm}} = \frac{\nu_c + \nu_s}{4} \mp \frac{U}{2\pi}, \quad \frac{\nu_{\pm}}{2} K_{\pm} = \frac{\nu_c + \nu_s}{4}, \quad (3.28)$$

$$\Delta\nu = \frac{\nu_c - \nu_s}{2}, \quad g_B = \frac{B}{2\pi\delta}, \quad g_U = \frac{\Delta}{2\pi\delta}, \quad \Delta = \frac{U}{\pi\delta}. \quad (3.29)$$

The first set of equations in 3.28 solve for $\nu_{\pm}$ and $K_{\pm}$, which are:

$$\nu_{\pm}(\mu, \alpha, h, U) = \frac{k_{F_c} + k_{F_s}}{2m} \sqrt{1 \mp \frac{2mU}{\pi(k_{F_c} + k_{F_s})}}, \quad (3.30)$$

$$K_{\pm}(\mu, \alpha, h, U) = \frac{1}{\sqrt{1 \mp \frac{2mU}{\pi(k_{F_c} + k_{F_s})}}}. \quad (3.31)$$

The second set of equations (equations 3.29) define the strengths of the deviations from the harmonic oscillator, this is, g_B and g_U , where $g_B/g_U = B/\Delta$. Δ must not be confused with $\Delta\nu$ and also the latter must not with $\Delta \cdot \nu$. Δ represents the one-dimensional superconductive gap under the bosonization theory while $\Delta\nu$ is a distinctive parameter of the Zeeman coupling of strength h . In fact, from the equation 3.24 it can be seen that the mean velocity $\bar{v}$ and difference $\Delta\nu$ parameters emerge from the distinction between the Fermi momenta that h introduces, which involves a whole different study aside of the topology and more related with quantum magnetism of Fermi lattices [56, 57]. Eventually, these solutions can be discarded as they have nothing to do with the expected topological solutions.

If the cosine products are expanded, the Hamiltonian is now:

$$\begin{aligned} \hat{H} &= \frac{\nu_+}{2} \int dx \left[\frac{1}{K_+} \left(\partial_x \hat{\phi}_+(x) \right)^2 + K_+ \left(\partial_x \hat{\theta}_+(x) \right)^2 \right] \\ &+ \frac{\nu_-}{2} \int dx \left[\frac{1}{K_-} \left(\partial_x \hat{\phi}_-(x) \right)^2 + K_- \left(\partial_x \hat{\theta}_-(x) \right)^2 \right] \\ &+ \Delta\nu \int dx \left[\partial_x \hat{\phi}_+(x) \partial_x \hat{\phi}_-(x) + \partial_x \hat{\theta}_+(x) \partial_x \hat{\theta}_-(x) \right] \\ &- g_B \int dx \cos \left[\sqrt{2\pi} \left(\hat{\phi}_-(x) + \hat{\theta}_-(x) \right) + (k_{F_c} - k_{F_s} + 2k_0)x \right] \end{aligned} \quad (3.32)$$

$$- g_B \int dx \cos \left[\sqrt{2\pi} \left(\hat{\phi}_-(x) - \hat{\theta}_-(x) \right) + (k_{F_c} - k_{F_s} - 2k_0)x \right] \quad (3.33)$$

$$- g_B \int dx \cos \left[\sqrt{2\pi} \left(\hat{\phi}_+(x) + \hat{\theta}_-(x) \right) + (k_{F_c} + k_{F_s} + 2k_0)x \right] \quad (3.34)$$

$$- g_B \int dx \cos \left[\sqrt{2\pi} \left(\hat{\phi}_+(x) - \hat{\theta}_-(x) \right) + (k_{F_c} + k_{F_s} - 2k_0)x \right] \quad (3.35)$$

$$- g_U \int dx \cos \left[\sqrt{8\pi} \hat{\phi}_-(x) + (k_{F_c} - k_{F_s})x \right], \quad (3.36)$$

which is the most general presented, as general as it is difficult to solve. There are particular configurations of the parameters that lead to specific phenomena, but the central problem is around the possibility to solve it, and it is known that the solution is not exact and simple due to the cosine contributions [25]. For a simplification of the bosonization the processes of the cosines are reviewed in the next section.

3.1 Cosine processes

The following is to get to know the processes that the latter cosines introduce, which could change critically the physics of the harmonic oscillator [25, 34, 35, 45, 55]. Next, all processes are described by doing a refermionization of the terms, and then one can decide which of them might be considered or not according to the topology and the bosonization formalism, where the full calculations are given in section C.3 of appendix C:

a) The cosine from equation 3.32 satisfy

$$\begin{aligned} & \cos \left[\sqrt{2\pi} \left(\hat{\phi}_-(x) + \hat{\theta}_-(x) \right) + (k_{F_c} - k_{F_s} + 2k_0)x \right] \\ &= \frac{\delta}{2} e^{i(k_{F_c} - k_{F_s} + 2k_0)x} \int_{\substack{p \sim -k_{F_c} - k_0 \\ q \sim -k_{F_s} + k_0}} dp dq \hat{\psi}_{L_c}^\dagger(p) \hat{\psi}_{L_s}(q) + h.c., \end{aligned} \quad (3.37)$$

so it takes one particle from branch L_s to the branch L_c around the Fermi level with a transferred momentum of $|k_{F_c} - k_{F_s} + 2k_0|$. The inverse process is given in its conjugated hermitian, $h.c.$

b) The cosine from equation 3.33 gives

$$\begin{aligned} & \cos \left[\sqrt{2\pi} \left(\hat{\phi}_-(x) - \hat{\theta}_-(x) \right) + (k_{F_c} - k_{F_s} - 2k_0)x \right] \\ &= \frac{\delta}{2} e^{i(k_{F_c} - k_{F_s} - 2k_0)x} \int_{\substack{p \sim k_{F_s} + k_0 \\ q \sim k_{F_c} - k_0}} dp dq \hat{\psi}_{R_s}^\dagger(p) \hat{\psi}_{R_c}(q) + h.c., \end{aligned} \quad (3.38)$$

which is a process similar to the latter but between the right branches and a transferred momentum of $|k_{F_c} - k_{F_s} - 2k_0|$.

These two processes involve cosines of fields of the same quantum number $-$, and the jumps are between equal chirality but not equal helicity, that in this work are refereed as **lateral flips**, illustrated in the next figure.

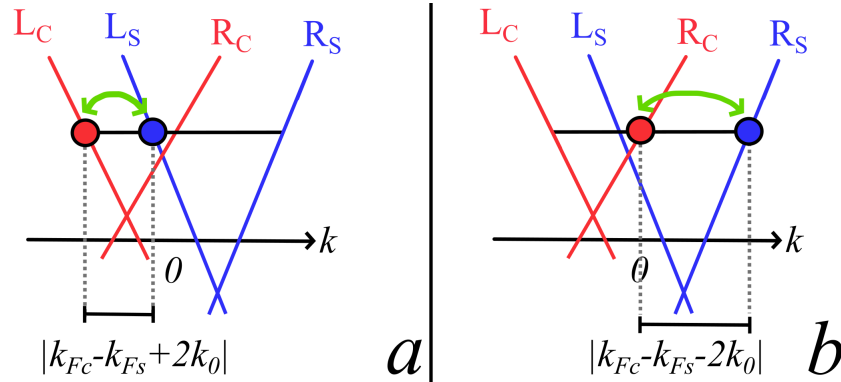


Figure 3.4: Lateral flips introduced by the Zeeman coupling, where the flip is illustrated with the green arrows and the transferred momentum is shown below the curves.

c) The cosine of equation 3.34 has the form

$$\begin{aligned} & \cos \left[\sqrt{2\pi} \left(\hat{\phi}_+(x) + \hat{\theta}_-(x) \right) + (k_{F_c} + k_{F_s} + 2k_0)x \right] \\ &= \frac{\delta}{2} e^{i(k_{F_c} + k_{F_s} + 2k_0)x} \int_{\substack{p \sim -k_{F_c} - k_0 \\ q \sim k_{F_s} + k_0}} dp dq \hat{\psi}_{L_c}^\dagger(p) \hat{\psi}_{R_s}(q) + h.c. \end{aligned} \quad (3.39)$$

This process involves an exchange among the external branches, with a transferred momentum of $|k_{F_c} + k_{F_s} + 2k_0|$.

d) The last g_B cosine in equation 3.35 is given as

$$\begin{aligned} & \cos \left[\sqrt{2\pi} \left(\hat{\phi}_+(x) - \hat{\theta}_-(x) \right) + (k_{F_c} + k_{F_s} - 2k_0)x \right] \\ &= \frac{\delta}{2} e^{i(k_{F_c} + k_{F_s} - 2k_0)x} \int_{\substack{p \sim -k_{F_s} + k_0 \\ q \sim k_{F_c} - k_0}} dp dq \hat{\psi}_{L_s}^\dagger(p) \hat{\psi}_{R_c}(q) + h.c., \end{aligned} \quad (3.40)$$

which is the opposite of the latter: an exchange among the internal branches (the closest opposite branches, that are L_s and R_c), with a transferred momentum of $|k_{F_c} + k_{F_s} - 2k_0|$.

These processes involve cosines of fields with an opposite quantum number, so the jumps are between different chirality, referred to in this work as **external and internal flips**. The processes of equation 3.40 is related to the zero energy modes, since there is an internal flip of zero-energy cost according to the crossing point of the two dispersion relations. The illustration of these processes is given in the next figure.

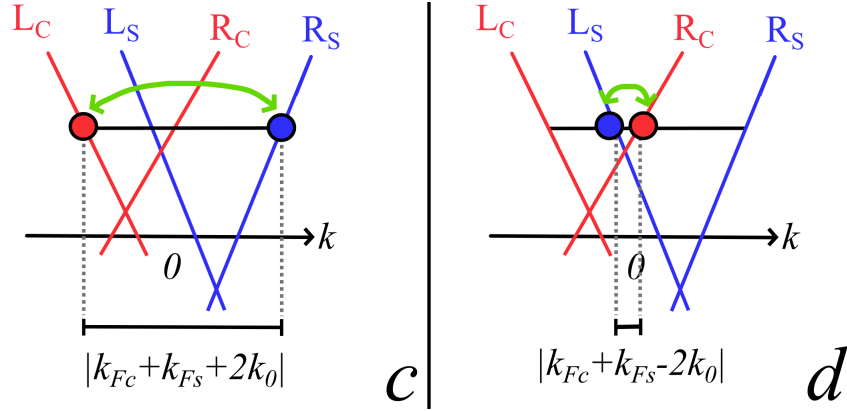


Figure 3.5: External and internal flips introduced by the Zeeman coupling. It can be seen that for the case d) there could be a configuration where the process happens at zero energy, exactly at the crossing point.

e) The g_U cosine in the last line of the Hamiltonian at the equation 3.36 satisfy

$$\begin{aligned} & \cos \left[\sqrt{8\pi} \hat{\phi}_-(x) + (k_{F_c} - k_{F_s})x \right] \\ &= \frac{\delta^2}{2} e^{i(k_{F_c} - k_{F_s})x} \int_{\substack{p \sim -k_{F_c} - k_0 \\ q \sim k_{F_s} + k_0 \\ l \sim k_{F_c} - k_0 \\ u \sim -k_{F_s} + k_0}} dp dq dl du \hat{\psi}_{L_c}^\dagger(p) \hat{\psi}_{R_s}^\dagger(q) \hat{\psi}_{R_c}(l) \hat{\psi}_{L_s}(u) + h.c., \end{aligned} \quad (3.41)$$

which is a pairing process, moving two fermions from internal and opposite branches to the external branches and still opposite, with a momentum transfer of $|k_{F_c} - k_{F_s}|$.

The illustration on the latter process is given in the next figure.

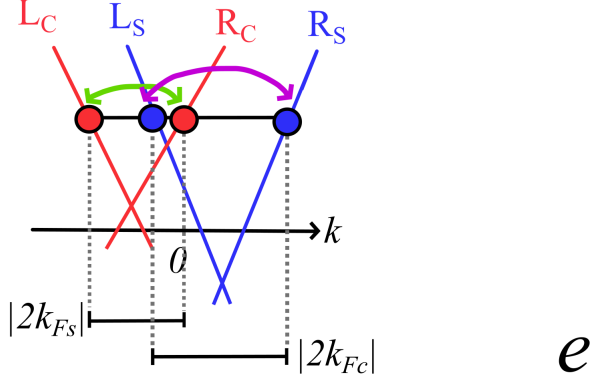


Figure 3.6: Pairing process introduced by the attractive interaction, commonly called the BCS channel in one dimension. In fact, as long as the transferred momentum is $|k_{F_c} - k_{F_s}|$, it could be done as shown in this figure or via the combination of the flipping processes a) and b), but the unique process given by the attractive interactions is the no-flip one illustrated here.

By definition $k_{F_{c,s}} > 0$, $\mu > 0$ and $\alpha > 0$, thus the cosine process from equation 3.34 always oscillate and does not contribute ($k_{F_c} + k_{F_s} + 2k_0 > 0$) [25–28]. The only way one cosine does not oscillate for a gas system is when its phase is zero (not the case of the discrete lattice [25]), and the possible configurations are based on those phases to become zero (see the explanation of why some cosines contribute or not after the equation A.97 in the appendix A). In this case, there are four configurations:

1. Consider $\mu < \frac{1}{2}m\alpha^2$, $h = \pm 2\alpha\sqrt{2m\mu}$, where $k_{F_c} + k_{F_s} = 2k_0$ is satisfied. Then the last cosine of g_B does not oscillate, but the others do. Also, $k_{F_c} \neq k_{F_s}$ so the $\Delta\nu$ term contributes. The Hamiltonian in this case is:

$$\begin{aligned}
 \hat{H} &= \frac{\nu_+}{2} \int dx \left[\frac{1}{K_+} \left(\partial_x \hat{\phi}_+(x) \right)^2 + K_+ \left(\partial_x \hat{\theta}_+(x) \right)^2 \right] \\
 &+ \frac{\nu_-}{2} \int dx \left[\frac{1}{K_-} \left(\partial_x \hat{\phi}_-(x) \right)^2 + K_- \left(\partial_x \hat{\theta}_-(x) \right)^2 \right] \\
 &+ \Delta\nu \int dx \left[\partial_x \hat{\phi}_+(x) \partial_x \hat{\phi}_-(x) + \partial_x \hat{\theta}_+(x) \partial_x \hat{\theta}_-(x) \right] \\
 &- g_B \int dx \cos \left[\sqrt{2\pi} \left(\hat{\phi}_+(x) - \hat{\theta}_-(x) \right) \right].
 \end{aligned} \tag{3.42}$$

2. Also could it be $\mu > \frac{1}{2}m\alpha^2$, $h = \pm 2\alpha\sqrt{2m\mu}$, where $k_{F_c} - k_{F_s} = \mp 2k_0$ is given. The first g_B cosine contributes for $h > 0$ and the second for $h < 0$, the others do not. Still

$k_{F_c} \neq k_{F_s}$ so $\Delta\nu$ appears too. The Hamiltonian is:

$$\begin{aligned}\hat{H} &= \frac{\nu_+}{2} \int dx \left[\frac{1}{K_+} \left(\partial_x \hat{\phi}_+(x) \right)^2 + K_+ \left(\partial_x \hat{\theta}_+(x) \right)^2 \right] \\ &+ \frac{\nu_-}{2} \int dx \left[\frac{1}{K_-} \left(\partial_x \hat{\phi}_-(x) \right)^2 + K_- \left(\partial_x \hat{\theta}_-(x) \right)^2 \right] \\ &+ \Delta\nu \int dx \left[\partial_x \hat{\phi}_+(x) \partial_x \hat{\phi}_-(x) + \partial_x \hat{\theta}_+(x) \partial_x \hat{\theta}_-(x) \right] \\ &- g_B \int dx \cos \left[\sqrt{2\pi} \left(\hat{\phi}_-(x) \pm \hat{\theta}_-(x) \right) \right].\end{aligned}\quad (3.43)$$

3. Other case is when $\mu > 0$, $h = 0$, which implies $k_{F_c} = k_{F_s}$. No g_B cosine contributes but the g_U cosine does. As the Fermi momenta of C and S are equal, then $\Delta\nu = 0$ and the Hamiltonian is given as:

$$\begin{aligned}\hat{H} &= \frac{\nu_+}{2} \int dx \left[\frac{1}{K_+} \left(\partial_x \hat{\phi}_+(x) \right)^2 + K_+ \left(\partial_x \hat{\theta}_+(x) \right)^2 \right] \\ &+ \frac{\nu_-}{2} \int dx \left[\frac{1}{K_-} \left(\partial_x \hat{\phi}_-(x) \right)^2 + K_- \left(\partial_x \hat{\theta}_-(x) \right)^2 \right] \\ &- g_U \int dx \cos \left[\sqrt{8\pi} \hat{\phi}_-(x) \right].\end{aligned}\quad (3.44)$$

4. The last case is when $\mu = 0$, $h = 0$, which implies $k_{F_c} = k_{F_s} = k_0$. The last g_B cosine contributes along the one from g_U with $\Delta\nu = 0$. Thus, the Hamiltonian is:

$$\begin{aligned}\hat{H} &= \frac{\nu_+}{2} \int dx \left[\frac{1}{K_+} \left(\partial_x \hat{\phi}_+(x) \right)^2 + K_+ \left(\partial_x \hat{\theta}_+(x) \right)^2 \right] \\ &+ \frac{\nu_-}{2} \int dx \left[\frac{1}{K_-} \left(\partial_x \hat{\phi}_-(x) \right)^2 + K_- \left(\partial_x \hat{\theta}_-(x) \right)^2 \right] \\ &- g_B \int dx \cos \left[\sqrt{2\pi} \left(\hat{\phi}_+(x) - \hat{\theta}_-(x) \right) \right] \\ &- g_U \int dx \cos \left[\sqrt{8\pi} \hat{\phi}_-(x) \right].\end{aligned}\quad (3.45)$$

The cases 1 and 2 have the big problem that the BCS channel of the attractive interaction cannot contribute if the time-reversal symmetry of the helical system is broken ($h \neq 0$), so a phase transition cannot occur [25]. In fact, looking at the case 4, in the strong interaction limit of $g_B \rightarrow \infty$, the fields $\hat{\phi}_+(x)$ and $\hat{\theta}_-(x)$ could be trapped, so the field $\hat{\phi}_-(x)$ oscillate and g_U does not contribute, and as $\partial_x \hat{\phi}_+(x) = 0$ and $\partial_x \hat{\theta}_-(x) = 0$ notice that the system makes a transition to the case 1 (see section B.8 of appendix B). On the opposite, if $g_U \rightarrow \infty$ then g_B does not contribute and the system makes a transition to the case 3. Hence, it is not necessary to solve 1 and 3 separately, when in terms of the sought phase the case 4 has both.

Thus, the only relevant model for this work is the case 4, where the parameters reduce to:

$$k_{F_c}, k_{F_s} = \sqrt{2m \cdot 0 + m^2 \alpha^2 \pm m \cdot 0} = \sqrt{m^2 \alpha^2} = m\alpha = k_0, \quad (3.46)$$

$$\nu_{\pm}(\alpha, U) = \frac{k_0}{m} \sqrt{1 \mp \frac{mU}{\pi k_0}} = \alpha \sqrt{1 \mp \frac{U}{\pi \alpha}}, \quad (3.47)$$

$$K_{\pm}(\alpha, U) = \frac{1}{\sqrt{1 \mp \frac{mU}{\pi k_0}}} = \frac{1}{\sqrt{1 \mp \frac{U}{\pi \alpha}}}, \quad (3.48)$$

and the dispersion relation changes to the one illustrated in the next figure.

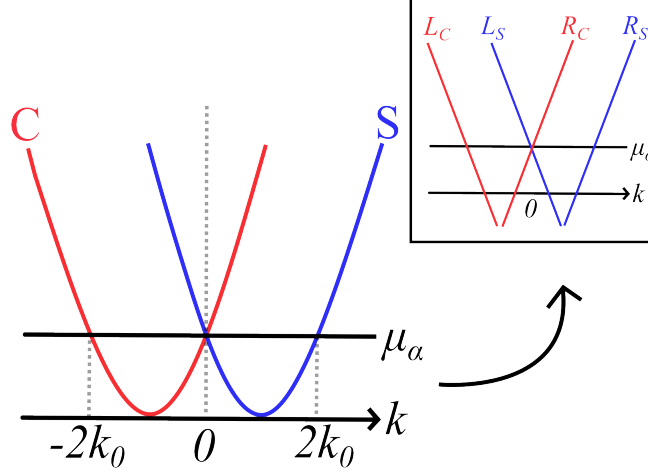


Figure 3.7: Dispersion relation of the Hamiltonian in the case 4. The external chiralities cut the Fermi surface μ_α in $\pm 2k_0$ (symmetric Fermi momenta for those chiralities that form the 2-band, see figure 3.2, which means there is a spacial inversion symmetry), while the internal chiralities cut it in $\pm 0 = 0$ (the same inversion symmetry for the 0-band). Notice the cone form of the linearized dispersion relation with the L_s and R_c bands that touch exactly at the Fermi level in $k = 0$, which is a distinction of non-trivial topology in many systems, given by these helical modes or with modes in other methods [16, 17, 20, 21, 49–53].

As the bosonization for the model is defined, with the extra condition of the case 4, the following is to define the correlations and bosonize them as well.

3.2 Correlations of the model

The fundamental spin correlations are presented with full arguments and calculations in the appendix B. What it is done in this section is to define the operators, bosonize them and calculate the correlations in analogy with the analysis in that appendix.

First, the charge density operator is exactly the same in any of the bases, as $\hat{n}_\uparrow + \hat{n}_\downarrow = \hat{n}_c + \hat{n}_s$, that is, the total density of excitations do not change by a rotation [1–3, 26]. If the operators have an extra notation of CS , for example, the charge density of helical modes could be denoted $\hat{\rho}_{cs}(x)$, so $\hat{\rho}_{cs}(x) = \hat{\rho}(x)$ (see the definition in equation B.11).

Second, the spin density operators are changed according to the frame rotation (recall figure 2.1), so it is satisfied:

$$\hat{\sigma}_{cs}^z(x) = \hat{\sigma}^x(x), \quad \hat{\sigma}_{cs}^x(x) = \hat{\sigma}^z(x), \quad \hat{\sigma}_{cs}^y(x) = -\hat{\sigma}^y(x). \quad (3.49)$$

The $\hat{\sigma}_{cs}(x)$ operators refer to a helical density, in complete analogy with the spin density, so they are defined in terms of the fermionic operators $\hat{\psi}_{c,s}(x)$ exactly in the same way as the

spin density operators are defined in equations B.12, B.13 and B.14. In many works they are usually called **helical spin densities** [58–60]. Although it is true that the system could support helical density fluctuations, the equivalencies between those fluctuations and the spin fluctuations in the equation 3.49 make it possible to do an analysis in terms of the spin.

Third and last, in the case of the pairing operators, it can be shown that the full on-site pairing operator $\hat{O}(x) = \hat{\psi}_\uparrow^\dagger(x)\hat{\psi}_\downarrow^\dagger(x)$ satisfy:

$$\begin{aligned}\hat{O}(x) &= \hat{\psi}_\uparrow^\dagger(x)\hat{\psi}_\downarrow^\dagger(x) = \frac{1}{\sqrt{2}} \left(\hat{\psi}_c^\dagger(x) + \hat{\psi}_s^\dagger(x) \right) \frac{1}{\sqrt{2}} \left(\hat{\psi}_c^\dagger(x) - \hat{\psi}_s^\dagger(x) \right) \\ &= \frac{1}{2} \hat{\psi}_c^\dagger(x)\hat{\psi}_c^\dagger(x) - \frac{1}{2} \hat{\psi}_c^\dagger(x)\hat{\psi}_s^\dagger(x) + \frac{1}{2} \hat{\psi}_s^\dagger(x)\hat{\psi}_c^\dagger(x) - \frac{1}{2} \hat{\psi}_s^\dagger(x)\hat{\psi}_s^\dagger(x) \\ &= \frac{1}{2} \hat{\psi}_s^\dagger(x)\hat{\psi}_c^\dagger(x) + \frac{1}{2} \hat{\psi}_s^\dagger(x)\hat{\psi}_c^\dagger(x) = \hat{\psi}_s^\dagger(x)\hat{\psi}_c^\dagger(x) = \hat{O}_{cs}(x),\end{aligned}\quad (3.50)$$

if the helical pairing operator $\hat{O}_{cs}(x)$ is defined as in the last line, which shows that the spin pairing and helical pairing are completely equivalent. Although that is true, in many systems with these states the helical spin pairings are reported, and their apparition on a system is completely related to topological superconducting phases and Majorana modes in higher dimensions [61–63]. According to the equivalence, the four pairing operators for the helical modes are defined exactly with the same form of the spin pairings in equations B.15, B.16, B.17 and B.18 of the appendix B, only the quantum numbers are changed as previously explained. Also, it is important to notice that due to the SOC, some pairings, if can be established, would have momentum because the operators have to be defined with an additional phase, in contrast with the Fermi gas without SOC in the appendix B. The possible pairings are:

$$\begin{aligned}R_c(k=0), L_s(k=0), &\rightarrow R_c L_s(k=0), \\ L_c(k=-2k_0), R_s(k=2k_0), &\rightarrow L_c R_s(k=0), \\ R_c(k=0), L_c(k=-2k_0), &\rightarrow R_c L_c(k=-2k_0), \\ R_s(k=2k_0), L_s(k=0), &\rightarrow R_s L_s(k=2k_0).\end{aligned}\quad (3.51)$$

Thus, the helical spin-singlet pairings (HSS) and axial helical spin-triplet pairings (HTS- z) are zero-momentum pairings still, while the planar helical spin-triplet pairings (HTS- x and HTS- y) are non-zero-momentum pairings with $k = \pm 2k_0$. Finally, with all the latter discussed, the operators for the correlations are defined as:

$$\hat{\rho}_{cs}(x) = \hat{\psi}_c^\dagger(x)\hat{\psi}_c(x) + \hat{\psi}_s^\dagger(x)\hat{\psi}_s(x), \quad (3.52)$$

$$\hat{\sigma}_{cs}^z(x) = \hat{\psi}_c^\dagger(x)\hat{\psi}_c(x) - \hat{\psi}_s^\dagger(x)\hat{\psi}_s(x), \quad (3.53)$$

$$\hat{\sigma}_{cs}^x(x) = \hat{\psi}_c^\dagger(x)\hat{\psi}_s(x) + \hat{\psi}_s^\dagger(x)\hat{\psi}_c(x), \quad (3.54)$$

$$\hat{\sigma}_{cs}^y(x) = i \left(\hat{\psi}_s^\dagger(x)\hat{\psi}_c(x) - \hat{\psi}_c^\dagger(x)\hat{\psi}_s(x) \right), \quad (3.55)$$

$$\hat{O}_{HSS}(x) = \hat{\psi}_{R_c}^\dagger(x)\hat{\psi}_{L_s}^\dagger(x) - \hat{\psi}_{L_c}^\dagger(x)\hat{\psi}_{R_s}^\dagger(x), \quad (3.56)$$

$$\hat{O}_{HTS}^z(x) = \hat{\psi}_{R_c}^\dagger(x)\hat{\psi}_{L_s}^\dagger(x) + \hat{\psi}_{L_c}^\dagger(x)\hat{\psi}_{R_s}^\dagger(x), \quad (3.57)$$

$$\hat{O}_{HTS}^x(x) = e^{i2k_0x}\hat{\psi}_{R_c}^\dagger(x)\hat{\psi}_{L_c}^\dagger(x) + e^{-i2k_0x}\hat{\psi}_{L_s}^\dagger(x)\hat{\psi}_{R_s}^\dagger(x), \quad (3.58)$$

$$\hat{O}_{HTS}^y(x) = i \left(e^{-i2k_0x}\hat{\psi}_{L_s}^\dagger(x)\hat{\psi}_{R_s}^\dagger(x) - e^{i2k_0x}\hat{\psi}_{R_c}^\dagger(x)\hat{\psi}_{L_c}^\dagger(x) \right). \quad (3.59)$$

The next step is to bosonize these operators, which is done following the bosonized operators in equations B.21, B.22, B.23, B.24, B.25, B.26, B.27 and B.28, while comparing the Hamiltonian in equation B.10 and of the case 4. The bosonized operators are:

$$\hat{\rho}_{cs}(x) = \sqrt{\frac{2}{\pi}} \partial_x \hat{\phi}_+(x) + \frac{2}{\pi\delta} \cos \left[\sqrt{2\pi} \hat{\phi}_+(x) + 2k_0 x \right] \cos \left[\sqrt{2\pi} \hat{\phi}_-(x) \right], \quad (3.60)$$

$$\hat{\sigma}_{cs}^z(x) = \sqrt{\frac{2}{\pi}} \partial_x \hat{\phi}_-(x) - \frac{2}{\pi\delta} \sin \left[\sqrt{2\pi} \hat{\phi}_+(x) + 2k_0 x \right] \sin \left[\sqrt{2\pi} \hat{\phi}_-(x) \right], \quad (3.61)$$

$$\begin{aligned} \hat{\sigma}_{cs}^x(x) = & \frac{2}{\pi\delta} \cos \left[\sqrt{2\pi} \hat{\theta}_-(x) + 2k_0 x \right] \cos \left[\sqrt{2\pi} \hat{\phi}_-(x) \right] \\ & + \frac{2}{\pi\delta} \cos \left[\sqrt{2\pi} \hat{\theta}_-(x) + 2k_0 x \right] \cos \left[\sqrt{2\pi} \hat{\phi}_+(x) + 2k_0 x \right], \end{aligned} \quad (3.62)$$

$$\begin{aligned} \hat{\sigma}_{cs}^y(x) = & \frac{2}{\pi\delta} \sin \left[\sqrt{2\pi} \hat{\theta}_-(x) + 2k_0 x \right] \cos \left[\sqrt{2\pi} \hat{\phi}_-(x) \right] \\ & + \frac{2}{\pi\delta} \sin \left[\sqrt{2\pi} \hat{\theta}_-(x) + 2k_0 x \right] \cos \left[\sqrt{2\pi} \hat{\phi}_+(x) + 2k_0 x \right], \end{aligned} \quad (3.63)$$

$$\hat{O}_{HSS}(x) = \frac{1}{\pi\delta} e^{i\sqrt{2\pi}\hat{\theta}_+(x)} \cos \left[\sqrt{2\pi} \hat{\phi}_-(x) \right], \quad (3.64)$$

$$\hat{O}_{HTS}^z(x) = -\frac{i}{\pi\delta} e^{i\sqrt{2\pi}\hat{\theta}_+(x)} \sin \left[\sqrt{2\pi} \hat{\phi}_-(x) \right], \quad (3.65)$$

$$\hat{O}_{HTS}^x(x) = \frac{1}{\pi\delta} e^{i\sqrt{2\pi}\hat{\theta}_+(x)} \cos \left[\sqrt{2\pi} \hat{\theta}_-(x) + 2k_0 x \right], \quad (3.66)$$

$$\hat{O}_{HTS}^y(x) = \frac{1}{\pi\delta} e^{i\sqrt{2\pi}\hat{\theta}_+(x)} \sin \left[\sqrt{2\pi} \hat{\theta}_-(x) + 2k_0 x \right]. \quad (3.67)$$

The last step is to calculate the correlations, which are the expected values of the products of the latter operators, as shown and explained in the appendix B. According to the results in sections B.3, B.4 and B.5, the correlations are given as:

$$\langle \hat{\rho}_{cs}(r) \hat{\rho}_{cs}(0) \rangle = \frac{1}{(\pi\delta)^2} \cos(2k_0 x) \left(\frac{\delta}{r} \right)^{K_+ + K_-}, \quad (3.68)$$

$$\langle \hat{\sigma}_{cs}^z(r) \hat{\sigma}_{cs}^z(0) \rangle = \frac{1}{(\pi\delta)^2} \cos(2k_0 x) \left(\frac{\delta}{r} \right)^{K_+ + K_-}, \quad (3.69)$$

$$\langle \sigma_{cs}^x(r) \sigma_{cs}^x(0) \rangle = \frac{1}{(\pi\delta)^2} \cos(2k_0 x) \left(\frac{\delta}{r} \right)^{K_- + \frac{1}{K_-}}, \quad (3.70)$$

$$\langle \sigma_{cs}^y(r) \sigma_{cs}^y(0) \rangle = \frac{1}{(\pi\delta)^2} \cos(2k_0 x) \left(\frac{\delta}{r} \right)^{K_- + \frac{1}{K_-}}, \quad (3.71)$$

$$\langle \hat{O}_{HSS}^\dagger(r) \hat{O}_{HSS}(0) \rangle = \frac{2}{(2\pi\delta)^2} \left(\frac{\delta}{r} \right)^{K_- + \frac{1}{K_+}}, \quad (3.72)$$

$$\langle \hat{O}_{HTS}^{z\dagger}(r) \hat{O}_{HTS}^z(0) \rangle = \frac{2}{(2\pi\delta)^2} \left(\frac{\delta}{r} \right)^{K_- + \frac{1}{K_+}}, \quad (3.73)$$

$$\langle \hat{O}_{HTS}^{x\dagger}(r) \hat{O}_{HTS}^x(0) \rangle = \frac{2}{(2\pi\delta)^2} \cos(2k_0 x) \left(\frac{\delta}{r} \right)^{\frac{1}{K_-} + \frac{1}{K_+}}, \quad (3.74)$$

$$\langle \hat{O}_{HTS}^{y\dagger}(r) \hat{O}_{HTS}^y(0) \rangle = \frac{2}{(2\pi\delta)^2} \cos(2k_0 x) \left(\frac{\delta}{r} \right)^{\frac{1}{K_-} + \frac{1}{K_+}}, \quad (3.75)$$

with r the distance from the origin, so the property of the operator is evaluated in two different points and with the correlation it is seen how the ground state changes with distance, with $r \gg \delta$. This calculation is preliminary, since the cosines of the Hamiltonian are not considered for it (see the introduction of appendix B). Part of the next analysis is how to introduce them, that it is done via the strong interaction limit as it was mentioned before. Hence, according to the calculations of this chapter, the analysis on the system can be started, where the implications on the functional form of the correlations are reviewed, the corresponding response is determined and the phases are unveiled, which is the subject of the next chapter.

Chapter 4

Analysis and results

The resultant correlations reaffirm the one-dimensional nature of the system as their decay is given as a non-universal power law [25], and depend on the Luttinger parameters [25, 32, 40, 41], that according to the definitions in the equation 3.48, depend on the strength of the two-particle interaction U and the SOC strength α . The behavior is completely expected, as the correlation diverges if the point approach the origin ($r \rightarrow 0$) and converge to zero as the point separate from the origin ($r \neq 0$). The trigonometric phases introduce fluctuations in which the correlation could be negative or positive. Those fluctuations indicate that in some regions of the system the property decreases or increases lengthwise, while the others with no phase have a constant ratio of change [36, 40, 41]. The fact that these correlations decay differently for some values of the Luttinger parameters can show a change in the behavior of the system for some critical values of the parameters in the model. It means that some correlations will decay faster than others, so the information of some quantities spread longer (are more correlated), indicating a possible transition [36, 40, 41]. This can be seen in figure 4.1, where a graph of the latter correlations is made.

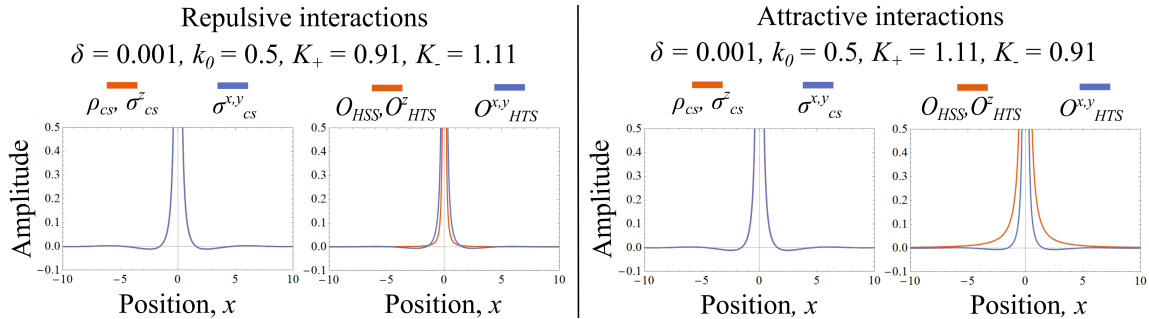


Figure 4.1: Change in the behavior of the correlations (amplitude) in terms of the position x ($r = |x|$), in the limit of repulsive interactions $U < 0$ (top) and attractive interactions $U > 0$ (bottom). In the repulsive case, the helical spin-singlet and axial helical spin-triplet pairing correlations decays faster than the other correlations, so the charge density, helical spin density, and planar helical spin-triplet pairing correlations are more relevant. In the attractive case the most relevant decay is the helical spin-singlet pairing, where the system prefer to enter a one-dimensional BCS superconducting phase.

The latter transition has been vastly explored, so the work focuses only on $U > 0$. The fact is,

if only $U > 0$ is considered, then the transition must be defined via another parameter, which in this system is the magnetic field B . The transition can be shown by including the cosine terms in the strong interaction limit, and the analysis and results on that approach are given in the following sections.

4.1 Strong interaction limit of the model

To see the effect of the cosines on the correlations, it must be noticed from the Hamiltonian in the case 4 that the cosines of those fields are not the most suitable to do such analysis. According to the section B.8 of appendix B, the limit $g_U, g_B \rightarrow \infty$ means that the field on the respective cosine gets pinned (it takes a constant value when it is trapped in one of the minima of the cosine) [25]. For g_U , $\hat{\phi}_-(x)$ is pinned and $\hat{\theta}_-(x)$ oscillates rapidly, so the g_B cosine is safely integrated falling into the case 2. For g_B , $\hat{\phi}_+(x) - \hat{\theta}_-(x)$ is pinned, and there could be many configurations for both fields. They could oscillate slowly at a rate that their sum up as a constant, they could oscillate rapidly and still their sum is a constant, or both get pinned, but from the point of view of the method, there is an **indeterminacy**. What it is concluded is that the spin-charge separation is not suitable while introducing the cosines, and so another transformation must be applied as reported in similar works [17, 25, 45, 56].

At this point introducing another transformation would mean the spin-charge separation was useless (+ and - fields), since all the latter correlations were written in terms of those fields, but it is shown that in fact, that writing is still correct to do the analysis. The transformation is found seeking two things: the first rewrite $\hat{\phi}_+(x) - \hat{\theta}_-(x)$ into only one field, and the second is that the fields remain canonical (see equation B.67). Also, the fields are renormalized by a symmetry argument so the harmonic oscillator, associated with a massless Klein-Gordon equation, are now associated to double sine-Gordon model [64, 65]. The new basis is denoted $\square \blacksquare$ and is proposed as:

$$\begin{aligned}\hat{\phi}_{\square}(x) &= \sqrt{2}\hat{\phi}_+(x), \quad \hat{\theta}_{\square}(x) = \frac{1}{\sqrt{2}} \left(\hat{\phi}_-(x) + \hat{\theta}_+(x) \right), \\ \hat{\phi}_{\blacksquare}(x) &= \frac{1}{\sqrt{2}} \left(\hat{\phi}_+(x) - \hat{\theta}_-(x) \right), \quad \hat{\theta}_{\blacksquare}(x) = -\sqrt{2}\hat{\phi}_-(x),\end{aligned}\tag{4.1}$$

$$\begin{aligned}\hat{\phi}_+(x) &= \frac{1}{\sqrt{2}}\hat{\phi}_{\square}(x), \quad \hat{\theta}_+(x) = \frac{1}{\sqrt{2}} \left(\hat{\theta}_{\blacksquare}(x) + 2\hat{\theta}_{\square}(x) \right), \\ \hat{\phi}_-(x) &= -\frac{1}{\sqrt{2}}\hat{\theta}_{\blacksquare}(x), \quad \hat{\theta}_-(x) = \frac{1}{\sqrt{2}} \left(\hat{\phi}_{\square}(x) - 2\hat{\phi}_{\blacksquare}(x) \right).\end{aligned}\tag{4.2}$$

Under this basis the processes are described by the 0-band and 2-band. It can easily be shown that the commutation relations are preserved, according to the commutation relation between the $\pm$ fields established by the equation B.67, this is:

$$\begin{aligned}[\hat{\phi}_{\square}(x), \hat{\theta}_{\square}(y)] &= \left[\sqrt{2}\hat{\phi}_+(x), \frac{1}{\sqrt{2}} \left(\hat{\phi}_-(y) + \hat{\theta}_+(y) \right) \right] = [\hat{\phi}_+(x), \hat{\phi}_-(y)] + [\hat{\phi}_+(x), \hat{\theta}_+(y)] \\ &= 0 - \frac{i}{2} \text{sgn}(x - y) = -\frac{i}{2} \text{sgn}(x - y),\end{aligned}\tag{4.3}$$

$$\begin{aligned}[\hat{\phi}_{\blacksquare}(x), \hat{\theta}_{\blacksquare}(y)] &= \left[\frac{1}{\sqrt{2}} \left(\hat{\phi}_+(x) - \hat{\theta}_-(x) \right), -\sqrt{2}\hat{\phi}_-(y) \right] = [\hat{\theta}_-(x), \hat{\phi}_-(y)] - [\hat{\phi}_+(x), \hat{\phi}_-(y)] \\ &= -[\hat{\phi}_-(y), \hat{\theta}_-(x)] - 0 = \frac{i}{2} \text{sgn}(y - x) = -\frac{i}{2} \text{sgn}(x - y),\end{aligned}\tag{4.4}$$

while the others are zero. Once again the transformation is applied to the Hamiltonian so the Luttinger parameters are derived, where the extended calculation is presented in section C.4 of appendix C. The result is:

$$\begin{aligned}
\hat{H} = & \int dx \left[\left(\frac{\nu_+}{4K_+} + \frac{\nu_- K_-}{4} \right) \left(\partial_x \hat{\phi}_\square(x) \right)^2 + \nu_+ K_+ \left(\partial_x \hat{\theta}_\square(x) \right)^2 \right] \\
& + \int dx \left[\nu_- K_- \left(\partial_x \hat{\phi}_\blacksquare(x) \right)^2 + \left(\frac{\nu_+ K_+}{4} + \frac{\nu_-}{4K_-} \right) \left(\partial_x \hat{\theta}_\blacksquare(x) \right)^2 \right] \\
& + \int dx \left[\nu_+ K_+ \partial_x \hat{\theta}_\square(x) \partial_x \hat{\theta}_\blacksquare(x) - \nu_- K_- \partial_x \hat{\phi}_\square(x) \partial_x \hat{\phi}_\blacksquare(x) \right] \\
& - g_B \int dx \cos \left[\sqrt{4\pi} \hat{\phi}_\blacksquare(x) \right] - g_U \int dx \cos \left[\sqrt{4\pi} \hat{\theta}_\blacksquare(x) \right].
\end{aligned} \tag{4.5}$$

The Luttinger parameters and velocities have to be defined via the equations:

$$\frac{\nu_\square}{2K_\square} = \frac{\nu_+}{4K_+} + \frac{\nu_- K_-}{4}, \quad \frac{\nu_\square K_\square}{2} = \nu_+ K_+, \tag{4.6}$$

$$\frac{\nu_\blacksquare}{2K_\blacksquare} = \nu_- K_-, \quad \frac{\nu_\blacksquare K_\blacksquare}{2} = \frac{\nu_+ K_+}{4} + \frac{\nu_-}{4K_-}. \tag{4.7}$$

It can be noticed that, in contrast with the equation 3.28, the Luttinger parameters in this case satisfy inverse roles. There is a separation between the behavior of the $\square$ and $\blacksquare$ fields. The solutions are:

$$K_\square = \frac{\sqrt{2}}{\sqrt{1 - \frac{U}{2\pi\alpha}}}, \quad \nu_\square = \alpha\sqrt{2}\sqrt{1 - \frac{U}{2\pi\alpha}}, \tag{4.8}$$

$$K_\blacksquare = \frac{1}{\sqrt{2}}\sqrt{1 + \frac{U}{2\pi\alpha}}, \quad \nu_\blacksquare = \alpha\sqrt{2}\sqrt{1 + \frac{U}{2\pi\alpha}}. \tag{4.9}$$

From the latter solutions, for example in the case of attractive interactions, $K_\square > \sqrt{2}$ and $K_\blacksquare > 1/\sqrt{2}$ (while $K_+ > 1$ and $K_- < 1$). This means that now U does not control a phase transition anymore, so the transformation shows that the system has moved on from the repulsive and attractive transition to another one which is expected to be a topological transition.

On the other hand, there is a coupling term between the fields that is proportional to $\nu_+ K_+ = \nu_- K_- = \alpha$. This is expected since the 0-band and 2-band get coupled via the SOC. This term can be discarded in the strong interaction limit, as if $g_B, g_U \rightarrow \infty$, that means when $g_U > g_B$, then $\partial_x \hat{\theta}_\blacksquare(x) = 0$ and $\hat{\phi}_\blacksquare(x)$ as $\partial_x \hat{\phi}_\blacksquare(x)$ oscillate, and the coupling term can be safely integrated. If $g_B > g_U$, then $\partial_x \hat{\phi}_\blacksquare(x) = 0$ and $\hat{\theta}_\blacksquare(x)$ as $\partial_x \hat{\theta}_\blacksquare(x)$ oscillate, and the term is safely integrated again. Then, the criticality is around $g_U = g_B$, which would correspond to the topological transition, and if that is the case, the coupling does not contribute in the strong interaction limit (except for the critical point $g_U = g_B$, so it is said that both fields $\hat{\phi}_\blacksquare(x)$ and $\hat{\theta}_\blacksquare(x)$ are gapped [45]). With all the latter considerations, the critical Hamiltonian

is:

$$\begin{aligned}
\hat{H} = & \frac{\nu_{\square}}{2} \int dx \left[\frac{1}{K_{\square}} \left(\partial_x \hat{\phi}_{\square}(x) \right)^2 + K_{\square} \left(\partial_x \hat{\theta}_{\square}(x) \right)^2 \right] \\
& + \frac{\nu_{\blacksquare}}{2} \int dx \left[\frac{1}{K_{\blacksquare}} \left(\partial_x \hat{\phi}_{\blacksquare}(x) \right)^2 + K_{\blacksquare} \left(\partial_x \hat{\theta}_{\blacksquare}(x) \right)^2 \right] \\
& - g_B \int dx \cos \left[\sqrt{4\pi} \hat{\phi}_{\blacksquare}(x) \right] - g_U \int dx \cos \left[\sqrt{4\pi} \hat{\theta}_{\blacksquare}(x) \right], \tag{4.10}
\end{aligned}$$

which is exactly the same result as in [45, 66, 67], given by all the arguments up to this point, and in general for a Hamiltonian with this criticality [25, 64, 65], which is the double sine-Gordon model. Under that model the following is to see how the cosines affect the correlations in the limit, which will define the relevant correlations for the transition $g_U = g_B$. At last, the susceptibilities are calculated for those correlations, which is in fact what shows the topologic transitions in the next sections.

4.2 Susceptibilities of the model

In the next analysis the response of the system to an external force is calculated via the susceptibilities, which can show the behavior of the system in the two critical limits, $\Delta > B$ and $B > \Delta$ for $g_U, g_B \rightarrow \infty$ (since $g_B/g_U = B/\Delta$). In there, the transition and its topological nature can be concluded. It must be recalled that the susceptibilities are the Fourier transform of the correlations, where a fully general calculation is given in section B.6 of appendix B. In the next subsections those results are used to directly show the susceptibilities for the coupled and interacting Fermi gas.

4.2.1 System dominated by the superconducting gap

For $\Delta > B$ the g_B cosine does not contribute and the system falls in the case 3. In that sense the analysis can be done with that Hamiltonian and the $\pm$ basis, which is why it is still useful. According to the equations B.112, B.113 and B.113, as $U > 0$, the energy is minimized if the field is trapped into one of the cosine maxima:

$$\hat{\theta}_{\blacksquare}(x) = \frac{2\pi n}{\sqrt{4\pi}} \rightarrow \hat{\phi}_{-}(x) = -\frac{\pi n}{\sqrt{2\pi}}, \tag{4.11}$$

where n is any integer and the transformation in the equation 4.1 was used. So, $\hat{\phi}_{-}(x)$ is constant and $\hat{\theta}_{-}(x)$ oscillates, and the corresponding trigonometric functions satisfy:

$$\left\langle \cos \left[\sqrt{2\pi} \hat{\phi}_{-}(x) \right] \right\rangle = A, \quad \left\langle \sin \left[\sqrt{2\pi} \hat{\phi}_{-}(x) \right] \right\rangle = 0, \tag{4.12}$$

$$\left\langle \cos \left[a \hat{\theta}_{-}(x) \right] \right\rangle = \left\langle \sin \left[a \hat{\theta}_{-}(x) \right] \right\rangle = 0, \tag{4.13}$$

where A and a are constants. The correlations involving a trigonometric function of $\hat{\theta}_{-}(x)$ vanish, either if there is a phase or not, because in the end the oscillating factor is $e^{\pm ia\hat{\theta}_{-}(x)}$. On the other hand, the correlations with a sine function of $\hat{\phi}_{-}(x)$ vanish and with a cosine

function are scaled by the constant A , so their K_- contribution to the power law disappear. Hence, the only non-vanishing correlations are:

$$\langle \hat{\rho}_{cs}(r) \hat{\rho}_{cs}(0) \rangle = \frac{A^2}{(\pi\delta)^2} \cos(2k_0 x) \left(\frac{\delta}{r} \right)^{K_+}, \quad (4.14)$$

$$\langle \hat{O}_{HSS}^\dagger(r) \hat{O}_{HSS}(0) \rangle = \frac{2A^2}{(2\pi\delta)^2} \left(\frac{\delta}{r} \right)^{\frac{1}{K_+}}, \quad (4.15)$$

This result means that, under dominant attractive interactions there is no order in spin or helical spin-triplet pairings since the system will prefer to be in the one-dimensional BCS superconducting phase and also a possible charge density order, where in general the system will prefer the superconducting order due to the strong attractive interactions [20, 21, 25, 66, 68]. This is exactly the same result of the superconducting limit of the Fermi gas with repulsive-attractive transition [25], so it is identified as the non-topological phase for this system. The susceptibilities for the latter correlations are:

$$\chi_{\rho_{cs}}(p) \propto \left(\frac{1}{|p - 2k_0|} \right)^{2-K_+} + \left(\frac{1}{|p + 2k_0|} \right)^{2-K_+}, \quad (4.16)$$

$$\chi_{HSS}(p) \propto \left(\frac{1}{|p|} \right)^{2-\frac{1}{K_+}}. \quad (4.17)$$

The susceptibilities show that, under strong attractive interactions, there could be an order in charge density at the excitation momentum of $p = \pm 2k_0$, or a superconducting order with helical spin-singlet pairs of $p = 0$ momentum.

4.2.2 System dominated by the Zeeman gap

For $B > \Delta$ the g_U cosine does not contribute in the Hamiltonian of equation 4.10. With $B > 0$, the energy is minimized if the field is trapped in one of the cosine maxima:

$$\hat{\phi}_{\blacksquare}(x) = \frac{2\pi n}{\sqrt{4\pi}}, \quad (4.18)$$

so $\hat{\phi}_{\blacksquare}(x)$ is constant and $\hat{\theta}_{\blacksquare}(x)$ oscillates. According to the transformation in the equation 4.1, as $\hat{\phi}_-(x)$ is proportional to $\hat{\theta}_{\blacksquare}(x)$ then $\hat{\phi}_-(x)$ oscillates too, and it is satisfied:

$$\left\langle \cos \left[\sqrt{4\pi} \hat{\phi}_{\blacksquare}(x) \right] \right\rangle = A, \quad \left\langle \sin \left[\sqrt{4\pi} \hat{\phi}_{\blacksquare}(x) \right] \right\rangle = 0, \quad (4.19)$$

$$\left\langle \cos \left[a \hat{\theta}_{\blacksquare}(x) \right] \right\rangle = \left\langle \sin \left[a \hat{\theta}_{\blacksquare}(x) \right] \right\rangle = 0, \quad (4.20)$$

$$\left\langle \cos \left[a \hat{\phi}_-(x) \right] \right\rangle = \left\langle \sin \left[a \hat{\phi}_-(x) \right] \right\rangle = 0. \quad (4.21)$$

As in this case the trigonometric functions of $\hat{\phi}_-(x)$ vanish, one could try to say that all the correlations where the operators have a trigonometric function of $\hat{\phi}_-(x)$ vanish, but in fact **one must be very careful** because the transformation must be considered in every step. For example, if one revisits the operators for the planar helical spin-triplet pairing in equations 3.66 and 3.67, there is not $\hat{\phi}_-(x)$ so it looks like they do not vanish, but it is not true: one

should separate the trigonometric functions into exponential ones so the result has exponential contributions of the field $\hat{\theta}_+(x) \pm \hat{\theta}_-(x)$, where it satisfies:

$$\hat{\theta}_+(x) \pm \hat{\theta}_-(x) \propto \hat{\theta}_{\blacksquare}(x) + 2\hat{\theta}_{\square}(x) \pm \hat{\phi}_{\square}(x) \mp \hat{\phi}_{\blacksquare}(x), \quad (4.22)$$

so indeed there is a contribution of the $\hat{\theta}_{\blacksquare}(x)$ field. As in this limit of the theory the fields $\hat{\phi}_{\square}(x)$ and $\hat{\theta}_{\square}(x)$ vary slowly and $\hat{\phi}_{\blacksquare}(x)$ is constant, the whole field oscillates due to the oscillation of the $\hat{\theta}_{\blacksquare}(x)$ field, and the planar helical spin-triplet correlation vanishes. This example shows that one should expand all the operators, transform them and evaluate the ones that do not have $\hat{\theta}_{\blacksquare}(x)$ contributions. This does not have to be done explicitly, one just should look at the product of exponential functions and recognize the two fields that will add or subtract, apply the transformation, and see if there is no net $\hat{\theta}_{\blacksquare}(x)$ field. In the end, the non-vanishing correlations will be the ones that when transformed have a term $\hat{\theta}_{\blacksquare}(x) - \hat{\theta}_{\blacksquare}(x) = 0$, which means that although the fields on their own oscillate, the oscillations annihilate each other when brought together (in normal order) [26–28].

With the latter analysis, the only non-vanishing correlations are the planar helical spin density, the helical spin-singlet pairing, and axial helical spin-triplet pairing. This means that a superconducting order could happen for both $\Delta > B$ and $B > \Delta$ according to the strong attractive interactions, but for enough strong Zeeman coupling a spin order could happen as well, which is expected according to [20, 25, 66, 68]. The explicit calculation of the operators and the correlations are given in the section C.5 of the appendix C, where the results are:

$$\langle \hat{\sigma}_{cs}^x(r) \hat{\sigma}_{cs}^x(0) \rangle = 2 \left(\frac{A}{2\pi\delta} \right)^2 \cos(4k_0x) \left(\frac{\delta}{r} \right)^{2K_{\square}} + \left(\frac{A}{2\pi\delta} \right)^2, \quad (4.23)$$

$$\langle \hat{\sigma}_{cs}^y(r) \hat{\sigma}_{cs}^y(0) \rangle = 2 \left(\frac{A}{2\pi\delta} \right)^2 \cos(4k_0x) \left(\frac{\delta}{r} \right)^{2K_{\square}} + \left(\frac{A}{2\pi\delta} \right)^2, \quad (4.24)$$

$$\langle \hat{O}_{HSS}^{\dagger}(r) \hat{O}_{HSS}(0) \rangle = \frac{1}{(2\pi\delta)^2} \left(\frac{1}{r} \right)^{\frac{2}{K_{\square}}}, \quad (4.25)$$

$$\langle \hat{O}_{HTS}^{z\dagger}(r) \hat{O}_{HTS}^z(0) \rangle = \frac{1}{(2\pi\delta)^2} \left(\frac{1}{r} \right)^{\frac{2}{K_{\square}}}. \quad (4.26)$$

The resultant correlations show a crucial difference among the correlations for $\Delta > B$ and $B > \Delta$, and it is **the constant contribution that does not depend on the Luttinger parameters** in the planar helical spin density correlations. Such contribution has never appeared in any of the other revisited cases, and it is vital for the analysis of the topological properties of this system. This is shown by calculating the susceptibilities of this case, which are:

$$\chi_{\sigma_{cs}}^x(p) = \chi_{\sigma_{cs}}^y(p) \propto \left(\frac{1}{|p - 4k_0|} \right)^{2-2K_{\square}} + \left(\frac{1}{|p + 4k_0|} \right)^{2-2K_{\square}} + \delta(p), \quad (4.27)$$

$$\chi_{HSS}(p) = \chi_{HTS}^z(p) \propto \left(\frac{1}{|p|} \right)^{2-\frac{2}{K_{\square}}}. \quad (4.28)$$

The delta peak is the contribution from the constant correlation which manifest the topological nature of the case $B > \Delta$, it means, this case is called the topological phase of the system. The susceptibilities can be analyzed in the next section according to the expanded analysis of the phases in the appendix B.

4.3 Analysis of the susceptibilities

The correlations and susceptibilities are illustrated in the figures 4.2 and 4.3.

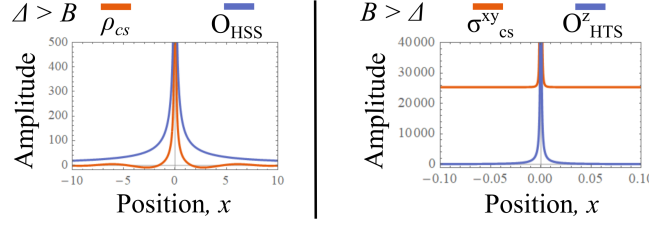


Figure 4.2: Functional form of the correlations under the two critical cases. For $\Delta > B$, the correlations decay as power laws where the helical spin-singlet pairing decay slower, and for $B > \Delta$ there is a strong dominance of the planar helical spin density due to the constant contribution. These results are obtained for $\delta = 0.001$, $\alpha = 0.5$ and $U = 0.4$.

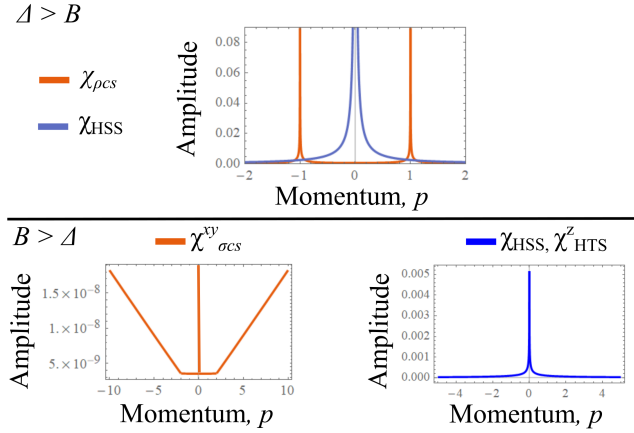


Figure 4.3: Functional form of the susceptibilities. In the case $\Delta > B$ the curves show the two power-law peaks of the charge density excitations around $\pm 2k_0$ and the power-law peak of the zero-momentum helical spin-singlet pairing. For $B > \Delta$ the orders of amplitude between the planar helical spin density and helical spin pairings are too different, so the curves are shown in different figures. The planar helical spin density at the left shows two non-peak branches (non-diverging), while the peak at the center is the delta-like peak. The helical spin-singlet and axial spin-triplet pairings at the right show a power-law peak that decays faster than the pairing peak for $\Delta > B$. These results are obtained for $\delta = 0.001$, $\alpha = 0.5$ and $U = 0.4$.

From figure 4.2 it is clear that the helical spin-singlet pairing correlation decay slower than the charge density correlation, indicating that if there is a possible ordering, then a 1D-superconducting order is more likely than a charge density order. For $B > \Delta$ there is a strong dominance of the planar helical spin density due to the constant contribution, while the helical spin-singlet pairing decays faster than for $\Delta > B$, which is expected since the Zeeman gap reduces the effect of the superconducting gap. What is interpreted from the constant contribution is that the spin excitations are always correlated for $B > \Delta$ (inside the Zeeman gap where the internal-flip process occurs at the crossing point), because although $r \rightarrow \infty$, there is still a finite contribution. This can happen if the particles are delocalized in space, such that there is no definite space for the particles to really separate and not correlate. Hence,

the system makes a transition at $B = \Delta$ from a localized to a delocalized state, where this kind of transition to a highly delocalized state is characteristic of a topological phase transition reported in many other works under the same conditions [20, 21, 45, 66, 67, 69, 70]. Although the constant contribution is indeed greater, it decays faster than the helical spin-singlet pairing correlation, which indicates that a superconducting order could happen in the presence of the delocalized spin excitation, defining a novel superconducting state for $B > \Delta$.

From figure 4.3, both susceptibilities are diverging for $\Delta > B$, so the system is in a CDW phase or a one-dimensional superconducting phase (see section B.7 of appendix B for a definition and illustration of the CDW phase), which is in complete accordance with other systems of similar characteristics [25, 45, 66]. For $B > \Delta$ the power-law contribution of the planar helical spin density is non-diverging, so there is no helical spin density order (planar SDW). The only peak that appears in that susceptibility is the delta one, where **that peak implies that in the region $B > \Delta$ there is quasiparticle-like excitation of zero energy** [17, 18, 71], as the peak corresponds to the internal spin-flip which has zero energy cost. These are the zero energy modes of the system (Majoranas), such that in this region there is non-trivial topology physics. On the other hand, the susceptibilities for the helical spin-singlet and helical spin-triplet pairings are diverging, so there is an order in either spin-singlet pairs ($|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle$) or spin-triplet pairs ($|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle$), but both could be established a topological phase, so the novel superconductivity is a **one-dimensional topological superconductivity**. Although the decay for both pairings are equal, since $B > \Delta$ the system will prefer to be in the triplet state.

It has been shown that this system can support a quantum phase transition between a phase without Majorana fermions (trivial topology), and a phase with Majorana fermions (non-trivial topology), that under $B > \Delta$ the superconducting phase is protected by the robustness of the topological phase [14, 16, 17]. In the following, a constructed phase diagram is given according to the divergence of the susceptibilities, being one of the most relevant results of the present work.

4.4 Phase diagram

The phases described in the latter analysis are given by some specific values of the attractive interaction amplitude ($U = 0.4$) and SOC amplitude ($\alpha = 0.5$), which are not general. Next, the task is to define all the possible values of U and α considering all possible parameters and in this way establishing all possible phases, which we usually display as a phase diagram. As discussed in the section B.7 in the appendix B, the susceptibility can diverge at the specific low-energy excitation if the power law is positive. Thus, the conditions for the phases to appear are:

- There is a CDW phase if $2 - K_+ > 0 \rightarrow K_+ < 2$.
- There is 1D-superconducting phase if $2 - \frac{1}{K_+} > 0 \rightarrow K_+ > \frac{1}{2}$.
- There is a topological planar SDW phase if $2 - 2K_\square > 0 \rightarrow K_\square < 1$.
- There is a topological 1D-superconducting phase, either of spin-singlet or spin-triplet pairs, if $2 - \frac{2}{K_\square} > 0 \rightarrow K_\square > 1$.

As $U, \alpha > 0$, then $K_+ > 1$ and $K_\square > \sqrt{2}$, which means the only non-possible achievable phase is the planar SDW phase of non-trivial topology for any values of U and α . Instead, a

superconducting phase is always established, but for $\Delta > B$ is a superconducting phase of trivial topology and for $B > \Delta$ is a superconducting phase of non-trivial topology. Although both Luttinger parameters depend on U and α , a more fundamental dependence is the ratio $\gamma = U/\pi\alpha$, so in the phases the parameter satisfies:

$$K_+ = \frac{1}{\sqrt{1 - \frac{U}{\pi\alpha}}} = \frac{1}{\sqrt{1 - \gamma}} < 2 \rightarrow 1 - \gamma > \frac{1}{4} \rightarrow \gamma < \frac{3}{4}, \quad (4.29)$$

while the others are given for any $\gamma < 1$ (as the K_+ diverges when γ approaches 1). The summary of the latter conditions is displayed in the phase diagram presented in figure 4.4.

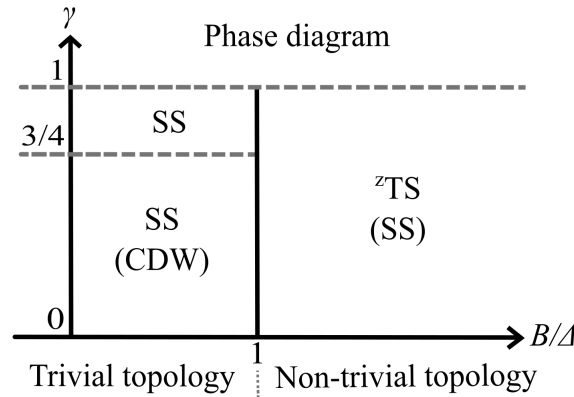


Figure 4.4: Phase diagram for the one-dimensional Fermi gas with SOC, Zeeman coupling and attractive interactions under the strong interaction limit. The most probable phase appears first in any of the regions, while the less probable appears in parenthesis down the main phase. The SOC coupling introduced the possibility for the CDW to happen, but when the attractive interactions are strong enough as $U > \frac{3}{4}\pi\alpha$ then the system becomes the one-dimensional superconductor.

In figure 4.4, the many regions of the parameters are illustrated so when they are tuned, the system should respond by ordering in the specific phases described previously. To finish, it must be said that the quasiparticle-like excitation was associated with the Majoranas but that has not been shown explicitly. In the framework of the Majorana fermions, these excitations appear lengthwise at certain edges of the nanowire or system, which is illustrated for discrete lattices, while it is not so straightforward with this continuous system. The method for the edges to appear lies in the modulation of the phase transition along the system's longitude, which is achieved precisely with the modulation of the chemical potential μ presented at the start of the work. This specific feature is discussed in the next section where the behavior of the Majorana fermions is exploited.

4.5 Display of the Majorana fermions

The reason why the modulation of the chemical potential can provoke the phase transition lies on the analysis of the several cases presented for the Hamiltonian in the chapter 3, where it is shown that the transition between the case 3 and 4 is around $\mu = 0$. Hence, $\mu = 0$ has the

same effect of $B > \Delta$ and $\mu \neq 0$ the same effect of $\Delta > B$. This connection can be explained if one revisits the dispersion relation in figure 3.7, in which $\mu = 0$ implies being exactly at the verge between the two bands, while $\mu \neq 0$ means the chemical potential moves away from the zero energy excitation. The problem with the topological setting in this way is that if B , Δ and μ are fixed then **there is only one point in the dispersion relation where the state appears**, so it is an unstable point. But, if the chemical potential is modulated for example with a harmonic trap $V(x) = m\omega^2 x^2/2$, for fixed B and Δ the system makes a transition from the region without the excitation when entering the Zeeman gap as shown in the next figure.

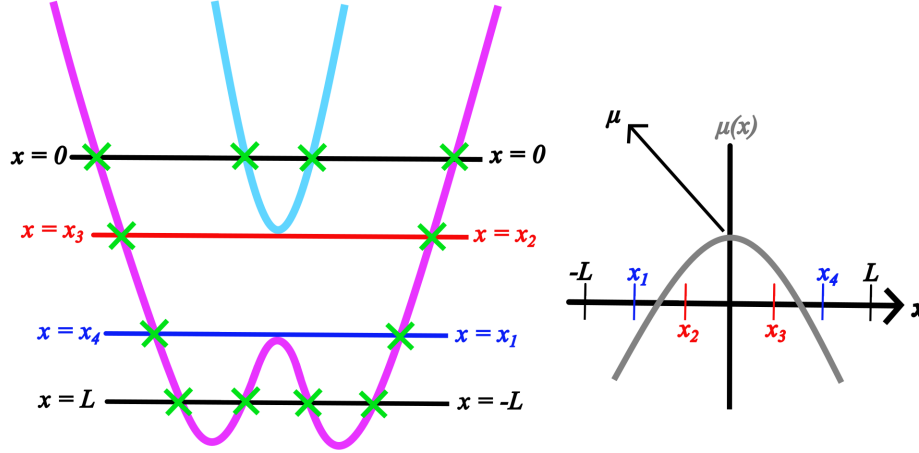


Figure 4.5: Dispersion relation of the system with $B > 0$. On the left, the horizontal lines represent the chemical potential at the point x of the wire, which cut the dispersion relation in the green marks. Between the points, x_1 and x_2 , and x_3 and x_4 , the chemical potential is inside the gap. On the right, there is the space variation of the chemical potential, which is parabolic due to the harmonic trap, where the maximum peak μ is the established constant chemical potential of the system.

A crucial characteristic of the topological states is displayed in figure 4.5. The figure shows that the Majorana fermions carry with themselves a degeneration of the ground state related to a symmetry. When the chemical potential enters the gap, **the time-reversal symmetry is restored**, where the Fermi level cut the curve in two points, while outside the inversion symmetry is broken as it cuts the curve in four points. Exactly at the edges, there are two cuts but there is a virtual third cut of zero energy that corresponds to the Majorana fermions [46]. With the modulation of the chemical potential the topological states not only appear at $\mu = 0$ but in a region between $\mu(x_1) = \mu(x_4)$ and $\mu(x_3) = \mu(x_2)$, where the points are given as (see section C.6 of appendix C):

$$x_1 = -\sqrt{\frac{2\mu - m\alpha^2 + B}{m\omega^2}}, \quad x_2 = -\sqrt{\frac{2\mu - m\alpha^2 - B}{m\omega^2}}, \quad (4.30)$$

$$x_3 = \sqrt{\frac{2\mu - m\alpha^2 - B}{m\omega^2}}, \quad x_4 = \sqrt{\frac{2\mu - m\alpha^2 + B}{m\omega^2}}, \quad (4.31)$$

where $\mu > \mu_\alpha + B/2$ is the fixed and external chemical potential. Thus, from all the possible cases of the Hamiltonian, the topological regions only exist for an horizontal short region in a h vs. μ figure, shown in figure 4.6.

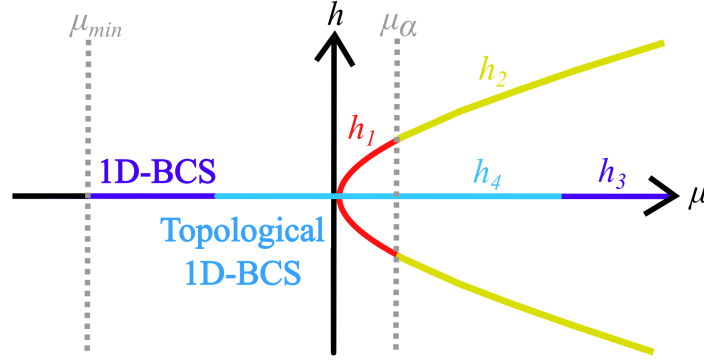


Figure 4.6: Relation between the on-plane Zeeman coupling h and the full chemical potential μ for fixed α . The dispersion relations is bounded from below with $\mu_{min} = -B^2/8m\alpha^2$, where there are two non-topological regions outside the gap, with a one-dimensional superconducting phase (1D-BCS, blue), which is for the case 3 (h_3). In the middle, there is the topological region inside the gap, with a one-dimensional superconducting phase is established (topological 1D-BCS, cyan), which is for the case 4 (h_4). The red and yellow curves are the physics for $h \neq 0$ (cases 1 and 2, h_1 and h_2 respectively), that disrupt the superconducting phases.

A final image of the system with the phases displayed over its extension is given in figure 4.7.

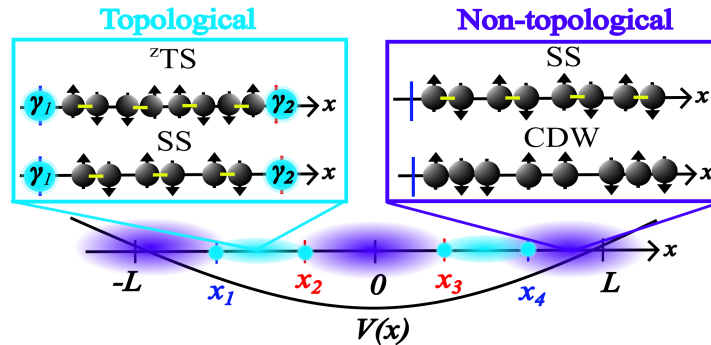


Figure 4.7: Display of the different phases lengthwise with the harmonic trap at the bottom. In the cyan-colored regions there are the topological phases, that when expanded in the upper box, there could be a triplet pairing or singlet pairing between the fermions (black balls), between the Majorana fermions γ_1 and γ_2 at the edges of the topological regions. The short yellow band shows when the fermions are paired. On the other hand, there are the non-topological phases in the blue-colored regions, that when expanded in the top box there could be a singlet pairing or a charge density wave, with no Majorana excitations between these fermions.

Notice that the phases are described with the spin degrees of freedom, because it was shown that the helical charge density and helical spin pairing are equivalent. In the SS case, the particles arrange in singlet pairs with zero net spin lengthwise, while in the zTS case the singlet pairs array in such a way that locally there is a net spin of $s = 1$. The existence of the Majorana fermions has been proven in this system theoretically, where expected experimental realizations could be based in figure 1.1, hopefully in the near future with little modifications presented during this work.

Chapter 5

Conclusions and perspectives

The system is modeled in accordance with the multiple mentions of the topological states in the literature, where the preliminary couplings with their parameters are the spin-orbit coupling α , the planar Zeeman coupling h , the axial Zeeman coupling B and on-site attractive interactions between opposite spins U . It is concluded that, for the purpose of establishing topological states, the planar Zeeman coupling is not necessary as it was shown that this coupling breaks the interplay between the attractive interactions and the axial Zeeman coupling, reported vastly in the literature. This is because the planar Zeeman coupling breaks the time inversion symmetry that can be preserved using the helical basis, so it is not convenient to change the basis but keep that broken symmetry where in fact is the responsible of the phase transition. Instead, its inclusion constitutes a different physics analysis, like it motivates the research on the FFLO phase, for example, where it introduces the spin imbalance that can originate non-zero momentum pairings. The problem with it is that in gases it drops the contribution of the BCS channel of the on-site attractive interactions which allows the superconducting phase for this system. This coupling can be introduced to have a spin imbalance, but somehow the time-reversal symmetry must be restored, and many works in literature have suggested using long-range interactions.

When $h = 0$ is considered, it is concluded that true phases can be established between the interplay of g_U and g_B in the strong interaction limit (more specifically, between the interplay of $B \propto g_B$ and $\Delta \propto g_U$). This is because, under the proper basis of the axial Zeeman coupling, $h = 0$ enables both coupling and interaction to contribute with their full channels (cosines), where those cosines are evaluated in canonical conjugated fields. This interplay, either with bosonization or not, has been found in many systems that display Majorana fermions, like for example the interplay between two kinds of hopping in lattices under chain couplings. In this work, such interplay has been shown using the bosonization formalism.

On the other hand, as the strong interaction limit is used, the phases are given by the correlations and susceptibilities under the LRT, where their behavior concludes that the perturbative approaches fail to describe correctly these one-dimensional system. In those approximations, related to the Fermi liquid theory, the decay is finite (well-defined peaks where a finite region of excitations contribute to the spectra), meaning that those collective systems are modeled as free quasiparticles. This cannot happen in one dimension, due to their strong collectivity. Thus, under the low-energy approximation, the susceptibilities show that any excitation has a finite response to an external coupling, where the density phase is reached

at the exchange momentum between the particles in the Fermi surface, and for the pairing excitations it happens at the net momentum of both fermions in the Fermi surface. The response is less and less relevant as the momentum separates from the excitation momentum of the Fermi surface, just as stated from the low-energy approximation, but **it is never zero**, it always has a contribution that shows how strongly correlated the one-dimensional system is when the fermions can interact.

From the eight possible orders considered in the present Thesis, it is concluded that in the strong interaction limit only three of them can be established, which are the charge density wave, the topological one-dimensional triplet superconducting phase, and the topological or non-topological singlet superconducting phase, which have been arranged in a proper phase diagram displayed in figure 4.4 as a key result of the present work. The strong attractive interactions do not allow the system to order in spin density although the strong Zeeman gap exists. This because it is more relevant in two finite region, while the superconducting gap is more relevant in the rest of the physical parameters. Similarly, due to $h = 0$, the planar triplet-pairing states cannot happen because the planar spin symmetry is not broken, as it is shown by the correlations of the planar spin densities (they are equal). Within the latter phases, the topological phase existence is concluded from the quasiparticle-like excitation that appears in the $B > \Delta$ region, which is associated with the flipping excitation between the internal fermions (fermions in the 0-band), which is of zero energy due to the time-reversal symmetry exactly at the middle of the dispersion relation. This is usually described as a ghost state because of its degeneracy nature of the state. It is of zero energy but indeed exists and has been shown that it has a definite excitation. So, it is concluded that although the Hamiltonian separates in $\hat{H} = \hat{H} + 0$, that 0 contribution has a proper excitation which provides striking physics to the system.

In the end, it is concluded that the system displays two topological regions under the modulation of the chemical potential a harmonic trap, but that can also be achieved if a modulation of the tuning parameter B/Δ is used experimentally. This system can be used to take advantage of the protected states in those regions, for example as protected qubits, and also to introduce protected superconductivity in any device that could disrupt such phase with quantum perturbations.

5.1 Perspectives

There are three things that can be revisited in order to look for novel physics:

- To consider $h \neq 0$, so the physics of the cases 1 and 2 can be explored, where in the literature those cases are reported for magnetic phases which could allow the study of one-dimensional magnetization dynamics.
- To introduce thermal fluctuations. The given results are at zero temperature and are valid if thermal fluctuations are not relevant compared to quantum ones, which does not always happen. A more realistic system could have comparable thermal fluctuations, so an experimental setup could be brought closer to the modeled system.
- To consider a general interplay between g_B and g_U , that is, to also consider them to be finite, so the fields are not gapped and for example, a renormalization approach can be used. There could be found finite critical values for the Zeeman gap and superconducting gap, that are reported in the literature in more complex phase diagrams.

Appendix A

Quantum field theory of the bosonization

Following the Tomonaga-Luttinger model (see equation 1.9), the excitations are defined in terms of the chiral fermions $\hat{\psi}_R$ and $\hat{\psi}_L$ either in position or momentum space. To consider the time evolution, the coordinate space is of position and time, $\hat{\psi}_{R,L}(x, t)$, and the conjugate space of momentum and frequency, $\hat{\psi}_{R,L}(p, \omega)$, where a more general Fourier transform is [1–3]:

$$\hat{\psi}_{R,L}(x, t) = \frac{1}{\sqrt{2\pi}} \int dp d\omega e^{ipx} e^{-i\omega t} \hat{\psi}_{R,L}(p, \omega). \quad (\text{A.1})$$

The normalization factor is not changed since next is shown that only the momentum variable is needed [26, 28]. The Fourier transform in equation 1.3 suppress the time variable, so $\hat{\psi}_{R,L}(p) = \hat{\psi}_{R,L}(p, t)$. Using the Heisenberg image, the time evolution of the chiral fermions is:

$$\hat{\psi}_{R,L}(x, t) = e^{i\hat{H}t} \hat{\psi}_{R,L}(x, 0) e^{-i\hat{H}t}, \quad (\text{A.2})$$

with has $\hat{H}$ of the Tomonaga-Luttinger model. Hence, the Hamilton equation is:

$$\partial_t \hat{\psi}_{R,L}(x, t) = i \left[\hat{H}, \hat{\psi}_{R,L}(x, t) \right]. \quad (\text{A.3})$$

The commutator can be solved by replacing the Hamiltonian and the Fourier transform with a suppressed time variable, that for R chiral fermions is done as:

$$\begin{aligned} \left[\hat{H}, \hat{\psi}_R(x, t) \right] &= \hat{H} \hat{\psi}_R(x, t) - \hat{\psi}_R(x, t) \hat{H} \\ &= \int dp \nu_F p \left(\hat{\psi}_R^\dagger(p, t) \hat{\psi}_R(p, t) - \hat{\psi}_L^\dagger(p, t) \hat{\psi}_L(p, t) \right) \frac{1}{\sqrt{2\pi}} \int dq e^{iqx} \hat{\psi}_R(q, t) \\ &\quad - \frac{1}{\sqrt{2\pi}} \int dq e^{iqx} \hat{\psi}_R(q, t) \int dp \nu_F p \left(\hat{\psi}_R^\dagger(p, t) \hat{\psi}_R(p, t) - \hat{\psi}_L^\dagger(p, t) \hat{\psi}_L(p, t) \right) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\sqrt{2\pi}} \left[\int dp dq \nu_F p e^{iqx} \hat{\psi}_R^\dagger(p, t) \hat{\psi}_R(p, t) \hat{\psi}_R(q, t) - \int dp dq \nu_F p e^{iqx} \hat{\psi}_L^\dagger(p, t) \hat{\psi}_L(p, t) \hat{\psi}_R(q, t) \right. \\
&\quad \left. - \int dp dq \nu_F p e^{iqx} \hat{\psi}_R(q, t) \hat{\psi}_R^\dagger(p, t) \hat{\psi}_R(p, t) + \int dp dq \nu_F p e^{iqx} \hat{\psi}_R(q, t) \hat{\psi}_L^\dagger(p, t) \hat{\psi}_L(p, t) \right]. \tag{A.4}
\end{aligned}$$

The next step is to recall the anticommutation relations for fermions. For same chirality the fermions anticommute normally, this is $\{\hat{\psi}_{R,L}(p), \hat{\psi}_{R,L}^\dagger(q)\} = \delta(p-q)$ and $\{\hat{\psi}_{R,L}^\dagger(p), \hat{\psi}_{R,L}^\dagger(q)\} = 0$ [1–3, 25]. In the case of different chirality, the fermions do not have the same momentum by any means (except at zero momentum, which is not considered), so they anticommute $\{\hat{\psi}_R^\dagger(p), \hat{\psi}_L^\dagger(q)\} = 0$. So, these anticommutations are used in the following way. In the first integral:

$$\begin{aligned}
\hat{\psi}_R^\dagger(p, t) \hat{\psi}_R(p, t) \hat{\psi}_R(q, t) &= -\hat{\psi}_R^\dagger(p, t) \hat{\psi}_R(q, t) \hat{\psi}_R(p, t) \\
&= -(\delta(p-q) - \hat{\psi}_R(q, t) \hat{\psi}_R^\dagger(p, t)) \hat{\psi}_R(p, t) \\
&= -\delta(p-q) \hat{\psi}_R(p, t) + \hat{\psi}_R(q, t) \hat{\psi}_R^\dagger(p, t) \hat{\psi}_R(p, t), \tag{A.5}
\end{aligned}$$

and in the second:

$$\hat{\psi}_L^\dagger(p, t) \hat{\psi}_L(p, t) \hat{\psi}_R(q, t) = -\hat{\psi}_L^\dagger(p, t) \hat{\psi}_R(q, t) \hat{\psi}_L(p, t) = \hat{\psi}_R(q, t) \hat{\psi}_L^\dagger(p, t) \hat{\psi}_L(p, t). \tag{A.6}$$

Replacing the anticommutations:

$$\begin{aligned}
&[\hat{H}, \hat{\psi}_R(x, t)] \\
&= \frac{1}{\sqrt{2\pi}} \left[\int dp dq \nu_F p e^{iqx} \left[-\delta(p-q) \hat{\psi}_R(p, t) + \hat{\psi}_R(q, t) \hat{\psi}_R^\dagger(p, t) \hat{\psi}_R(p, t) \right] \right. \\
&\quad - \int dp dq \nu_F p e^{iqx} \hat{\psi}_R(q, t) \hat{\psi}_L^\dagger(p, t) \hat{\psi}_L(p, t) \\
&\quad \left. - \int dp dq \nu_F p e^{iqx} \hat{\psi}_R(q, t) \hat{\psi}_R^\dagger(p, t) \hat{\psi}_R(p, t) + \int dp dq \nu_F p e^{iqx} \hat{\psi}_R(q, t) \hat{\psi}_L^\dagger(p, t) \hat{\psi}_L(p, t) \right] \\
&= \frac{1}{\sqrt{2\pi}} \left[- \int dp \nu_F p e^{ipx} \hat{\psi}_R(p, t) \right. \\
&\quad \left. \int dp dq \nu_F p e^{iqx} \hat{\psi}_R(q, t) \hat{\psi}_R^\dagger(p, t) \hat{\psi}_R(p, t) - \int dp dq \nu_F p e^{iqx} \hat{\psi}_R(q, t) \hat{\psi}_R^\dagger(p, t) \hat{\psi}_R(p, t) \right] \\
&= -\frac{\nu_F}{\sqrt{2\pi}} \int dp p e^{ipx} \hat{\psi}_R(p, t) = -\frac{\nu_F}{\sqrt{2\pi}} \int dp (-i\partial_x e^{ipx}) \hat{\psi}_R(p, t) \\
&= i\nu_F \partial_x \frac{1}{\sqrt{2\pi}} \int dp e^{ipx} \hat{\psi}_R(p, t) = i\nu_F \partial_x \hat{\psi}_R(x, t), \tag{A.7}
\end{aligned}$$

where $p e^{ipx} = -i\partial_x e^{ipx}$ and the derivative is outside the integral by the differentiation under integral sign [26]. Replacing the commutator in equation A.3, and recalling that for L fermions

the velocity changes from ν_F to $-\nu_F$, the evolution for the chiral fermions is:

$$\partial_t \hat{\psi}_R(x, t) = -\nu_F \partial_x \hat{\psi}_R(x, t) \rightarrow (\partial_t + \nu_F \partial_x) \hat{\psi}_R(x, t) = 0, \quad (\text{A.8})$$

$$\partial_t \hat{\psi}_L(x, t) = \nu_F \partial_x \hat{\psi}_L(x, t) \rightarrow (\partial_t - \nu_F \partial_x) \hat{\psi}_L(x, t) = 0. \quad (\text{A.9})$$

The latter equations establish the independence of the chiralities. For the R fermions (equivalent to L fermions but with $-\nu_F$), if the Fourier transform of equation A.1 is replaced in equation A.8, thus:

$$\begin{aligned} (\partial_t + \nu_F \partial_x) \hat{\psi}_R(x, t) &= (\partial_t + \nu_F \partial_x) \frac{1}{\sqrt{2\pi}} \int dp d\omega e^{ipx} e^{-i\omega t} \hat{\psi}_R(p, \omega) \\ &= \frac{i}{\sqrt{2\pi}} \int dp d\omega (\nu_F p - \omega) e^{ipx} e^{-i\omega t} \hat{\psi}_R(p, \omega) = 0, \end{aligned} \quad (\text{A.10})$$

so the solution is not trivial if $\nu_F p - \omega = 0 \rightarrow \omega = \nu_F p$. This condition is satisfied proposing $\hat{\psi}_R(p, \omega) = \delta(\nu_F p - \omega) \hat{\psi}_R(p)$ [28], and when replaced in the Fourier transform of equation A.1 for R and L fermions, they satisfy:

$$\begin{aligned} \hat{\psi}_R(x, t) &= \frac{1}{\sqrt{2\pi}} \int dp d\omega e^{ipx} e^{-i\omega t} \hat{\psi}_R(p, \omega) \\ &= \frac{1}{\sqrt{2\pi}} \int dp d\omega e^{ipx} e^{-i\omega t} \delta(\nu_F p - \omega) \hat{\psi}_R(p) = \frac{1}{\sqrt{2\pi}} \int dp e^{ipx} e^{-i\nu_F p t} \hat{\psi}_R(p) \\ &= \frac{1}{\sqrt{2\pi}} \int dp e^{ip(x - \nu_F t)} \hat{\psi}_R(p) = \hat{\psi}_R(x - \nu_F t), \end{aligned} \quad (\text{A.11})$$

$$\hat{\psi}_L(x, t) = \hat{\psi}_L(x + \nu_F t). \quad (\text{A.12})$$

The latter conditions show that the fermions depend on independent variables in $1+1$ spaces (like a two-dimensional Minkowski space [25, 72, 73]), so they belong to independent spaces where the operators commute, $[\hat{\psi}_R^{(\dagger)}(r), \hat{\psi}_L^{(\dagger)}(r')] = 0$ ($r = [x, \nu_F t]$ is the vector position in the $1+1$ space), as presented in chapter 1 (if used in the derivation of equation A.7, the calculation becomes simpler). Notice that the position space in fact depends on space and time, but as the variable number reduces to only one ($x \pm \nu_F t$), hence x or r is widely used in the fields although it is not only x , just to avoid extra notation. In almost all of this document the notation is given in x variable but time is introduced explicitly when needed. These results are crucial to the diagonalization of the Hamiltonian in terms of densities and currents [25], and it is because of the following two features.

First, **the separability of the space**. The grounds state is built by reiterated creation of fermions in their respective branches, where the correlations for such fermions become trivial and are not mutually affected by different chirality [25]. This means that:

$$\langle \hat{\psi}_R^\dagger(p) \hat{\psi}_R(p') \rangle = \langle \hat{\psi}_L(p) \hat{\psi}_L^\dagger(p') \rangle = Y(-p) \delta(p - p'), \quad (\text{A.13})$$

$$\langle \hat{\psi}_L^\dagger(p) \hat{\psi}_L(p') \rangle = \langle \hat{\psi}_R(p) \hat{\psi}_R^\dagger(p') \rangle = Y(p) \delta(p - p'), \quad (\text{A.14})$$

where the notation of $Y(p)$ is the step function in the momentum variable. The latter correlations simply mean that R fermions are annihilated and L fermions are created only if

momentum is negative, as negative momentum is the filled part of the R dispersion relation while it is empty for L fermions. On the other hand, R fermions are created and L fermions annihilated only if momentum is positive. The delta function appears because the ket in the correlation must have the same occupation of the bra so the inner product $\langle | \rangle$ is not zero. An illustration of these correlations is given in the next figure.

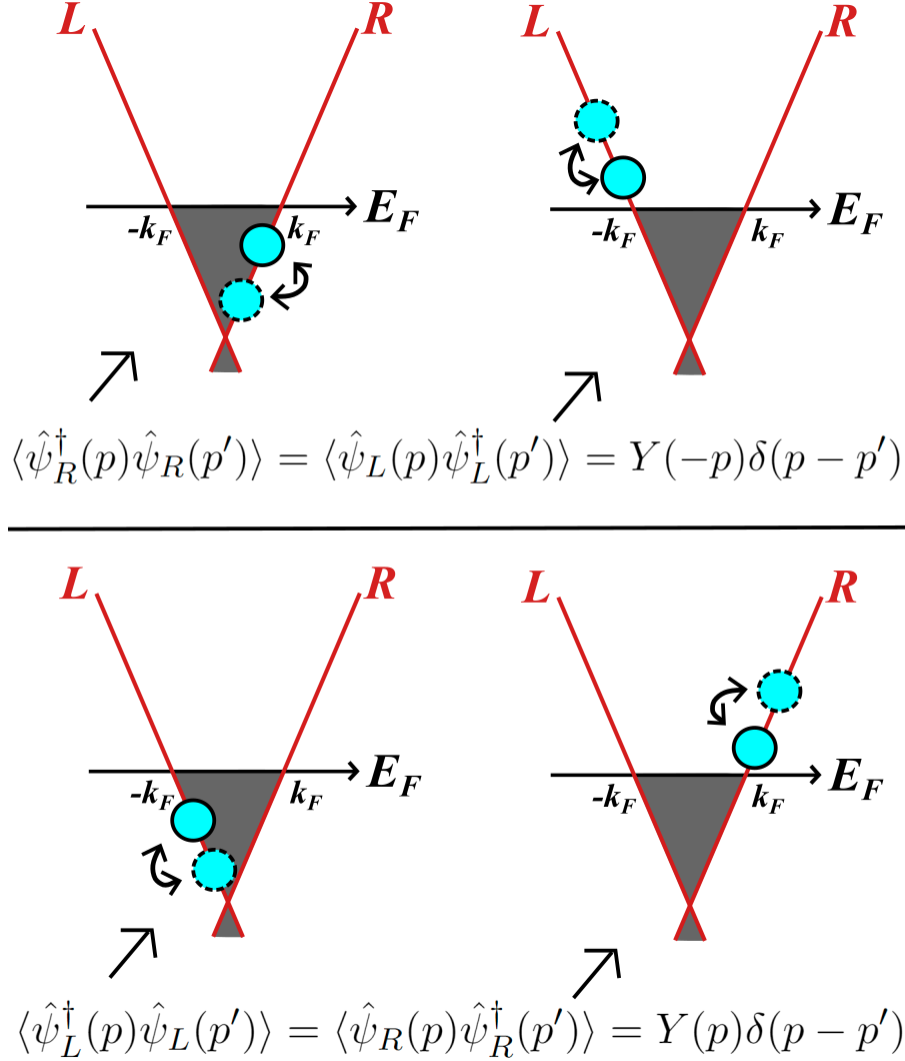


Figure A.1: Possible own-branch scattering for chiral fermions. In the upper part, there is the scattering at negative momentum while in the lower part, there is the positive momentum one. It must be clear that the sign of momentum is related to the definition $p = k \pm k_F$, so the negative momentum is the one at the left side to the respective Fermi momenta of the branches, and the positive is at the right side.

The second feature is **the specific form of the equations of motion for the annihilation and creation operators**. Those equations are fundamental for the physics of the density and current operators in an hydrodynamics theory, where the quantum field theory is sustained on the *continuity equation* [25, 26, 28], as it is shown in the next section.

A.1 Density and current formulation

The chiral current densities are defined as [25, 42]:

$$\hat{J}_{R,L}(r) = :\hat{\psi}_{R,L}^\dagger(r)\hat{\psi}_{R,L}(r):, \quad (\text{A.15})$$

with the explicit normal ordering, which is exploited later. Although it has a form of a charge density, it is said a current due to the inherent directionality of the operators in this case. It can be shown that the chiral densities satisfy the same equations A.8 and A.9 as follows:

$$\begin{aligned} (\partial_t + \nu_F \partial_x) \hat{J}_R(r) &= (\partial_t + \nu_F \partial_x) :\hat{\psi}_R^\dagger(r) \hat{\psi}_R(r): \\ &= (\partial_t + \nu_F \partial_x) \left[\hat{\psi}_R^\dagger(r) \hat{\psi}_R(r) - \langle \hat{\psi}_R^\dagger(r) \hat{\psi}_R(r) \rangle \right] \\ &= (\partial_t + \nu_F \partial_x) \hat{\psi}_R^\dagger(r) \hat{\psi}_R(r) \\ &= \left[(\partial_t + \nu_F \partial_x) \hat{\psi}_R^\dagger(r) \right] \hat{\psi}_R(r) + \hat{\psi}_R^\dagger(r) \left[(\partial_t + \nu_F \partial_x) \hat{\psi}_R(r) \right] \\ &= \left[(\partial_t + \nu_F \partial_x) \hat{\psi}_R(r) \right]^\dagger \hat{\psi}_R(r) + \hat{\psi}_R^\dagger(r) \cdot 0 \\ &= 0 \cdot \hat{\psi}_R(r) = 0, \end{aligned} \quad (\text{A.16})$$

$$(\partial_t - \nu_F \partial_x) \hat{J}_L(r) = 0. \quad (\text{A.17})$$

On the other hand, the net charge and current densities of the system are defined as:

$$\hat{\rho}(r) = \hat{J}_R(r) + \hat{J}_L(r), \quad (\text{A.18})$$

$$\hat{J}(r) = \nu_F \left(\hat{J}_R(r) - \hat{J}_L(r) \right), \quad (\text{A.19})$$

from where the continuity equation is derived in the following way. First, when applying the derivative in position:

$$\partial_x \hat{J}(r) = \partial_x \left[\nu_F \left(\hat{J}_R(r) - \hat{J}_L(r) \right) \right] = \nu_F \partial_x \hat{J}_R(r) - \nu_F \partial_x \hat{J}_L(r),$$

so by using equations the latter results for the chiral currents, $\nu_F \partial_x \hat{J}_R(r) = -\partial_t \hat{J}_R(r)$ and $-\nu_F \partial_x \hat{J}_L(r) = -\partial_t \hat{J}_L(r)$ are replaced, and:

$$\begin{aligned} \partial_x \hat{J}(r) &= \nu_F \partial_x \hat{J}_R(r) - \nu_F \partial_x \hat{J}_L(r) = -\partial_t \hat{J}_R(r) - \partial_t \hat{J}_L(r) = -\partial_t \left(\hat{J}_R(r) + \hat{J}_L(r) \right) \rightarrow \\ &\partial_x \hat{J}(r) + \partial_t \rho(r) = 0, \end{aligned} \quad (\text{A.20})$$

which is the expected continuity equation. It is important to notice that the operators which diagonalize the Tomonaga-Luttinger model in momentum are the chiral fermions, and their equations of motion lead to the continuity equation in $\hat{J}$ and $\hat{\rho}$ operators, so it means those are the real current and charge densities that summarize the collective behavior of the one-dimensional system [25]. Hence, in position space those are the fundamental excitations, this is, **the Hamiltonian must be quadratic in charge and current densities**. The quadratic

form is written with the squared currents $\hat{J}_R(x)^2$ and $\hat{J}_L(x)^2$, where for the product of two normal ordered operators they are given as [28]:

$$\begin{aligned}
\hat{J}_{R,L}(x)^2 &= :\hat{\psi}_{R,L}^\dagger(x)\hat{\psi}_{R,L}(x)::\hat{\psi}_{R,L}^\dagger(x)\hat{\psi}_{R,L}(x): \\
&= :\hat{\psi}_{R,L}^\dagger(x)\hat{\psi}_{R,L}(x)\hat{\psi}_{R,L}^\dagger(x)\hat{\psi}_{R,L}(x): + \langle\hat{\psi}_{R,L}^\dagger(x)\hat{\psi}_{R,L}^\dagger(x)\rangle:\hat{\psi}_{R,L}(x)\hat{\psi}_{R,L}(x): \\
&\quad + \langle\hat{\psi}_{R,L}^\dagger(x)\hat{\psi}_{R,L}(x)\rangle:\hat{\psi}_{R,L}(x)\hat{\psi}_{R,L}^\dagger(x): + \langle\hat{\psi}_{R,L}(x)\hat{\psi}_{R,L}^\dagger(x)\rangle:\hat{\psi}_{R,L}^\dagger(x)\hat{\psi}_{R,L}(x): \\
&\quad + \langle\hat{\psi}_{R,L}(x)\hat{\psi}_{R,L}(x)\rangle:\hat{\psi}_{R,L}^\dagger(x)\hat{\psi}_{R,L}^\dagger(x): + \langle\hat{\psi}_{R,L}^\dagger(x)\hat{\psi}_{R,L}^\dagger(x)\rangle\langle\hat{\psi}_{R,L}(x)\hat{\psi}_{R,L}(x)\rangle \\
&\quad + \langle\hat{\psi}_{R,L}^\dagger(x)\hat{\psi}_{R,L}(x)\rangle\langle\hat{\psi}_{R,L}(x)\hat{\psi}_{R,L}^\dagger(x)\rangle \\
&= :\hat{\psi}_{R,L}^\dagger(x)\hat{\psi}_{R,L}(x)\hat{\psi}_{R,L}^\dagger(x)\hat{\psi}_{R,L}(x): \\
&\quad + \langle\hat{\psi}_{R,L}^\dagger(x)\hat{\psi}_{R,L}(x)\rangle:\hat{\psi}_{R,L}(x)\hat{\psi}_{R,L}^\dagger(x): \\
&\quad + \langle\hat{\psi}_{R,L}(x)\hat{\psi}_{R,L}^\dagger(x)\rangle:\hat{\psi}_{R,L}^\dagger(x)\hat{\psi}_{R,L}(x): \\
&\quad + \langle\hat{\psi}_{R,L}^\dagger(x)\hat{\psi}_{R,L}(x)\rangle\langle\hat{\psi}_{R,L}(x)\hat{\psi}_{R,L}^\dagger(x)\rangle. \tag{A.21}
\end{aligned}$$

The problem with the latter definition is that the last term diverges since a fermion is infinitely correlated with itself in the same point, which is a common problem in the quantum field theory. Thus, a useful technique is the *point splitting* [25–28, 42], where the squared chiral currents are defined for instance as:

$$\hat{J}_{R,L}(x)^2 = \lim_{\epsilon \rightarrow 0} \hat{J}_{R,L}(x - \epsilon)\hat{J}_{R,L}(x + \epsilon). \tag{A.22}$$

Using the latter definition, the expansion is now:

$$\begin{aligned}
\hat{J}_{R,L}(x)^2 &= \lim_{\epsilon \rightarrow 0} :\hat{\psi}_{R,L}^\dagger(x - \epsilon)\hat{\psi}_{R,L}(x - \epsilon)::\hat{\psi}_{R,L}^\dagger(x + \epsilon)\hat{\psi}_{R,L}(x + \epsilon): \\
&= \lim_{\epsilon \rightarrow 0} \hat{\psi}_{R,L}^\dagger(x - \epsilon)\hat{\psi}_{R,L}(x - \epsilon)\hat{\psi}_{R,L}^\dagger(x + \epsilon)\hat{\psi}_{R,L}(x + \epsilon): \\
&\quad + \langle\hat{\psi}_{R,L}^\dagger(x - \epsilon)\hat{\psi}_{R,L}^\dagger(x + \epsilon)\rangle:\hat{\psi}_{R,L}(x - \epsilon)\hat{\psi}_{R,L}(x + \epsilon): \\
&\quad + \langle\hat{\psi}_{R,L}^\dagger(x - \epsilon)\hat{\psi}_{R,L}(x + \epsilon)\rangle:\hat{\psi}_{R,L}(x - \epsilon)\hat{\psi}_{R,L}^\dagger(x + \epsilon): \\
&\quad + \langle\hat{\psi}_{R,L}(x - \epsilon)\hat{\psi}_{R,L}^\dagger(x + \epsilon)\rangle:\hat{\psi}_{R,L}^\dagger(x - \epsilon)\hat{\psi}_{R,L}(x + \epsilon): \\
&\quad + \langle\hat{\psi}_{R,L}(x - \epsilon)\hat{\psi}_{R,L}(x + \epsilon)\rangle:\hat{\psi}_{R,L}^\dagger(x - \epsilon)\hat{\psi}_{R,L}^\dagger(x + \epsilon): \\
&\quad + \langle\hat{\psi}_{R,L}^\dagger(x - \epsilon)\hat{\psi}_{R,L}^\dagger(x + \epsilon)\rangle\langle\hat{\psi}_{R,L}(x - \epsilon)\hat{\psi}_{R,L}(x + \epsilon)\rangle \\
&\quad + \langle\hat{\psi}_{R,L}^\dagger(x - \epsilon)\hat{\psi}_{R,L}(x + \epsilon)\rangle\langle\hat{\psi}_{R,L}(x - \epsilon)\hat{\psi}_{R,L}^\dagger(x + \epsilon)\rangle \\
&= - \lim_{\epsilon \rightarrow 0} \langle\hat{\psi}_{R,L}^\dagger(x + \epsilon)\hat{\psi}_{R,L}(x - \epsilon)\rangle:\hat{\psi}_{R,L}^\dagger(x - \epsilon)\hat{\psi}_{R,L}(x + \epsilon): \\
&\quad - \langle\hat{\psi}_{R,L}^\dagger(x - \epsilon)\hat{\psi}_{R,L}(x + \epsilon)\rangle:\hat{\psi}_{R,L}^\dagger(x + \epsilon)\hat{\psi}_{R,L}(x - \epsilon): \\
&\quad - \langle\hat{\psi}_{R,L}^\dagger(x - \epsilon)\hat{\psi}_{R,L}(x + \epsilon)\rangle\langle\hat{\psi}_{R,L}^\dagger(x + \epsilon)\hat{\psi}_{R,L}(x - \epsilon)\rangle. \tag{A.23}
\end{aligned}$$

The normal ordered product of four fields and terms with two annihilation (creation) operators vanish as the splitting ϵ approaches zero. The correlations for the fermionic fields in position

space are calculated for both branches replacing the Fourier transform:

$$\begin{aligned}
\langle \hat{\psi}_R^\dagger(x) \hat{\psi}_R(y) \rangle &= \left\langle \frac{1}{\sqrt{2\pi}} \int dp e^{-ipx} \hat{\psi}_R^\dagger(p) \frac{1}{\sqrt{2\pi}} \int dq e^{iqy} \hat{\psi}_R(q) \right\rangle \\
&= \frac{1}{2\pi} \int dp dq e^{-ipx} e^{iqy} \langle \hat{\psi}_R^\dagger(p) \hat{\psi}_R(q) \rangle = \frac{1}{2\pi} \int dp dq e^{-ipx} e^{iqy} Y(-p) \delta(p-q) \\
&= \frac{1}{2\pi} \int_{-\infty}^0 dp e^{-ipx} e^{ipy} = \frac{1}{2\pi} \int_0^\infty dp e^{ip(x-y)}, \tag{A.24}
\end{aligned}$$

where the latter integral is not well defined. To calculate it properly, a convergence *cutoff* δ is introduced manually (the same cutoff in the definition of the bosonization identity of equation 1.12) [25, 42], by introducing the function $e^{-\delta|p|}$ in the argument where such integral converges to the latter integral in the limit $\delta \rightarrow 0^+$, this is:

$$\begin{aligned}
\langle \hat{\psi}_R^\dagger(x) \hat{\psi}_R(y) \rangle &= \frac{1}{2\pi} \int_0^\infty dp e^{ip(x-y)} = \lim_{\delta \rightarrow 0^+} \frac{1}{2\pi} \int_0^\infty dp e^{-\delta|p|} e^{ip(x-y)} \\
&= \lim_{\delta \rightarrow 0^+} \frac{1}{2\pi} \left(\int_0^\infty dp e^{-\delta p} \cos[p(x-y)] + i \int_0^\infty dp e^{-\delta p} \sin[p(x-y)] \right). \tag{A.25}
\end{aligned}$$

The resultant two integrals are solved via reiterated integration by parts and expecting a cyclic integral, so both are:

$$\int_0^\infty dp e^{-\delta p} \cos(px) = \frac{\delta}{\delta^2 + x^2}, \tag{A.26}$$

$$\int_0^\infty dp e^{-\delta p} \sin(px) = \frac{x}{\delta^2 + x^2}. \tag{A.27}$$

Replacing both in the correlation, it gives:

$$\begin{aligned}
\langle \hat{\psi}_R^\dagger(x) \hat{\psi}_R(y) \rangle &= \lim_{\delta \rightarrow 0^+} \frac{1}{2\pi} \left(\frac{\delta}{\delta^2 + (x-y)^2} + i \frac{x-y}{\delta^2 + (x-y)^2} \right) = \lim_{\delta \rightarrow 0^+} \frac{1}{2\pi} \frac{\delta + i(x-y)}{\delta^2 + (x-y)^2}, \\
&= \lim_{\delta \rightarrow 0^+} \frac{1}{2\pi} \frac{1}{\delta - i(x-y)} = \lim_{\delta \rightarrow 0^+} \frac{i}{2\pi[(x-y) + i\delta]}. \tag{A.28}
\end{aligned}$$

where it is noticed that $z = \delta + i(x-y)$ is a complex number, so $|z|^2 = z^* z = \delta^2 + (x-y)^2$ and then $\delta + i(x-y)/\delta^2 + (x-y)^2 = z/z^* z = 1/z^* = 1/\delta - i(x-y)$. On the other hand, the limit is not trivial to evaluate because if $x = y$ the correlation diverges as said previously, but if they are different simply $\delta = 0$ works. The latter case is the one in the squared current of equation A.23 since ϵ approaches zero but is not zero, so:

$$\langle \hat{\psi}_R^\dagger(x) \hat{\psi}_R(y) \rangle = \frac{i}{2\pi(x-y)}, \quad x \neq y. \tag{A.29}$$

The case for the L branch is analogous to the R branch, the only change is a sign in the argument, this is:

$$\begin{aligned}
\langle \hat{\psi}_L^\dagger(x) \hat{\psi}_L(y) \rangle &= \left\langle \frac{1}{\sqrt{2\pi}} \int dp e^{-ipx} \hat{\psi}_L^\dagger(p) \frac{1}{\sqrt{2\pi}} \int dq e^{iqy} \hat{\psi}_L(q) \right\rangle \\
&= \frac{1}{2\pi} \int dp dq e^{-ipx} e^{iqy} \langle \hat{\psi}_L^\dagger(p) \hat{\psi}_L(q) \rangle = \frac{1}{2\pi} \int dp dq e^{-ipx} e^{iqy} Y(p) \delta(p-q) \\
&= \frac{1}{2\pi} \int_0^\infty dp e^{-ipx} e^{ipy} = \frac{1}{2\pi} \int_0^\infty dp e^{ip(y-x)}, \tag{A.30}
\end{aligned}$$

So it is the same result for the R branch but with $y - x$ instead of $x - y$, thus:

$$\langle \hat{\psi}_L^\dagger(x) \hat{\psi}_L(y) \rangle = -\frac{i}{2\pi(x-y)}, \quad x \neq y. \quad (\text{A.31})$$

Finally, the squared currents are:

$$\begin{aligned} \hat{J}_R(x)^2 &= -\lim_{\epsilon \rightarrow 0} \langle \hat{\psi}_R^\dagger(x+\epsilon) \hat{\psi}_R(x-\epsilon) \rangle : \hat{\psi}_R^\dagger(x-\epsilon) \hat{\psi}_R(x+\epsilon) : \\ &\quad - \langle \hat{\psi}_R^\dagger(x-\epsilon) \hat{\psi}_R(x+\epsilon) \rangle : \hat{\psi}_R^\dagger(x+\epsilon) \hat{\psi}_R(x-\epsilon) : \\ &\quad - \langle \hat{\psi}_R^\dagger(x-\epsilon) \hat{\psi}_R(x+\epsilon) \rangle \langle \hat{\psi}_R^\dagger(x+\epsilon) \hat{\psi}_R(x-\epsilon) \rangle \\ &= -\lim_{\epsilon \rightarrow 0} \left(\frac{i}{2\pi(x+\epsilon-x+\epsilon)} \right) : \hat{\psi}_R^\dagger(x-\epsilon) \hat{\psi}_R(x+\epsilon) : \\ &\quad - \left(\frac{i}{2\pi(x-\epsilon-x-\epsilon)} \right) : \hat{\psi}_R^\dagger(x+\epsilon) \hat{\psi}_R(x-\epsilon) : \\ &\quad - \left(\frac{i}{2\pi(x-\epsilon-x-\epsilon)} \right) \left(\frac{i}{2\pi(x+\epsilon-x+\epsilon)} \right) \\ &= -\lim_{\epsilon \rightarrow 0} \frac{i}{2\pi(2\epsilon)} : \hat{\psi}_R^\dagger(x-\epsilon) \hat{\psi}_R(x+\epsilon) : \\ &\quad + \frac{i}{2\pi(2\epsilon)} : \hat{\psi}_R^\dagger(x+\epsilon) \hat{\psi}_R(x-\epsilon) : - \frac{i}{(2\pi(2\epsilon))^2} \\ &\approx \frac{i}{2\pi} \lim_{\epsilon \rightarrow 0} \frac{1}{2\epsilon} : \left(\hat{\psi}_R^\dagger(x+\epsilon) \hat{\psi}_R(x-\epsilon) - \hat{\psi}_R^\dagger(x-\epsilon) \hat{\psi}_R(x+\epsilon) \right) :, \end{aligned} \quad (\text{A.32})$$

where the linear order in ϵ is the only relevant and retained. The next step is to recognize the definition of the derivative in terms of a limit, that is:

$$\frac{df(x)}{dx} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x-h)}{2h}. \quad (\text{A.33})$$

Such form can be achieved introducing $\hat{\psi}_R^\dagger(x-\epsilon) \hat{\psi}_R(x-\epsilon) - \hat{\psi}_R^\dagger(x-\epsilon) \hat{\psi}_R(x+\epsilon)$ in the result A.32:

$$\begin{aligned} \hat{J}_R(x)^2 &\approx \frac{i}{2\pi} \lim_{\epsilon \rightarrow 0} \frac{1}{2\epsilon} : \left(\hat{\psi}_R^\dagger(x+\epsilon) \hat{\psi}_R(x-\epsilon) - \hat{\psi}_R^\dagger(x-\epsilon) \hat{\psi}_R(x-\epsilon) \right. \\ &\quad \left. + \hat{\psi}_R^\dagger(x-\epsilon) \hat{\psi}_R(x-\epsilon) - \hat{\psi}_R^\dagger(x-\epsilon) \hat{\psi}_R(x+\epsilon) \right) : \\ &= \frac{i}{2\pi} : \left[\lim_{\epsilon \rightarrow 0} \frac{\hat{\psi}_R^\dagger(x+\epsilon) - \hat{\psi}_R^\dagger(x-\epsilon)}{2\epsilon} \hat{\psi}_R(x-\epsilon) \right. \\ &\quad \left. - \hat{\psi}_R^\dagger(x-\epsilon) \lim_{\epsilon \rightarrow 0} \frac{\hat{\psi}_R(x+\epsilon) - \hat{\psi}_R(x-\epsilon)}{2\epsilon} \right] : \\ &= \frac{i}{2\pi} : \left(\partial_x \hat{\psi}_R^\dagger(x) \hat{\psi}_R(x) - \hat{\psi}_R^\dagger(x) \partial_x \hat{\psi}_R(x) \right) :. \end{aligned} \quad (\text{A.34})$$

The latter procedure shows how to deal correctly with the divergence and obtain the relevant term which is clouded by higher and divergent irrelevant terms. For the L branch the only

change is a global minus sign, so:

$$\hat{J}_L(x)^2 \approx -\frac{i}{2\pi} : \left(\partial_x \hat{\psi}_L^\dagger(x) \hat{\psi}_L(x) - \hat{\psi}_L^\dagger(x) \partial_x \hat{\psi}_L(x) \right) :. \quad (\text{A.35})$$

With the latter quadratic forms it can be proved that the Hamiltonian is diagonal in those currents. For that, the first thing to do is to change the space of the Hamiltonian in equation 1.9 from momentum to position, where according to the second quantized one-particle operator in equation 1.1 and the representation of momentum $k = -i\partial_x$, the Hamiltonian is [25]:

$$\hat{H} = -i\nu_F \int dx \left[\hat{\psi}_R^\dagger(x) \partial_x \hat{\psi}_R(x) - \hat{\psi}_L^\dagger(x) \partial_x \hat{\psi}_L(x) \right]. \quad (\text{A.36})$$

Using the product rule for derivatives, it is shown that:

$$\begin{aligned} \hat{H} &= -i\nu_F \int dx \left[\hat{\psi}_R^\dagger(x) \partial_x \hat{\psi}_R(x) - \hat{\psi}_L^\dagger(x) \partial_x \hat{\psi}_L(x) \right] \\ &= -i\nu_F \int dx \left[\partial_x \left(\hat{\psi}_R^\dagger(x) \hat{\psi}_R(x) \right) - \partial_x \hat{\psi}_R^\dagger(x) \hat{\psi}_R(x) \right. \\ &\quad \left. - \partial_x \left(\hat{\psi}_L^\dagger(x) \hat{\psi}_L(x) \right) + \partial_x \hat{\psi}_L^\dagger(x) \hat{\psi}_L(x) \right] \\ &= i\nu_F \int dx \left[\partial_x \hat{\psi}_R^\dagger(x) \hat{\psi}_R(x) - \partial_x \hat{\psi}_L^\dagger(x) \hat{\psi}_L(x) \right] \\ &\quad - i\nu_F \int dx \left[\partial_x \left(\hat{\psi}_R^\dagger(x) \hat{\psi}_R(x) \right) - \partial_x \left(\hat{\psi}_L^\dagger(x) \hat{\psi}_L(x) \right) \right] \\ &= i\nu_F \int dx \left[\partial_x \hat{\psi}_R^\dagger(x) \hat{\psi}_R(x) - \partial_x \hat{\psi}_L^\dagger(x) \hat{\psi}_L(x) \right], \end{aligned} \quad (\text{A.37})$$

due to the vanishing of the last integral in the third line because the fields are bounded, so at the extremes of evaluation the fields vanish [25–28]. Replacing the chiral currents in equations A.34 and A.35 on the Hamiltonian of equation A.37, then:

$$\begin{aligned} \hat{H} &= i\nu_F \int dx \left[\partial_x \hat{\psi}_R^\dagger(x) \hat{\psi}_R(x) - \partial_x \hat{\psi}_L^\dagger(x) \hat{\psi}_L(x) \right] \\ &= i\nu_F \int dx \left[\hat{\psi}_R^\dagger(x) \partial_x \hat{\psi}_R(x) - i2\pi \hat{J}_R^2(x) - i2\pi \hat{J}_L^2(x) - \hat{\psi}_L^\dagger(x) \partial_x \hat{\psi}_L(x) \right] \\ &= i\nu_F \int dx \left[\hat{\psi}_R^\dagger(x) \partial_x \hat{\psi}_R(x) - \hat{\psi}_L^\dagger(x) \partial_x \hat{\psi}_L(x) \right] + 2\pi\nu_F \int dx \left[\hat{J}_R^2(x) + \hat{J}_L^2(x) \right] \\ &= -\hat{H} + 2\pi\nu_F \int dx \left[\hat{J}_R^2(x) + \hat{J}_L^2(x) \right] \rightarrow \\ &\quad \hat{H} = \pi\nu_F \int dx \left[\hat{J}_R^2(x) + \hat{J}_L^2(x) \right]. \end{aligned} \quad (\text{A.38})$$

This last result is one of the most important equations in the bosonization technique, where it is shown that the real collective current fields diagonalize the Hamiltonian. To finish the construction, the bosonic fields are derived according to those currents and their mode decomposition is shown in the next section.

A.2 Formulation in bosonic canonical fields

The quadratic form of the Hamiltonian A.38 gives a lot of insight into the bosonic nature of the system represented in this way, since the fermionic densities commute as bosonic operators [26–28]. Although it is like it, the bosonic annihilation and creation operators have not appeared explicitly, and this is to be done in this section. Those operators are used to define the canonical fields according to the mode decomposition, and the corresponding Klein-Gordon equation can be proposed (quantum field theory in the harmonic oscillator). First, one has to compute the commutators between the density operators in momentum representation, where the inverse Fourier transform of the chiral currents is:

$$\hat{J}_{R,L}(p) = \frac{1}{\sqrt{2\pi}} \int dx e^{-ipx} \hat{J}_{R,L}(x). \quad (\text{A.39})$$

Using the definition A.15 and replacing the Fourier transform of the respective chiral fermionic fields, then:

$$\begin{aligned} \hat{J}_{R,L}(p) &= \frac{1}{\sqrt{2\pi}} \int dx e^{-ipx} \hat{J}_{R,L}(x) = \frac{1}{\sqrt{2\pi}} \int dx e^{-ipx} \hat{\psi}_{R,L}^\dagger(x) \hat{\psi}_{R,L}(x) \\ &= \frac{1}{\sqrt{2\pi}} \int dx e^{-ipx} \frac{1}{\sqrt{2\pi}} \int dq e^{-iqx} \hat{\psi}_{R,L}^\dagger(q) \frac{1}{\sqrt{2\pi}} \int dk e^{ikx} \hat{\psi}_{R,L}(k) \\ &= \frac{1}{\sqrt{2\pi}} \int dp dk \left(\frac{1}{2\pi} \int dx e^{-ix(p+q-k)} \right) \hat{\psi}_{R,L}^\dagger(q) \hat{\psi}_{R,L}(k) \\ &= \frac{1}{\sqrt{2\pi}} \int dq dk \delta(p+q-k) \hat{\psi}_{R,L}^\dagger(q) \hat{\psi}_{R,L}(k) = \frac{1}{\sqrt{2\pi}} \int dk \hat{\psi}_{R,L}^\dagger(k-p) \hat{\psi}_{R,L}(k). \end{aligned} \quad (\text{A.40})$$

Hence, the Fourier component of the currents is nothing else than a collection of all possible hole and particle excitations [25]. The latter definition allows showing that:

$$\begin{aligned} \hat{J}_{R,L}^\dagger(p) &= \frac{1}{\sqrt{2\pi}} \int dk \hat{\psi}_{R,L}^\dagger(k) \hat{\psi}_{R,L}(k-p) \\ &= \frac{1}{\sqrt{2\pi}} \int dk \hat{\psi}_{R,L}^\dagger(k+p) \hat{\psi}_{R,L}(k) = \hat{J}_{R,L}(-p), \end{aligned} \quad (\text{A.41})$$

which is a usual characteristic of the modes that decompose the canonical fields in the classical Klein-Gordon equation [26–28]. It is expected that these densities commute somehow as the bosonic annihilation and creation operators. To see it, a first commutator between the chiral currents is computed as follows. First, the latter expression is replaced in the commutator:

$$\begin{aligned} [\hat{J}_{R,L}(p), \hat{J}_{R,L}(q)] &= \left[\frac{1}{\sqrt{2\pi}} \int dk \hat{\psi}_{R,L}^\dagger(k-p) \hat{\psi}_{R,L}(k), \frac{1}{\sqrt{2\pi}} \int dk' \hat{\psi}_{R,L}^\dagger(k'-q) \hat{\psi}_{R,L}(k') \right] \\ &= \frac{1}{2\pi} \int dk dk' [\hat{\psi}_{R,L}^\dagger(k-p) \hat{\psi}_{R,L}(k), \hat{\psi}_{R,L}^\dagger(k'-q) \hat{\psi}_{R,L}(k')]. \end{aligned} \quad (\text{A.42})$$

In this step the inner commutator is calculated using the rule for the product of operators,

this is:

$$\begin{aligned}
\left[\hat{\psi}_{R,L}^\dagger(k-p) \hat{\psi}_{R,L}(k), \hat{\psi}_{R,L}^\dagger(k'-q) \hat{\psi}_{R,L}(k') \right] &= \hat{\psi}_{R,L}^\dagger(k-p) \left[\hat{\psi}_{R,L}(k), \hat{\psi}_{R,L}^\dagger(k'-q) \right] \hat{\psi}_{R,L}(k') \\
&+ \left[\hat{\psi}_{R,L}^\dagger(k-p), \hat{\psi}_{R,L}^\dagger(k'-q) \right] \hat{\psi}_{R,L}(k) \hat{\psi}_{R,L}(k') \\
&+ \hat{\psi}_{R,L}^\dagger(k'-q) \hat{\psi}_{R,L}^\dagger(k-p) \left[\hat{\psi}_{R,L}(k), \hat{\psi}_{R,L}(k') \right] \\
&+ \hat{\psi}_{R,L}^\dagger(k'-q) \left[\hat{\psi}_{R,L}^\dagger(k-p), \hat{\psi}_{R,L}(k') \right] \hat{\psi}_{R,L}(k)
\end{aligned} \tag{A.43}$$

where the simpler commutators are computed using the anticommutation relations for the fermions:

$$\begin{aligned}
\left[\hat{\psi}_{R,L}(k), \hat{\psi}_{R,L}^\dagger(k'-q) \right] &= \hat{\psi}_{R,L}(k) \hat{\psi}_{R,L}^\dagger(k'-q) - \hat{\psi}_{R,L}^\dagger(k'-q) \hat{\psi}_{R,L}(k) \\
&= \delta(k' - q - k) - 2\hat{\psi}_{R,L}^\dagger(k'-q) \hat{\psi}_{R,L}(k),
\end{aligned} \tag{A.44}$$

$$\begin{aligned}
\left[\hat{\psi}_{R,L}^\dagger(k-p), \hat{\psi}_{R,L}^\dagger(k'-q) \right] &= \hat{\psi}_{R,L}^\dagger(k-p) \hat{\psi}_{R,L}^\dagger(k'-q) - \hat{\psi}_{R,L}^\dagger(k'-q) \hat{\psi}_{R,L}^\dagger(k-p) \\
&= -2\hat{\psi}_{R,L}^\dagger(k'-q) \hat{\psi}_{R,L}^\dagger(k-p),
\end{aligned} \tag{A.45}$$

$$\begin{aligned}
\left[\hat{\psi}_{R,L}(k), \hat{\psi}_{R,L}(k') \right] &= \hat{\psi}_{R,L}(k) \hat{\psi}_{R,L}(k') - \hat{\psi}_{R,L}(k') \hat{\psi}_{R,L}(k) \\
&= -2\hat{\psi}_{R,L}(k') \hat{\psi}_{R,L}(k),
\end{aligned} \tag{A.46}$$

$$\begin{aligned}
\left[\hat{\psi}_{R,L}^\dagger(k-p), \hat{\psi}_{R,L}(k') \right] &= \hat{\psi}_{R,L}^\dagger(k-p) \hat{\psi}_{R,L}(k') - \hat{\psi}_{R,L}(k') \hat{\psi}_{R,L}^\dagger(k-p) \\
&= 2\hat{\psi}_{R,L}^\dagger(k-p) \hat{\psi}_{R,L}(k') - \delta(k-p-k').
\end{aligned} \tag{A.47}$$

Replacing the latter commutators in the fermionic commutator of equation A.43, it becomes:

$$\begin{aligned}
&\left[\hat{\psi}_{R,L}^\dagger(k-p) \hat{\psi}_{R,L}(k), \hat{\psi}_{R,L}^\dagger(k'-q) \hat{\psi}_{R,L}(k') \right] = \delta(k' - q - k) \hat{\psi}_{R,L}^\dagger(k-p) \hat{\psi}_{R,L}(k') \\
&- 2\hat{\psi}_{R,L}^\dagger(k-p) \hat{\psi}_{R,L}^\dagger(k'-q) \hat{\psi}_{R,L}(k) \hat{\psi}_{R,L}(k') - 2\hat{\psi}_{R,L}^\dagger(k'-q) \hat{\psi}_{R,L}^\dagger(k-p) \hat{\psi}_{R,L}(k) \hat{\psi}_{R,L}(k') \\
&- 2\hat{\psi}_{R,L}^\dagger(k'-q) \hat{\psi}_{R,L}^\dagger(k-p) \hat{\psi}_{R,L}(k') \hat{\psi}_{R,L}(k) + 2\hat{\psi}_{R,L}^\dagger(k'-q) \hat{\psi}_{R,L}^\dagger(k-p) \hat{\psi}_{R,L}(k') \hat{\psi}_{R,L}(k) \\
&- \delta(k-p-k') \hat{\psi}_{R,L}^\dagger(k'-q) \hat{\psi}_{R,L}(k) \\
&= \delta(k' - q - k) \hat{\psi}_{R,L}^\dagger(k-p) \hat{\psi}_{R,L}(k') - \delta(k-p-k') \hat{\psi}_{R,L}^\dagger(k'-q) \hat{\psi}_{R,L}(k).
\end{aligned} \tag{A.48}$$

The second and third lines vanish according to the anticommutation relations of the fermions. Replacing the latter result in the equation A.42, the commutator takes the form:

$$\begin{aligned}
\left[\hat{J}_{R,L}(p), \hat{J}_{R,L}(q) \right] &= \frac{1}{2\pi} \int dk dk' \left[\delta(k' - q - k) \hat{\psi}_{R,L}^\dagger(k-p) \hat{\psi}_{R,L}(k') \right. \\
&\quad \left. - \delta(k-p-k') \hat{\psi}_{R,L}^\dagger(k'-q) \hat{\psi}_{R,L}(k) \right] \\
&= \frac{1}{2\pi} \int dk \left[\hat{\psi}_{R,L}^\dagger(k-p) \hat{\psi}_{R,L}(k+q) - \hat{\psi}_{R,L}^\dagger(k-p-q) \hat{\psi}_{R,L}(k) \right].
\end{aligned} \tag{A.49}$$

In the last step there must be very caution since it might appear that the commutator vanishes using a simple change of variable for the integration (in fact it is an indeterminacy of the sort $\infty - \infty$). One has to recall the normal order, where the normal ordered operators vanish their integrals safely. When replaced:

$$\begin{aligned}
[\hat{J}_{R,L}(p), \hat{J}_{R,L}(q)] &= \frac{1}{2\pi} \int dk \left[:\hat{\psi}_{R,L}^\dagger(k-p)\hat{\psi}_{R,L}(k+q): - :\hat{\psi}_{R,L}^\dagger(k-p-q)\hat{\psi}_{R,L}(k): \right] \\
&+ \frac{1}{2\pi} \int dk \left[\langle \hat{\psi}_{R,L}^\dagger(k-p)\hat{\psi}_{R,L}(k+q) \rangle - \langle \hat{\psi}_{R,L}^\dagger(k-p-q)\hat{\psi}_{R,L}(k) \rangle \right] \\
&= \frac{1}{2\pi} \int dk [Y(p-k)\delta(k-p-k-q) - Y(p+q-k)\delta(k-p-q-k)] \\
&= \frac{\delta(p+q)}{2\pi} \int dk [Y(p-k) - Y(p+q-k)] \\
&= \frac{\delta(p+q)}{2\pi} \int dk [Y(p-k) - Y(-k)] \\
&= \frac{\delta(p+q)}{2\pi} \int_0^p dk = \frac{p}{2\pi} \delta(p+q) \rightarrow \\
&[\hat{J}_{R,L}(p), \hat{J}_{R,L}(q)] = \frac{p}{2\pi} \delta(p+q). \tag{A.50}
\end{aligned}$$

The integral in the fourth line is reduced to the one in the fifth line using the delta function, and the last integral is solved using the interval where the difference $Y(p-k) - Y(-k)$ is not zero. The first choice is when $Y(p-k) = 1$ and $Y(-k) = 0$ which happen in $0 < k < p$ if $p > 0$. If $p < 0$ the latter integral vanish but the interval with $Y(p-k) = 0$ and $Y(-k) = 1$ gives the same result as with $p > 0$. The similarity with the commutation relation for bosonic operators is clear, and in fact, using equation A.41 the operator $\hat{J}_{R,L}(q)$ can be changed into $\hat{J}_{R,L}^\dagger(-q)$ and $-q$ to q , so equation A.50 satisfies:

$$[\hat{J}_{R,L}(p), \hat{J}_{R,L}^\dagger(q)] = \frac{p}{2\pi} \delta(p-q), \tag{A.51}$$

which is closer to the commutation relation for bosons ($[\hat{b}(p), \hat{b}^\dagger(q)] = \delta(p-q)$, if $\hat{b}^{(\dagger)}(p)$ are bosonic annihilation (creation) operators in variable p). The two differences are that they cannot be bosonic annihilation and creation operators since equation A.50 is not zero for any p and q . Also, equation A.51 has a factor $p/2\pi$, which can be removed if the operators are defined with a factor $\sqrt{2\pi/|p|}$. Thus, the bosonic operators are defined according to the previous normalization and the chirality of the current densities as follows [26–28, 42]:

$$\hat{b}(p) = -i\sqrt{\frac{2\pi}{|p|}} \left(\hat{J}_R(p)Y(p) - \hat{J}_L(-p)Y(-p) \right), \tag{A.52}$$

$$\hat{b}^\dagger(p) = i\sqrt{\frac{2\pi}{|p|}} \left(\hat{J}_R^\dagger(p)Y(p) - \hat{J}_L^\dagger(-p)Y(-p) \right). \tag{A.53}$$

The definition is made so the bosonic operators satisfy the corresponding commutations, this is, $[\hat{b}(p), \hat{b}^\dagger(p')] = \delta(p-p')$, $[\hat{b}(p), \hat{b}(p')] = 0$ and $[\hat{b}^\dagger(p), \hat{b}^\dagger(p')] = 0$. The operator $\hat{b}^\dagger(p)$ creates an excitation in the R branch if $p > 0$ and in the L branch if $p < 0$. The factors such as i

and the minus sign help to represent the branches as opposite hermitian processes in which it was shown the system separates. Using $Y(p) + Y(-p) = 1$ and the equation A.41, the chiral currents in momentum are given in bosonic operators as:

$$\hat{J}_R(p) = i\sqrt{\frac{|p|}{2\pi}} \left(\hat{b}(p)Y(p) - \hat{b}^\dagger(-p)Y(-p) \right), \quad (\text{A.54})$$

$$\hat{J}_L(p) = -i\sqrt{\frac{|p|}{2\pi}} \left(\hat{b}(-p)Y(p) - \hat{b}^\dagger(p)Y(-p) \right), \quad (\text{A.55})$$

which can be used to write the Hamiltonian in equation A.38 in terms of the bosonic operators. First, the quadratic Hamiltonian in currents has to be written in momentum space:

$$\begin{aligned} \hat{H} &= \pi\nu_F \int dx \left[\hat{J}_R^2(x) + \hat{J}_L^2(x) \right] \\ &= \pi\nu_F \int dx \left[\frac{1}{2\pi} \int dp e^{ipx} \hat{J}_R(p) \int dq e^{iqx} \hat{J}_R(q) + \frac{1}{2\pi} \int dp e^{ipx} \hat{J}_L(p) \int dq e^{iqx} \hat{J}_L(q) \right] \\ &= \pi\nu_F \int dp dq \left[\delta(p+q) \hat{J}_R(p) \hat{J}_R(q) + \delta(p+q) \hat{J}_L(p) \hat{J}_L(q) \right] \\ &= \pi\nu_F \int dp \left[\hat{J}_R(p) \hat{J}_R(-p) + \hat{J}_L(p) \hat{J}_L(-p) \right]. \end{aligned} \quad (\text{A.56})$$

Second, the equations A.54 and A.55 are replaced, so the Hamiltonian becomes:

$$\begin{aligned} \hat{H} &= \pi\nu_F \int dp \left[i\sqrt{\frac{|p|}{2\pi}} \left(\hat{b}(p)Y(p) - \hat{b}^\dagger(-p)Y(-p) \right) i\sqrt{\frac{|p|}{2\pi}} \left(\hat{b}(-p)Y(-p) - \hat{b}^\dagger(p)Y(p) \right) \right. \\ &\quad \left. + i\sqrt{\frac{|p|}{2\pi}} \left(\hat{b}(-p)Y(p) - \hat{b}^\dagger(p)Y(-p) \right) i\sqrt{\frac{|p|}{2\pi}} \left(\hat{b}(p)Y(-p) - \hat{b}^\dagger(-p)Y(p) \right) \right] \\ &= -\frac{\nu_F}{2} \int dp |p| \left[\hat{b}(p)\hat{b}(-p)Y(p)Y(-p) - \hat{b}(p)\hat{b}^\dagger(p)Y(p)^2 - \hat{b}^\dagger(-p)\hat{b}(-p)Y(-p)^2 \right. \\ &\quad \left. + \hat{b}^\dagger(-p)\hat{b}^\dagger(p)Y(-p)Y(p) + \hat{b}(-p)\hat{b}(p)Y(p)Y(-p) - \hat{b}(-p)\hat{b}^\dagger(-p)Y(p)^2 \right. \\ &\quad \left. - \hat{b}^\dagger(p)\hat{b}(p)Y(-p)^2 + \hat{b}^\dagger(p)\hat{b}^\dagger(-p)Y(-p)Y(p) \right] \\ &= \frac{\nu_F}{2} \int dp |p| \left[\hat{b}(p)\hat{b}^\dagger(p)Y(p) + \hat{b}(-p)\hat{b}^\dagger(-p)Y(p) \right. \\ &\quad \left. + \hat{b}^\dagger(-p)\hat{b}(-p)Y(-p) + \hat{b}^\dagger(p)\hat{b}(p)Y(-p) \right], \end{aligned} \quad (\text{A.57})$$

where $Y(p)Y(-p) = 0$ and $Y(\pm p)^2 = Y(\pm p)$ are use. Third and last, the normal order must be recalled, where $:\hat{b}(p)\hat{b}^\dagger(p): = \hat{b}^\dagger(p)\hat{b}(p)$ and $:\hat{b}^\dagger(p)\hat{b}(p): = \hat{b}^\dagger(p)\hat{b}(p)$, and also $-p \rightarrow p$ is set in

some of the integrals in the following way:

$$\begin{aligned}
:\hat{H}: &= \frac{\nu_F}{2} \int dp |p| : \hat{b}(p) \hat{b}^\dagger(p) : Y(p) + \frac{\nu_F}{2} \int dp |p| : \hat{b}(-p) \hat{b}^\dagger(-p) : Y(p) \\
&\quad + \frac{\nu_F}{2} \int dp |p| : \hat{b}^\dagger(-p) \hat{b}(-p) : Y(-p) + \frac{\nu_F}{2} \int dp |p| : \hat{b}^\dagger(p) \hat{b}(p) : Y(-p) \\
&= \frac{\nu_F}{2} \int dp |p| \hat{b}^\dagger(p) \hat{b}(p) Y(p) + \frac{\nu_F}{2} \int dp |p| \hat{b}^\dagger(p) \hat{b}(p) Y(-p) \\
&\quad + \frac{\nu_F}{2} \int dp |p| \hat{b}^\dagger(p) \hat{b}(p) Y(p) + \frac{\nu_F}{2} \int dp |p| \hat{b}^\dagger(p) \hat{b}(p) Y(-p) \\
&= \frac{\nu_F}{2} \int dp |p| \hat{b}^\dagger(p) \hat{b}(p) (Y(p) + Y(-p)) \\
&\quad + \frac{\nu_F}{2} \int dp |p| \hat{b}^\dagger(p) \hat{b}(p) (Y(p) + Y(-p)) \\
&= \frac{\nu_F}{2} \int dp |p| \hat{b}^\dagger(p) \hat{b}(p) + \frac{\nu_F}{2} \int dp |p| \hat{b}^\dagger(p) \hat{b}(p) \\
&= \int dp \nu_F |p| \hat{b}^\dagger(p) \hat{b}(p). \tag{A.58}
\end{aligned}$$

The system is reinterpreted in an infinite amount of harmonic oscillators with a linear dispersion relation, like for example an analogy with light could be made, where its linear dispersion relation is $\epsilon(p) = cp$ for photons [72, 73], while for these bosonic excitations the speed is the Fermi velocity ν_F . The latter leads to the full construction of the quantum field theory via the Klein-Gordon equation (wave equation in this case, because a mass term does not appear like it happens with light), so the canonical fields $\hat{\phi}(x, t)$ and $\pi(x, t)$, which satisfy $[\hat{\phi}(x), \hat{\pi}(y)]_{t=0} = i\delta(x - y)$, are expanded as [26, 42]:

$$\hat{\phi}(x, t) = \int \frac{dp}{\sqrt{4\pi|p|}} \left[e^{ipx} e^{-i|p|\nu_F t} \hat{b}(p) + e^{-ipx} e^{i|p|\nu_F t} \hat{b}^\dagger(p) \right], \tag{A.59}$$

$$\hat{\pi}(x, t) = -i \int dp \sqrt{\frac{|p|}{4\pi}} \left[e^{ipx} e^{-i|p|\nu_F t} \hat{b}(p) - e^{-ipx} e^{i|p|\nu_F t} \hat{b}^\dagger(p) \right]. \tag{A.60}$$

The canonical Hamiltonian for the latter fields is trivial for a free Fermi gas (harmonic oscillator), but in a more general system, like an interacting or coupled gas, there must be specific relations between the fermionic fields and the bosonic fields, where the more fundamental is the bosonization identity presented in equation 1.12. Its derivation is based on the commutator between the chiral currents and the fermionic and bosonic fields. To go in that direction, first one must relate the chiral currents with the bosonic fields (as the chiral currents are directly related with the fermionic densities). To do so, the derivative $\partial_x \hat{\phi}(x)$ is computed:

$$\begin{aligned}
\partial_x \hat{\phi}(x) &= \int \frac{dp}{\sqrt{4\pi|p|}} \left[ipe^{ipx} \hat{b}(p) - ipe^{-ipx} \hat{b}^\dagger(p) \right] = \int \frac{ipdp}{\sqrt{4\pi|p|}} \left[e^{ipx} \hat{b}(p) - e^{-ipx} \hat{b}^\dagger(p) \right] \\
&= i \int dp \operatorname{sgn}(p) \sqrt{\frac{|p|}{4\pi}} \left[e^{ipx} \hat{b}(p) - e^{-ipx} \hat{b}^\dagger(p) \right],
\end{aligned}$$

and integrating over position coordinate to complete a Fourier transform, one first inverse relation between the bosonic operators and the bosonic fields is:

$$\begin{aligned} \frac{1}{\sqrt{2\pi}} \int dx e^{-iqx} \partial_x \hat{\phi}(x) &= i \int dp \operatorname{sgn}(p) \sqrt{\frac{|p|}{2}} \left[\delta(p-q) \hat{b}(p) - \delta(p+q) \hat{b}^\dagger(p) \right] \\ &= i \sqrt{\frac{|q|}{2}} \operatorname{sgn}(q) \left[\hat{b}(q) + \hat{b}^\dagger(-q) \right], \end{aligned} \quad (\text{A.61})$$

$$\hat{b}(q) + \hat{b}^\dagger(-q) = -i \frac{\operatorname{sgn}(q)}{\sqrt{\pi|q|}} \int dx e^{-iqx} \partial_x \hat{\phi}(x), \quad (\text{A.62})$$

where the sign function is defined as $\operatorname{sgn}(p) = p/|p|$ if $p \neq 0$. The reciprocal inverse relation is derived with the field $\hat{\pi}(x)$ in the following way:

$$\begin{aligned} \frac{1}{\sqrt{2\pi}} \int dx e^{-iqx} \hat{\pi}(x) &= -i \int dp \sqrt{\frac{|p|}{2}} \left[\delta(p-q) \hat{b}(p) - \delta(p+q) \hat{b}^\dagger(p) \right] \\ &= -i \sqrt{\frac{|q|}{2}} \left[\hat{b}(q) - \hat{b}^\dagger(-q) \right], \end{aligned} \quad (\text{A.63})$$

$$\hat{b}(q) - \hat{b}^\dagger(-q) = \frac{i}{\sqrt{\pi|q|}} \int dx e^{-iqx} \hat{\pi}(x). \quad (\text{A.64})$$

When adding or subtracting equations A.62 and A.64, the corresponding bosonic operators in terms of the fields are given as:

$$\hat{b}(p) = -\frac{i}{\sqrt{4\pi|p|}} \int dx e^{-ipx} \left[\operatorname{sgn}(p) \partial_x \hat{\phi}(x) - \hat{\pi}(x) \right], \quad (\text{A.65})$$

$$\hat{b}^\dagger(p) = \frac{i}{\sqrt{4\pi|p|}} \int dx e^{ipx} \left[\operatorname{sgn}(p) \partial_x \hat{\phi}(x) - \hat{\pi}(x) \right]. \quad (\text{A.66})$$

Finally, equaling respective equations A.52 and A.65, and A.53 and A.66, the equation for bosonic fields in terms of the chiral currents in momentum is obtained, and later the Fourier transform is applied, so it is shown that:

$$\hat{J}_R(x) = \frac{1}{2\sqrt{\pi}} \left(\partial_x \hat{\phi}(x) - \hat{\pi}(x) \right), \quad (\text{A.67})$$

$$\hat{J}_L(x) = \frac{1}{2\sqrt{\pi}} \left(\partial_x \hat{\phi}(x) + \hat{\pi}(x) \right). \quad (\text{A.68})$$

These equations connect the real densities of the system with the bosonic fields (recall the definition of $\hat{J}(x)$ and $\hat{\rho}(x)$), where the relations are:

$$\hat{\rho}(x) = \frac{1}{\sqrt{\pi}} \partial_x \hat{\phi}(x), \quad \hat{J}(x) = -\frac{\nu_F}{\sqrt{\pi}} \hat{\pi}(x), \quad (\text{A.69})$$

which means that the variation of the $\hat{\phi}(x)$ field gives the charge density of the system while $\hat{\pi}(x)$ the corresponding current density. Second, on the developing of the bosonization identity, there are other fields that are important: the field $\hat{\theta}(x)$, which is defined to write a symmetry

Hamiltonian in equation 1.13 with $\hat{\pi}(x) = \partial_x \hat{\theta}(x)$, and the form which satisfies this equation is:

$$\hat{\theta}(x, t) = - \int dp \frac{\text{sgn}(p)}{\sqrt{4\pi|p|}} \left[e^{ipx} e^{-i|p|\nu_F t} \hat{b}(p) + e^{-ipx} e^{i|p|\nu_F t} \hat{b}^\dagger(p) \right]. \quad (\text{A.70})$$

Other fields are the chiral bosonic fields $\hat{\phi}_{R,L}(x)$, that can be defined by separating the integration interval of $\hat{\phi}(x)$, where the R field is for excitations of positive momentum and L for negative [25, 42]:

$$\hat{\phi}_R(x, t) = \int_0^\infty \frac{dp}{\sqrt{4\pi|p|}} \left[e^{ipx} e^{-i|p|\nu_F t} \hat{b}(p) + e^{-ipx} e^{i|p|\nu_F t} \hat{b}^\dagger(p) \right], \quad (\text{A.71})$$

$$\hat{\phi}_L(x, t) = \int_{-\infty}^0 \frac{dp}{\sqrt{4\pi|p|}} \left[e^{ipx} e^{-i|p|\nu_F t} \hat{b}(p) + e^{-ipx} e^{i|p|\nu_F t} \hat{b}^\dagger(p) \right], \quad (\text{A.72})$$

and thus the $\hat{\phi}(x)$ and $\hat{\theta}(x)$ fields separates as:

$$\hat{\phi}(x) = \hat{\phi}_R(x) + \hat{\phi}_L(x), \quad \hat{\theta}(x) = \hat{\phi}_L(x) - \hat{\phi}_R(x). \quad (\text{A.73})$$

With these latter relations it is more clear that the bosonic excitations are correctly separated via the $\hat{\phi}(x)$ and $\hat{\theta}(x)$ fields, which is not straightforward with the $\hat{\pi}(x)$ field although by construction it is the respective canonical field of the theory, so $\hat{\theta}(x)$ it is said and works as an auxiliar canonical field. With the latter, the relation between the chiral fermionic densities and the chiral bosonic fields is derived as:

$$\hat{\phi}_R(x) = \frac{1}{2} \left(\hat{\phi}(x) - \hat{\theta}(x) \right), \quad \hat{\phi}_L(x) = \frac{1}{2} \left(\hat{\phi}(x) + \hat{\theta}(x) \right), \quad (\text{A.74})$$

$$\hat{\psi}_R^\dagger(x) \hat{\psi}_R(x) = \frac{1}{\sqrt{\pi}} \partial_x \hat{\phi}_R(x), \quad \hat{\psi}_L^\dagger(x) \hat{\psi}_L(x) = \frac{1}{\sqrt{\pi}} \partial_x \hat{\phi}_L(x), \quad (\text{A.75})$$

where it must be recalled that the latter densities are in normal order. These last equations are as important as the bosonization identity to bosonize a Hamiltonian and correlations. Now, the last and third step is to determine the bosonization identity by calculating the commutation relations using the chiral currents, chiral fermionic fields and chiral bosonic fields, all previously defined and related.

A.3 The bosonization identity

To disclosed the possible relation, let us calculate the commutator between the corresponding chiral currents and chiral fermionic fields as follows:

$$\begin{aligned} \left[\hat{J}_{R,L}(x), \hat{\psi}_{R,L}(y) \right] &= \left[\hat{\psi}_{R,L}^\dagger(x) \hat{\psi}_{R,L}(x), \hat{\psi}_{R,L}(y) \right] \\ &= \hat{\psi}_{R,L}^\dagger(x) \left[\hat{\psi}_{R,L}(x), \hat{\psi}_{R,L}(y) \right] + \left[\hat{\psi}_{R,L}^\dagger(x), \hat{\psi}_{R,L}(y) \right] \hat{\psi}_{R,L}(x) \\ &= -2\hat{\psi}_{R,L}^\dagger(x) \hat{\psi}_{R,L}(y) \hat{\psi}_{R,L}(x) + \left(2\hat{\psi}_{R,L}^\dagger(x) \hat{\psi}_{R,L}(y) - \delta(x-y) \right) \hat{\psi}_{R,L}(x) \\ &= -\delta(x-y) \hat{\psi}_{R,L}(x). \end{aligned} \quad (\text{A.76})$$

On the other hand, the commutator between the chiral current and chiral bosonic fields is:

$$\left[\hat{J}_{R,L}(x), \hat{\phi}_{R,L}(y) \right] = \left[\frac{1}{\sqrt{\pi}} \partial_x \hat{\phi}_{R,L}(x), \hat{\phi}_{R,L}(y) \right] = \frac{1}{\sqrt{\pi}} \partial_x \left[\hat{\phi}_{R,L}(x), \hat{\phi}_{R,L}(y) \right], \quad (\text{A.77})$$

where the commutator of the chiral bosonic fields is computed as follows:

$$\begin{aligned} \left[\hat{\phi}_R(x), \hat{\phi}_R(y) \right] &= \frac{1}{4\pi} \int_0^\infty \frac{dpdq}{\sqrt{|pq|}} \left[e^{ipx} \hat{b}(p) + e^{-ipx} \hat{b}^\dagger(p), e^{iqy} \hat{b}(q) + e^{-iqy} \hat{b}^\dagger(q) \right] \\ &= \frac{1}{4\pi} \int_0^\infty \frac{dpdq}{\sqrt{|pq|}} \left(e^{ipx} e^{-iqy} \delta(p-q) - e^{-ipx} e^{iqy} \delta(p-q) \right) \\ &= \frac{1}{4\pi} \int_0^\infty \frac{dp}{p} \left(e^{ip(x-y)} - e^{-ip(x-y)} \right) \\ &= \frac{i}{2\pi} \int_0^\infty \frac{dp}{p} \sin[p(x-y)] = \frac{i}{4\pi} \int_{-\infty}^\infty \frac{dp}{p} \sin[p(x-y)] \\ &= \frac{1}{8\pi} \int_{-\infty}^\infty \frac{dp}{p} e^{ip(x-y)} - \frac{1}{8\pi} \int_{-\infty}^\infty \frac{dp}{p} e^{-ip(x-y)} \\ &= \frac{1}{4\pi} \int_{-\infty}^\infty \frac{dp}{p} e^{ip(x-y)} = \frac{i}{4} \left(\frac{1}{2\pi} \int_{-\infty}^\infty dp \left(\frac{2}{ip} \right) e^{ip(x-y)} \right) \\ &= \frac{i}{4} \text{sgn}(x-y), \end{aligned} \quad (\text{A.78})$$

$$\left[\hat{\phi}_L(x), \hat{\phi}_L(y) \right] = -\frac{i}{4} \text{sgn}(x-y), \quad (\text{A.79})$$

where the commutator for L fields is analogous to the R fields but in the same steps a minus sign appears [74]. Replacing the commutator of chiral bosonic fields with the initial commutator, then:

$$\left[\hat{J}_R(x), \hat{\phi}_R(y) \right] = \frac{1}{\sqrt{\pi}} \partial_x \left(\frac{i}{4} \text{sgn}(x-y) \right) = \frac{i}{4\sqrt{\pi}} \partial_x \text{sgn}(x-y) = \frac{i}{2\sqrt{\pi}} \delta(x-y), \quad (\text{A.80})$$

$$\left[\hat{J}_L(x), \hat{\phi}_L(y) \right] = \frac{1}{\sqrt{\pi}} \partial_x \left(-\frac{i}{4} \text{sgn}(x-y) \right) = -\frac{i}{4\sqrt{\pi}} \partial_x \text{sgn}(x-y) = -\frac{i}{2\sqrt{\pi}} \delta(x-y) \quad (\text{A.81})$$

with $\partial_x \text{sgn}(x-y) = 2\delta(x-y)$. It can be seen that the commutators of equations A.76, A.80 and A.81 are similar except for one operator in equation A.76. This gives the idea of using the following identity:

$$\left[\hat{A}, e^{\hat{B}} \right] = \left[\hat{A}, \hat{B} \right] e^{\hat{B}}, \quad (\text{A.82})$$

for $\hat{A}$ and $\hat{B}$ operators that satisfy $[[\hat{A}, \hat{B}], \hat{B}] = 0$, like operators in which their commutator is a scalar, just as the previous two. The identity can be proved using a Taylor expansion for the exponential function, taking out the sum from the commutator and showing by induction that $[\hat{A}, \hat{B}^n] = [\hat{A}, \hat{B}] n \hat{B}^{n-1}$, and the Taylor expansion of the exponential is reconstructed. This identity is useful because if one uses an exponential function of the chiral bosonic fields, then

according to that identity and equations A.80 and A.81, the commutator will be identical to the one in equation A.76, this is:

$$\begin{aligned} \left[\hat{J}_R(x), e^{i2\sqrt{\pi}\hat{\phi}_R(y)} \right] &= i2\sqrt{\pi} \left[\hat{J}_R(x), \hat{\phi}_R(y) \right] e^{i2\sqrt{\pi}\hat{\phi}_R(y)} = i2\sqrt{\pi} \left(\frac{i}{2\sqrt{\pi}} \delta(x-y) \right) e^{i2\sqrt{\pi}\hat{\phi}_R(y)} \\ &= -\delta(x-y) e^{i2\sqrt{\pi}\hat{\phi}_R(y)}, \end{aligned} \quad (\text{A.83})$$

$$\left[\hat{J}_L(x), e^{-i2\sqrt{\pi}\hat{\phi}_L(y)} \right] = -\delta(x-y) e^{-i2\sqrt{\pi}\hat{\phi}_L(y)}. \quad (\text{A.84})$$

Given that the commutators are the same as in equation A.76, it is proposed that [25, 35, 42]:

$$\hat{\psi}_R(x) = C\hat{\eta}_R e^{i2\sqrt{\pi}\hat{\phi}_R(x)}, \quad \hat{\psi}_L(x) = C\hat{\eta}_L e^{-i2\sqrt{\pi}\hat{\phi}_L(x)}, \quad (\text{A.85})$$

in which C is a constant that has to be determined to put these definitions in accordance with the rest of the developed theory so far, and $\hat{\eta}_{R,L}$ are the Klein factors. The Klein factors have to be set in the definition because the exponential operator preserves the number of particles, it is not directly an annihilation or creation operator, which also has to recover the statistics of the fermionic fields. Finally, C could be found calculating the same correlation in equations A.29 and A.31 but using the latter identities. It can be shown that the constant is equal for both chiralities (as stated), so the calculation is done only for R fermions as follows:

$$\langle \hat{\psi}_R^\dagger(x) \hat{\psi}_R(y) \rangle = \langle C\hat{\eta}_R^\dagger e^{-i2\sqrt{\pi}\hat{\phi}_R(x)} C\hat{\eta}_R e^{i2\sqrt{\pi}\hat{\phi}_R(y)} \rangle = C^2 \langle e^{-i2\sqrt{\pi}\hat{\phi}_R(x)} e^{i2\sqrt{\pi}\hat{\phi}_R(y)} \rangle, \quad (\text{A.86})$$

where the Klein factors disappear as the factor $\hat{\eta}_R^\dagger \hat{\eta}_R$ does not change the number of particles. The product of exponentials is treated using the Baker–Campbell–Hausdorff formula, this is:

$$e^{\hat{A}} e^{\hat{B}} = e^{\hat{A}+\hat{B}} e^{\frac{1}{2}[\hat{A},\hat{B}]}, \quad (\text{A.87})$$

that is valid when the operators $\hat{A}$ and $\hat{B}$ have a scalar commutator, such as the case of the $\hat{\phi}_{R,L}(x)$ fields. When applied to the previous correlation, it holds:

$$\begin{aligned} \langle \hat{\psi}_R^\dagger(x) \hat{\psi}_R(y) \rangle &= C^2 \langle e^{-i2\sqrt{\pi}(\hat{\phi}_R(x)-\hat{\phi}_R(y))} \rangle e^{\frac{1}{2}[-i2\sqrt{\pi}\hat{\phi}_R(x), i2\sqrt{\pi}\hat{\phi}_R(y)]} \\ &= C^2 e^{i\frac{\pi}{2}\text{sgn}(x-y)} \langle e^{-i2\sqrt{\pi}(\hat{\phi}_R(x)-\hat{\phi}_R(y))} \rangle. \end{aligned} \quad (\text{A.88})$$

The exponential of the sign function can be solved if $x \neq y$, and it satisfies $e^{i\frac{\pi}{2}\text{sgn}(x-y)} = i\text{sgn}(x-y)$, so the expected solution for the correlation is the one in equation A.29. On the other hand, the expected value of the exponential operator is one of the terms that appear in the encouraged correlations of appendix B and the analysis of chapters 2 and 3. Hence, here the solution is merely presented but the calculation is given explicitly in the next appendix. It is:

$$\langle e^{-i2\sqrt{\pi}(\hat{\phi}_R(x)-\hat{\phi}_R(y))} \rangle = \frac{\delta}{|x-y|}, \quad x \neq y. \quad (\text{A.89})$$

Replacing both solved exponentials into the correlation for the fermionic operator, the final result is:

$$\begin{aligned}\langle \hat{\psi}_R^\dagger(x) \hat{\psi}_R(y) \rangle &= iC^2 \text{sgn}(x-y) \frac{\delta}{|x-y|} = iC^2 \frac{|x-y|}{x-y} \frac{\delta}{|x-y|} \\ &= C^2 \frac{i\delta}{x-y}.\end{aligned}\tag{A.90}$$

Then, from direct comparison with the equation A.29, it can be concluded that:

$$C^2 = \frac{1}{2\pi\delta} \rightarrow C = \frac{1}{\sqrt{2\pi\delta}}.\tag{A.91}$$

With that said, the bosonization identity is given as it was presented in chapter 1 in equation 1.12 for both chiralities. These are the fundamental identities used in the document with their respective argument, which is almost the most important result of the appendix. This part must be clear to understand the rest and agree with other works based in bosonization as [20, 25, 34, 35, 42, 45, 55, 66] that could use different notations, but the physics must be the same. As a final quote, the fact that in equation A.86 the Klein factors disappear is the argument to simply not consider them explicitly in the fundamental identity. This is the reason why in chapter 2 and the rest of the document they are not written explicitly.

At this point it could be confusing the fact that the limit $\delta \rightarrow 0^+$ is not considered in the bosonization identity, and the argument behind leaving the cutoff as a finite quantity comes from the fact that the interactions considered are on-site interactions, which means they happen at one point. This is not exactly physical since realistic interactions among particles are defined via a *scattering length* ϵ , which can be introduced and it is shown that it replaces the cutoff δ in the equations, being finite and real [25]. The limit $\epsilon \rightarrow 0$ can be recovered for the continuum, but on the basis ϵ is finite, not as δ in general which is introduced by hand. Hence, the cutoff is preserved as it represents a physical quantity, so its introduction is an ansatz made according to that knowledge.

The next section shows an example of how to proceed with the bosonization of a quartic term, which is relevant for the approximation of interactions at low energies in one dimension.

A.4 Bosonization of a quartic term

After the development of the identities, an example of one of the utilities of the bosonization is to reduce complex terms into simpler ones. To show it, consider a Fermi gas with spin 1/2 in which particles interact with intensity U when two opposite spins approach the same position, where the Hamiltonian term is [54]:

$$\hat{V} = U \int dx \hat{\psi}_\uparrow^\dagger(x) \hat{\psi}_\uparrow(x) \hat{\psi}_\downarrow^\dagger(x) \hat{\psi}_\downarrow(x).\tag{A.92}$$

This term can be bosonized if the pair of densities are bosonized individually and then they are multiplied. But before, it must be noticed that the system is in fact two copies of the spinless Fermi gas, so the bosonization identities will have two copies as well, one for spin $\uparrow$ and the other for spin $\downarrow$. This means that:

$$\hat{\psi}_\sigma(x) = e^{-ik_F\sigma x} \hat{\psi}_{L\sigma}(x) + e^{ik_F\sigma x} \hat{\psi}_{R\sigma}(x),\tag{A.93}$$

$$\hat{\psi}_{R\sigma}(x) = \frac{1}{\sqrt{2\pi\delta}} e^{i2\sqrt{\pi}\hat{\phi}_{R\sigma}(x)}, \quad \hat{\psi}_{L\sigma}(x) = \frac{1}{\sqrt{2\pi\delta}} e^{-i2\sqrt{\pi}\hat{\phi}_{L\sigma}(x)}, \quad (\text{A.94})$$

for $\sigma = \uparrow, \downarrow$, and where for a general case, the Fermi momenta are considered different for each spin species (that happens for example when an external magnetic field is applied to the system). Now, the densities are bosonized as follows:

$$\begin{aligned} \hat{\psi}_{\sigma}^{\dagger}(x) \hat{\psi}_{\sigma}(x) &= \left[e^{k_{F\sigma}x} \hat{\psi}_{L\sigma}^{\dagger}(x) + e^{-k_{F\sigma}x} \hat{\psi}_{R\sigma}^{\dagger}(x) \right] \left[e^{-k_{F\sigma}x} \hat{\psi}_{L\sigma}(x) + e^{k_{F\sigma}x} \hat{\psi}_{R\sigma}(x) \right] \\ &= \hat{\psi}_{L\sigma}^{\dagger}(x) \hat{\psi}_{L\sigma}(x) + \hat{\psi}_{R\sigma}^{\dagger}(x) \hat{\psi}_{R\sigma}(x) + e^{i2k_{F\sigma}} \hat{\psi}_{L\sigma}^{\dagger}(x) \hat{\psi}_{R\sigma}(x) + e^{-i2k_{F\sigma}} \hat{\psi}_{R\sigma}^{\dagger}(x) \hat{\psi}_{L\sigma}(x) \\ &= \frac{1}{\sqrt{\pi}} \partial_x \hat{\phi}_{\sigma}(x) + \frac{e^{i2k_{F\sigma}}}{2\pi\delta} e^{i2\sqrt{\pi}\hat{\phi}_{L\sigma}(x)} e^{i2\sqrt{\pi}\hat{\phi}_{R\sigma}(x)} + \frac{e^{-i2k_{F\sigma}}}{2\pi\delta} e^{-i2\sqrt{\pi}\hat{\phi}_{R\sigma}(x)} e^{-i2\sqrt{\pi}\hat{\phi}_{L\sigma}(x)} \\ &= \frac{1}{\sqrt{\pi}} \partial_x \hat{\phi}_{\sigma}(x) + \frac{e^{i2k_{F\sigma}}}{2\pi\delta} e^{i2\sqrt{\pi}\hat{\phi}_{\sigma}(x)} + \frac{e^{-i2k_{F\sigma}}}{2\pi\delta} e^{-i2\sqrt{\pi}\hat{\phi}_{\sigma}(x)} \\ &= \frac{1}{\sqrt{\pi}} \partial_x \hat{\phi}_{\sigma}(x) + \frac{1}{\pi\delta} \cos \left[\sqrt{4\pi}\hat{\phi}_{\sigma}(x) + 2k_{F\sigma}x \right], \end{aligned} \quad (\text{A.95})$$

where the first cosine function appears due to the products of operators with different chirality (which represent a jump from R branch to L branch and vice versa (of same spin), called *backscattering* process). On the other hand, the density term is precisely the collective diagonal excitations (derivative of the bosonic field), called *forward scattering* process. Next, the product of the densities can be done in the following way:

$$\begin{aligned} &\hat{\psi}_{\uparrow}^{\dagger}(x) \hat{\psi}_{\uparrow}(x) \hat{\psi}_{\downarrow}^{\dagger}(x) \hat{\psi}_{\downarrow}(x) \\ &= \left(\frac{1}{\sqrt{\pi}} \partial_x \hat{\phi}_{\uparrow}(x) + \frac{1}{\pi\delta} \cos \left[\sqrt{4\pi}\hat{\phi}_{\uparrow}(x) + 2k_{F\uparrow}x \right] \right) \\ &\quad \left(\frac{1}{\sqrt{\pi}} \partial_x \hat{\phi}_{\downarrow}(x) + \frac{1}{\pi\delta} \cos \left[\sqrt{4\pi}\hat{\phi}_{\downarrow}(x) + 2k_{F\downarrow}x \right] \right) \\ &= \frac{1}{\pi} \partial_x \hat{\phi}_{\uparrow}(x) \partial_x \hat{\phi}_{\downarrow}(x) + \frac{1}{\pi\delta\sqrt{\pi}} \partial_x \hat{\phi}_{\uparrow}(x) \cos \left[\sqrt{4\pi}\hat{\phi}_{\downarrow}(x) + 2k_{F\downarrow}x \right] \\ &\quad + \frac{1}{\pi\delta\sqrt{\pi}} \cos \left[\sqrt{4\pi}\hat{\phi}_{\uparrow}(x) + 2k_{F\uparrow}x \right] \partial_x \hat{\phi}_{\downarrow}(x) \\ &\quad + \frac{1}{(\pi\delta)^2} \cos \left[\sqrt{4\pi}\hat{\phi}_{\uparrow}(x) + 2k_{F\uparrow}x \right] \cos \left[\sqrt{4\pi}\hat{\phi}_{\downarrow}(x) + 2k_{F\downarrow}x \right]. \end{aligned} \quad (\text{A.96})$$

Replacing the product of fermionic densities and applying the identity $2 \cos(\alpha) \cos(\beta) = \cos(\alpha + \beta) + \cos(\alpha - \beta)$, the resultant interaction Hamiltonian term is:

$$\begin{aligned} \hat{V} &= \frac{U}{\pi} \int dx \partial_x \hat{\phi}_{\uparrow}(x) \partial_x \hat{\phi}_{\downarrow}(x) + \frac{U}{\pi\delta\sqrt{\pi}} \int dx \partial_x \hat{\phi}_{\uparrow}(x) \cos \left[\sqrt{4\pi}\hat{\phi}_{\downarrow}(x) + 2k_{F\downarrow}x \right] \\ &\quad + \frac{U}{\pi\delta\sqrt{\pi}} \int dx \cos \left[\sqrt{4\pi}\hat{\phi}_{\uparrow}(x) + 2k_{F\uparrow}x \right] \partial_x \hat{\phi}_{\downarrow}(x) \\ &\quad + \frac{2U}{(2\pi\delta)^2} \int dx \cos \left[\sqrt{4\pi} \left(\hat{\phi}_{\uparrow}(x) + \hat{\phi}_{\downarrow}(x) \right) + 2(k_{F\uparrow} + k_{F\downarrow})x \right] \\ &\quad + \frac{2U}{(2\pi\delta)^2} \int dx \cos \left[\sqrt{4\pi} \left(\hat{\phi}_{\uparrow}(x) - \hat{\phi}_{\downarrow}(x) \right) + 2(k_{F\uparrow} - k_{F\downarrow})x \right]. \end{aligned} \quad (\text{A.97})$$

In chapter 1 is said that the phases that had the Fermi momenta were rapidly oscillating in contrast to chiral fermionic operators. As the chiral fermionic operators relate directly with the bosonic fields via the bosonization identity, the bosonic fields are slowly varying in contrast with the phases that appear in the cosine ($2k_{F\sigma}$ or $k_{F\uparrow} \pm k_{F\downarrow}$) [25]. As the phases make the cosines oscillate rapidly, the integral vanishes safely according to the normal order [25–28]. The only way those integrals not vanish in the continuum is if the phase is zero (in contrast with proportional values of π when the periodicity of a lattice exist, for example). If one looks at the terms in the middle, they are always oscillating since the Fermi momenta cannot be zero, and also $k_{F\uparrow} + k_{F\downarrow}$ could not be zero as $k_{F\sigma} > 0$. This is the reason why those terms are never considered in the bosonization of the Hamiltonian, and for the gas in general. Hence, the interaction term reduces to [25]:

$$\hat{V} \approx \frac{U}{\pi} \int dx \partial_x \hat{\phi}_{\uparrow}(x) \partial_x \hat{\phi}_{\downarrow}(x) + \frac{2U}{(2\pi\delta)^2} \int dx \cos \left[\sqrt{4\pi} \left(\hat{\phi}_{\uparrow}(x) - \hat{\phi}_{\downarrow}(x) \right) + 2(k_{F\uparrow} - k_{F\downarrow})x \right] \quad (\text{A.98})$$

Also, if definitely $k_{F\uparrow} \neq k_{F\downarrow}$, then the other cosine will not contribute as well, and simply the interaction term is quadratic. The most interesting case is when the cosine stays, which happens if $k_{F\uparrow} = k_{F\downarrow} = k_F$, so the term is:

$$\hat{V} \approx \frac{U}{\pi} \int dx \partial_x \hat{\phi}_{\uparrow}(x) \partial_x \hat{\phi}_{\downarrow}(x) + \frac{2U}{(2\pi\delta)^2} \int dx \cos \left[\sqrt{4\pi} \left(\hat{\phi}_{\uparrow}(x) - \hat{\phi}_{\downarrow}(x) \right) \right], \quad (\text{A.99})$$

in where the analysis of the one-dimensional superconducting Fermi gas is based [25], so it is said the cosine term is the BCS channel of the interaction term, while the quadratic product of densities is the forward scattering channel [45]. The method has to deal with these trigonometric functions which change the system from a Klein-Gordon one to a *sine-Gordon* [25, 34, 35, 45]. From the point of view of the bosonization, it must be clear that the only thing that can be done is to analyze when those cosines could contribute or not. In fact, this work presents one approach to it, as many representatives of the theory could have other approaches in that sense, like [20, 21, 25, 34, 35, 42, 55, 66, 75] and certainly many others.

From the point of view of the correlations, in the following appendix they can be easily calculated for quadratic terms such as the ones presented in this appendix, but for the cosine functions it is impossible to get an exact solution [25]. Instead, there are two ways of dealing with them, that are presented in the next appendix and used in chapter 4, while in general all the useful correlations and susceptibilities in this work are generally presented and calculated by means of the bosonization already done in this appendix.

Appendix B

Generalities on correlations and susceptibilities

In this appendix the full derivation of the correlations and susceptibilities is presented, where the procedure is valid for any other bosonized Hamiltonian like the main one in chapter 2. We keep doing the analysis for the spin 1/2 Fermi gas presented in the previous appendix, where the expanded fermionic operators are given in equation A.93 and the bosonization identities in equations A.94. Adding the kinetic energy, the Hamiltonian is:

$$\hat{H} = \int dx \left[\hat{\psi}_{\uparrow}^{\dagger}(x) \left(-\frac{\partial_x^2}{2m} \right) \hat{\psi}_{\uparrow}(x) + \hat{\psi}_{\downarrow}^{\dagger}(x) \left(-\frac{\partial_x^2}{2m} \right) \hat{\psi}_{\downarrow}(x) - U \hat{\psi}_{\uparrow}^{\dagger}(x) \hat{\psi}_{\uparrow}(x) \hat{\psi}_{\downarrow}^{\dagger}(x) \hat{\psi}_{\downarrow}(x) \right]. \quad (\text{B.1})$$

The interaction term has changed into $-U$ thinking in an attractive interaction if $U > 0$ and a repulsive one if $U < 0$. According to equations 1.13 and A.99, the bosonized Hamiltonian is:

$$\begin{aligned} \hat{H} &= \frac{\nu_F}{2} \int dx \left[\left(\partial_x \hat{\phi}_{\uparrow}(x) \right)^2 + \hat{\pi}_{\uparrow}(x)^2 \right] + \frac{\nu_F}{2} \int dx \left[\left(\partial_x \hat{\phi}_{\downarrow}(x) \right)^2 + \hat{\pi}_{\downarrow}(x)^2 \right] \\ &- \frac{U}{\pi} \int dx \partial_x \hat{\phi}_{\uparrow}(x) \partial_x \hat{\phi}_{\downarrow}(x) - \frac{2U}{(2\pi\delta)^2} \int dx \cos \left[\sqrt{4\pi} \left(\hat{\phi}_{\uparrow}(x) - \hat{\phi}_{\downarrow}(x) \right) \right], \end{aligned} \quad (\text{B.2})$$

where $\nu_F = k_F/m$. From the form of the Hamiltonian, it can be seen that the basis $\uparrow\downarrow$ is not suitable as the fields remain coupled in the interaction term. Hence, it is said that the spin 1/2 Hamiltonian separates into charge (c) and spin (s) excitations [25] according to the next transformation:

$$\hat{\phi}_c(x) = \frac{1}{\sqrt{2}} \left(\hat{\phi}_{\uparrow}(x) + \hat{\phi}_{\downarrow}(x) \right), \quad \hat{\phi}_s(x) = \frac{1}{\sqrt{2}} \left(\hat{\phi}_{\uparrow}(x) - \hat{\phi}_{\downarrow}(x) \right), \quad (\text{B.3})$$

$$\hat{\phi}_{\uparrow}(x) = \frac{1}{\sqrt{2}} \left(\hat{\phi}_c(x) + \hat{\phi}_s(x) \right), \quad \hat{\phi}_{\downarrow}(x) = \frac{1}{\sqrt{2}} \left(\hat{\phi}_c(x) - \hat{\phi}_s(x) \right), \quad (\text{B.4})$$

and exactly the same for the $\hat{\pi}_{\sigma}(x)$ fields. When the latter transformation is applied to the

Hamiltonian, it takes the form:

$$\begin{aligned}
\hat{H} &= \frac{\nu_F}{2} \int dx \left[\left(\partial_x \hat{\phi}_c(x) \right)^2 + \hat{\pi}_c(x)^2 \right] + \frac{\nu_F}{2} \int dx \left[\left(\partial_x \hat{\phi}_s(x) \right)^2 + \hat{\pi}_s(x)^2 \right] \\
&- \frac{U}{2\pi} \int dx \left[\left(\partial_x \hat{\phi}_c(x) \right)^2 - \left(\partial_x \hat{\phi}_s(x) \right)^2 \right] - \frac{2U}{(2\pi\delta)^2} \int dx \cos \left[\sqrt{8\pi} \hat{\phi}_s(x) \right] \\
&= \int dx \left[\left(\frac{\nu_F}{2} - \frac{U}{2\pi} \right) \left(\partial_x \hat{\phi}_c(x) \right)^2 + \frac{\nu_F}{2} \hat{\pi}_c(x)^2 \right] \\
&+ \int dx \left[\left(\frac{\nu_F}{2} + \frac{U}{2\pi} \right) \left(\partial_x \hat{\phi}_s(x) \right)^2 + \frac{\nu_F}{2} \hat{\pi}_s(x)^2 \right] - \frac{2U}{(2\pi\delta)^2} \int dx \cos \left[\sqrt{8\pi} \hat{\phi}_s(x) \right].
\end{aligned} \tag{B.5}$$

Here the Hamiltonian is simplified by defining the *Luttinger parameters*, which are set when terms, such as the interaction term, alter the parameters that multiply the fields, so the diagonal form of the Hamiltonian is recovered by renormalizing the fields with those parameters [25]. The equations to define them are:

$$\frac{\nu_c}{2K_c} = \frac{\nu_F}{2} - \frac{U}{2\pi}, \quad \frac{\nu_c K_c}{2} = \frac{\nu_F}{2}, \tag{B.6}$$

$$\frac{\nu_s}{2K_s} = \frac{\nu_F}{2} + \frac{U}{2\pi}, \quad \frac{\nu_s K_s}{2} = \frac{\nu_F}{2}, \tag{B.7}$$

with $K_{s,c}$ the Luttinger parameters and $\nu_{c,s}$ the new velocities. The solutions are:

$$K_c = \frac{1}{\sqrt{1 - \frac{mU}{\pi k_F}}}, \quad K_s = \frac{1}{\sqrt{1 + \frac{mU}{\pi k_F}}}, \tag{B.8}$$

$$\nu_c = \frac{k_F}{m} \sqrt{1 - \frac{mU}{\pi k_F}}, \quad \nu_s = \frac{k_F}{m} \sqrt{1 + \frac{mU}{\pi k_F}}. \tag{B.9}$$

It must be noticed how U determines how much the Luttinger parameters alter the renormalization of the fields. If $U = 0$ the parameters are equal to one, so there is no renormalization, but if $U \neq 0$ there is a net effect of the interaction term in the normalization of the fields, which also alters the physical quantity they represent, such as the charge or spin of the system. Finally, recalling the fields $\hat{\theta}_{c,s}(x)$, the final bosonized Hamiltonian is:

$$\begin{aligned}
\hat{H} &= \frac{\nu_c}{2} \int dx \left[\frac{1}{K_c} \left(\partial_x \hat{\phi}_c(x) \right)^2 + K_c \left(\partial_x \hat{\theta}_c(x) \right)^2 \right] \\
&+ \frac{\nu_s}{2} \int dx \left[\frac{1}{K_s} \left(\partial_x \hat{\phi}_s(x) \right)^2 + K_s \left(\partial_x \hat{\theta}_s(x) \right)^2 \right] - \frac{2U}{(2\pi\delta)^2} \int dx \cos \left[\sqrt{8\pi} \hat{\phi}_s(x) \right].
\end{aligned} \tag{B.10}$$

The full diagonal form is reached with the explicit renormalization $\hat{\hat{\phi}}_{c,s}(x) = \hat{\phi}_c(x)/\sqrt{K_{c,s}}$ and $\hat{\hat{\theta}}_{c,s}(x) = \hat{\theta}_{c,s}(x)\sqrt{K_{c,s}}$. Although the excitations separate in charge and spin, only the charge part is described by the massless Klein-Gordon equation, so the $\hat{\hat{\phi}}_c(x)$ and $\hat{\hat{\theta}}_c(x)$ fields are expanded according to equations A.59 and A.70 respectively, with ν_c the velocity. The

spin part could not be in that way due to the cosine term, so the expansion of their fields is unknown and impossible to solve exactly. Instead, in the strong interaction limit applied later, the cosine is expanded around a minimum so the spin fields satisfy a massive Klein-Gordon equation. The next procedure consists in consider the mass term as a correction, and thus the same massless expansion is assumed for ν_s the velocity in case of the spin, the correlations are calculated in a standard form, and in the end the correlations are corrected by the massive part.

According to the discussion in the introduction, the eight operators of the correlations (the charge density $\hat{\rho}(x)$, the axial spin density $\hat{\sigma}^z(x)$, the planar spin densities $\hat{\sigma}^x(x)$ and $\hat{\sigma}^y(x)$, the spin-singlet pairing $\hat{O}_{SS}(x)$, axial spin-triplet pairing $\hat{O}_{TS}^z(x)$, and the planar spin-triplet pairings $\hat{O}_{TS}^x(x)$ and $\hat{O}_{TS}^y(x)$), are defined as follows [25]:

$$\hat{\rho}(x) = \hat{\psi}_\uparrow^\dagger(x)\hat{\psi}_\uparrow(x) + \hat{\psi}_\downarrow^\dagger(x)\hat{\psi}_\downarrow(x), \quad (\text{B.11})$$

$$\hat{\sigma}^z(x) = \hat{\psi}_\uparrow^\dagger(x)\hat{\psi}_\uparrow(x) - \hat{\psi}_\downarrow^\dagger(x)\hat{\psi}_\downarrow(x), \quad (\text{B.12})$$

$$\hat{\sigma}^x(x) = \hat{\psi}_\uparrow^\dagger(x)\hat{\psi}_\downarrow(x) + \hat{\psi}_\downarrow^\dagger(x)\hat{\psi}_\uparrow(x), \quad (\text{B.13})$$

$$\hat{\sigma}^y(x) = i \left(\hat{\psi}_\downarrow^\dagger(x)\hat{\psi}_\uparrow(x) - \hat{\psi}_\uparrow^\dagger(x)\hat{\psi}_\downarrow(x) \right), \quad (\text{B.14})$$

$$\hat{O}_{SS}(x) = \hat{\psi}_{R\uparrow}^\dagger(x)\hat{\psi}_{L\downarrow}^\dagger(x) - \hat{\psi}_{L\uparrow}^\dagger(x)\hat{\psi}_{R\downarrow}^\dagger(x), \quad (\text{B.15})$$

$$\hat{O}_{TS}^z(x) = \hat{\psi}_{R\uparrow}^\dagger(x)\hat{\psi}_{L\downarrow}^\dagger(x) + \hat{\psi}_{L\uparrow}^\dagger(x)\hat{\psi}_{R\downarrow}^\dagger(x), \quad (\text{B.16})$$

$$\hat{O}_{TS}^x(x) = \hat{\psi}_{R\uparrow}^\dagger(x)\hat{\psi}_{L\uparrow}^\dagger(x) + \hat{\psi}_{L\downarrow}^\dagger(x)\hat{\psi}_{R\downarrow}^\dagger(x), \quad (\text{B.17})$$

$$\hat{O}_{TS}^y(x) = i \left(\hat{\psi}_{L\downarrow}^\dagger(x)\hat{\psi}_{R\downarrow}^\dagger(x) - \hat{\psi}_{R\uparrow}^\dagger(x)\hat{\psi}_{L\uparrow}^\dagger(x) \right). \quad (\text{B.18})$$

The pairings are given in terms of the chiral fields since the complete pairing operator of two opposite spins $\hat{O}(x) = \hat{\psi}_\uparrow^\dagger(x)\hat{\psi}_\downarrow^\dagger(x)$ is expanded, and it have products of opposite branches, like in equation B.15, and of same branches like RR or LL . In the case $k_{F\uparrow} = k_{F\downarrow} = k_F$ the opposite branches have opposite momentum and those pairings have net zero momentum, but the ones of the same branches have $\pm 2k_F$ momentum. That momentum appears as a phase in the operators, so those pairings are less relevant than the zero-momentum pairings because their correlations decay faster than the others in the presence of attractive interactions [25]. The following is to bosonize the latter operators in the same way the interaction was bosonized in equation A.96, so first the opposite-spin densities are given as:

$$\begin{aligned} \hat{\psi}_\uparrow^\dagger(x)\hat{\psi}_\downarrow(x) &= \left[e^{ik_F x} \hat{\psi}_{L\uparrow}^\dagger(x) + e^{-ik_F x} \hat{\psi}_{R\uparrow}^\dagger(x) \right] \left[e^{-ik_F x} \hat{\psi}_{L\downarrow}(x) + e^{ik_F x} \hat{\psi}_{R\downarrow}(x) \right] \\ &= \hat{\psi}_{L\uparrow}^\dagger(x)\hat{\psi}_{L\downarrow}(x) + \hat{\psi}_{R\uparrow}^\dagger(x)\hat{\psi}_{R\downarrow}(x) + e^{i2k_F x} \hat{\psi}_{L\uparrow}^\dagger(x)\hat{\psi}_{R\downarrow}(x) + e^{-i2k_F x} \hat{\psi}_{R\uparrow}^\dagger(x)\hat{\psi}_{L\downarrow}(x) \\ &= \frac{1}{2\pi\delta} e^{i\sqrt{\pi}(\hat{\phi}_\uparrow(x) + \hat{\theta}_\uparrow(x))} e^{-i\sqrt{\pi}(\hat{\phi}_\downarrow(x) + \hat{\theta}_\downarrow(x))} + \frac{1}{2\pi\delta} e^{-i\sqrt{\pi}(\hat{\phi}_\uparrow(x) - \hat{\theta}_\uparrow(x))} e^{i\sqrt{\pi}(\hat{\phi}_\downarrow(x) - \hat{\theta}_\downarrow(x))} \\ &\quad + \frac{e^{i2k_F x}}{2\pi\delta} e^{i\sqrt{\pi}(\hat{\phi}_\uparrow(x) + \hat{\theta}_\uparrow(x))} e^{i\sqrt{\pi}(\hat{\phi}_\downarrow(x) - \hat{\theta}_\downarrow(x))} + \frac{e^{-i2k_F x}}{2\pi\delta} e^{-i\sqrt{\pi}(\hat{\phi}_\uparrow(x) - \hat{\theta}_\uparrow(x))} e^{-i\sqrt{\pi}(\hat{\phi}_\downarrow(x) + \hat{\theta}_\downarrow(x))} \\ &= \frac{1}{\pi\delta} e^{i\sqrt{2\pi}\hat{\theta}_s(x)} \cos \left[\sqrt{2\pi}\hat{\phi}_s(x) \right] + \frac{1}{\pi\delta} e^{i\sqrt{2\pi}\hat{\theta}_s(x)} \cos \left[\sqrt{2\pi}\hat{\phi}_c(x) + 2k_F x \right], \end{aligned} \quad (\text{B.19})$$

$$\hat{\psi}_\downarrow^\dagger(x)\hat{\psi}_\uparrow(x) = \frac{1}{\pi\delta} e^{-i\sqrt{2\pi}\hat{\theta}_s(x)} \cos \left[\sqrt{2\pi}\hat{\phi}_s(x) \right] + \frac{1}{\pi\delta} e^{-i\sqrt{2\pi}\hat{\theta}_s(x)} \cos \left[\sqrt{2\pi}\hat{\phi}_c(x) + 2k_F x \right]. \quad (\text{B.20})$$

Using the bosonized densities in equation A.95 as well, the bosonization of the operators is shown in the following lines:

$$\begin{aligned}\hat{\rho}(x) &= \frac{1}{\sqrt{\pi}}\partial_x\hat{\phi}_{\uparrow}(x) + \frac{1}{\pi\delta}\cos\left[\sqrt{4\pi}\hat{\phi}_{\uparrow}(x) + 2k_Fx\right] \\ &+ \frac{1}{\sqrt{\pi}}\partial_x\hat{\phi}_{\downarrow}(x) + \frac{1}{\pi\delta}\cos\left[\sqrt{4\pi}\hat{\phi}_{\downarrow}(x) + 2k_Fx\right] \\ &= \sqrt{\frac{2}{\pi}}\partial_x\hat{\phi}_c(x) + \frac{2}{\pi\delta}\cos\left[\sqrt{2\pi}\hat{\phi}_c(x) + 2k_Fx\right]\cos\left[\sqrt{2\pi}\hat{\phi}_s(x)\right],\end{aligned}\quad (\text{B.21})$$

$$\begin{aligned}\hat{\sigma}^z(x) &= \frac{1}{\sqrt{\pi}}\partial_x\hat{\phi}_{\uparrow}(x) + \frac{1}{\pi\delta}\cos\left[\sqrt{4\pi}\hat{\phi}_{\uparrow}(x) + 2k_Fx\right] \\ &- \frac{1}{\sqrt{\pi}}\partial_x\hat{\phi}_{\downarrow}(x) - \frac{1}{\pi\delta}\cos\left[\sqrt{4\pi}\hat{\phi}_{\downarrow}(x) + 2k_Fx\right] \\ &= \sqrt{\frac{2}{\pi}}\partial_x\hat{\phi}_s(x) - \frac{2}{\pi\delta}\sin\left[\sqrt{2\pi}\hat{\phi}_c(x) + 2k_Fx\right]\sin\left[\sqrt{2\pi}\hat{\phi}_s(x)\right],\end{aligned}\quad (\text{B.22})$$

$$\begin{aligned}\hat{\sigma}^x(x) &= \frac{1}{\pi\delta}e^{i\sqrt{2\pi}\hat{\theta}_s(x)}\cos\left[\sqrt{2\pi}\hat{\phi}_s(x)\right] + \frac{1}{\pi\delta}e^{i\sqrt{2\pi}\hat{\theta}_s(x)}\cos\left[\sqrt{2\pi}\hat{\phi}_c(x) + 2k_Fx\right] \\ &+ \frac{1}{\pi\delta}e^{-i\sqrt{2\pi}\hat{\theta}_s(x)}\cos\left[\sqrt{2\pi}\hat{\phi}_s(x)\right] + \frac{1}{\pi\delta}e^{-i\sqrt{2\pi}\hat{\theta}_s(x)}\cos\left[\sqrt{2\pi}\hat{\phi}_c(x) + 2k_Fx\right] \\ &= \frac{2}{\pi\delta}\cos\left[\sqrt{2\pi}\hat{\theta}_s(x)\right]\cos\left[\sqrt{2\pi}\hat{\phi}_s(x)\right] + \frac{2}{\pi\delta}\cos\left[\sqrt{2\pi}\hat{\theta}_s(x)\right]\cos\left[\sqrt{2\pi}\hat{\phi}_c(x) + 2k_Fx\right],\end{aligned}\quad (\text{B.23})$$

$$\begin{aligned}\hat{\sigma}^y(x) &= \frac{i}{\pi\delta}e^{-i\sqrt{2\pi}\hat{\theta}_s(x)}\cos\left[\sqrt{2\pi}\hat{\phi}_s(x)\right] + \frac{i}{\pi\delta}e^{-i\sqrt{2\pi}\hat{\theta}_s(x)}\cos\left[\sqrt{2\pi}\hat{\phi}_c(x) + 2k_Fx\right] \\ &- \frac{i}{\pi\delta}e^{i\sqrt{2\pi}\hat{\theta}_s(x)}\cos\left[\sqrt{2\pi}\hat{\phi}_s(x)\right] - \frac{i}{\pi\delta}e^{i\sqrt{2\pi}\hat{\theta}_s(x)}\cos\left[\sqrt{2\pi}\hat{\phi}_c(x) + 2k_Fx\right] \\ &= \frac{2}{\pi\delta}\sin\left[\sqrt{2\pi}\hat{\theta}_s(x)\right]\cos\left[\sqrt{2\pi}\hat{\phi}_s(x)\right] + \frac{2}{\pi\delta}\sin\left[\sqrt{2\pi}\hat{\theta}_s(x)\right]\cos\left[\sqrt{2\pi}\hat{\phi}_c(x) + 2k_Fx\right],\end{aligned}\quad (\text{B.24})$$

$$\begin{aligned}\hat{O}_{SS}(x) &= \frac{1}{2\pi\delta}e^{-i\sqrt{\pi}(\hat{\phi}_{\uparrow}(x)-\hat{\theta}_{\uparrow}(x))}e^{i\sqrt{\pi}(\hat{\phi}_{\downarrow}(x)+\hat{\theta}_{\downarrow}(x))} + \frac{1}{2\pi\delta}e^{i\sqrt{\pi}(\hat{\phi}_{\uparrow}(x)+\hat{\theta}_{\uparrow}(x))}e^{-i\sqrt{\pi}(\hat{\phi}_{\downarrow}(x)-\hat{\theta}_{\downarrow}(x))} \\ &= \frac{1}{\pi\delta}e^{i\sqrt{2\pi}\hat{\theta}_c(x)}\cos\left[\sqrt{2\pi}\hat{\phi}_s(x)\right],\end{aligned}\quad (\text{B.25})$$

$$\begin{aligned}\hat{O}_{TS}^z(x) &= \frac{1}{2\pi\delta}e^{-i\sqrt{\pi}(\hat{\phi}_{\uparrow}(x)-\hat{\theta}_{\uparrow}(x))}e^{i\sqrt{\pi}(\hat{\phi}_{\downarrow}(x)+\hat{\theta}_{\downarrow}(x))} - \frac{1}{2\pi\delta}e^{i\sqrt{\pi}(\hat{\phi}_{\uparrow}(x)+\hat{\theta}_{\uparrow}(x))}e^{-i\sqrt{\pi}(\hat{\phi}_{\downarrow}(x)-\hat{\theta}_{\downarrow}(x))} \\ &= -\frac{i}{\pi\delta}e^{i\sqrt{2\pi}\hat{\theta}_c(x)}\sin\left[\sqrt{2\pi}\hat{\phi}_s(x)\right],\end{aligned}\quad (\text{B.26})$$

$$\begin{aligned}\hat{O}_{TS}^x(x) &= \frac{1}{2\pi\delta}e^{-2i\sqrt{\pi}\hat{\phi}_{R\uparrow}(x)}e^{i\sqrt{\pi}\hat{\phi}_{L\uparrow}(x)} + \frac{1}{2\pi\delta}e^{i\sqrt{\pi}\hat{\phi}_{L\downarrow}(x)}e^{-i\sqrt{\pi}\hat{\phi}_{R\downarrow}(x)} \\ &= \frac{1}{\pi\delta}e^{i\sqrt{2\pi}\hat{\theta}_c(x)}\cos\left[\sqrt{2\pi}\hat{\theta}_s(x)\right],\end{aligned}\quad (\text{B.27})$$

$$\begin{aligned}\hat{O}_{TS}^y(x) &= \frac{i}{2\pi\delta}e^{i\sqrt{\pi}\hat{\phi}_{L\downarrow}(x)}e^{-i\sqrt{\pi}\hat{\phi}_{R\downarrow}(x)} - \frac{i}{2\pi\delta}e^{-2i\sqrt{\pi}\hat{\phi}_{R\uparrow}(x)}e^{i\sqrt{\pi}\hat{\phi}_{L\uparrow}(x)} \\ &= \frac{1}{\pi\delta}e^{i\sqrt{2\pi}\hat{\theta}_c(x)}\sin\left[\sqrt{2\pi}\hat{\theta}_s(x)\right].\end{aligned}\quad (\text{B.28})$$

These forms are standard in many physical systems described by the bosonization, the only differences are the phases in the trigonometric functions, such as the introduction of a Zeeman field where $k_{F\uparrow} \neq k_{F\downarrow}$, or the SOC where a phase $k_0 = m\alpha$ appears, which is reviewed in

chapter 2. With the bosonized operators, the correlations can be calculated using the mode expansion of the fields, but before two fundamental identities have to be proved.

B.1 Fundamental identities

There are two fundamental identities to use in the upcoming calculations: the first is the separability of the Hamiltonian, and the second is the exponential correlation for bosonic fields.

As the system has separated collective excitations into charge and spin, which means its Hamiltonian can be written as $\hat{H} = \hat{H}_c + \hat{H}_s$, then *the correlation of products of fields with different quantum number is equal to the product of the correlations for each one of them*, simply those two modes are not coupled and the expected values can be calculated separately [9]. This separation is not the only one given in the theory, for instance the main system of this work has its own separation, and it is quite different from the regular charge and spin due to the external couplings such as the Zeeman field or SOC [25]. To show this identity, suppose the correlation to calculate is $\langle \hat{A}_c \hat{B}_s \rangle$, where $\hat{A}_c$ is an operator with fields of the charge part and $\hat{B}_s$ with fields of the spin part. Using the Path Integral formulation of the correlation in equation 1.15 and the corresponding propagator in [26], the correlation is:

$$\begin{aligned}
 \langle \hat{A}_c \hat{B}_s \rangle &= \frac{1}{\mathcal{Z}} \int \prod_{\alpha=c,s} D\phi_\alpha D\pi_\alpha A_c B_s e^{\sum_{\alpha=c,s} \int dx \int_0^\beta d\tau (i\pi_\alpha(x,\tau) \partial_\tau \phi_\alpha(x,\tau) - H_\alpha(x,\tau))} \\
 &= \frac{1}{\mathcal{Z}} \int D\phi_c D\pi_c D\phi_s D\pi_s A_c B_s e^{\int dx \int_0^\beta d\tau (i\pi_c(x,\tau) \partial_\tau \phi_c(x,\tau) + i\pi_s(x,\tau) \partial_\tau \phi_s(x,\tau) - H_c(x,\tau) - H_s(x,\tau))} \\
 &= \frac{1}{\mathcal{Z}} \int D\phi_c D\pi_c A_c e^{\int dx \int_0^\beta d\tau (i\pi_c(x,\tau) \partial_\tau \phi_c(x,\tau) - H_c(x,\tau))} \\
 &\quad \int D\phi_s D\pi_s B_s e^{\int dx \int_0^\beta d\tau (i\pi_s(x,\tau) \partial_\tau \phi_s(x,\tau) - H_s(x,\tau))}, \tag{B.29}
 \end{aligned}$$

where the space-time dependent Hamiltonian is the term inside the integral of the bosonized Hamiltonian in equation B.10, $\hat{H} = \int dx d\tau \hat{H}(x, \tau)$. It can be shown that the partition function separates in the following way:

$$\begin{aligned}
 \mathcal{Z} &= \int D\phi_c D\pi_c D\phi_s D\pi_s e^{\int dx \int_0^\beta d\tau (i\pi_c(x,\tau) \partial_\tau \phi_c(x,\tau) + i\pi_s(x,\tau) \partial_\tau \phi_s(x,\tau) - H_c(x,\tau) - H_s(x,\tau))} \\
 &= \int D\phi_c D\pi_c e^{\int dx \int_0^\beta d\tau (i\pi_c(x,\tau) \partial_\tau \phi_c(x,\tau) - H_c(x,\tau))} \\
 &\quad \int D\phi_s D\pi_s e^{\int dx \int_0^\beta d\tau (i\pi_s(x,\tau) \partial_\tau \phi_s(x,\tau) - H_s(x,\tau))} = \mathcal{Z}_c \mathcal{Z}_s. \tag{B.30}
 \end{aligned}$$

Replacing the partition function and using the separability of the integrals, the $\langle \hat{A}_c \hat{B}_s \rangle$ correlation results in:

$$\begin{aligned}
 \langle \hat{A}_c \hat{B}_s \rangle &= \frac{1}{\mathcal{Z}_c} \int D\phi_c D\pi_c A_c e^{\int dx \int_0^\beta d\tau (i\pi_c(x,\tau) \partial_\tau \phi_c(x,\tau) - H_c(x,\tau))} \\
 &\quad \frac{1}{\mathcal{Z}_s} \int D\phi_s D\pi_s B_s e^{\int dx \int_0^\beta d\tau (i\pi_s(x,\tau) \partial_\tau \phi_s(x,\tau) - H_s(x,\tau))} \\
 &= \langle \hat{A}_c \rangle \langle \hat{B}_s \rangle, \blacksquare \tag{B.31}
 \end{aligned}$$

which is the expected separability of the correlation.

About the second property, since the wide usage of the bosonization identity in equation 1.12, the correlations are summarized in correlations of exponential functions. Those correlations can be calculated using the following identity [25, 26, 28, 42]:

$$\langle e^{f(\hat{\phi}, \hat{\theta})} \rangle = e^{\frac{1}{2} \langle f(\hat{\phi}, \hat{\theta})^2 \rangle}, \quad (\text{B.32})$$

that is proved using Taylor's expansion of the exponential function. The net expansion is the sum of expected values of the n power of the function $f(\hat{\phi}, \hat{\theta})$, but as it is a function of the bosonic fields, it can be expanded as $\int dp (\alpha_p \hat{b}(p) + \gamma_p \hat{b}^\dagger(p))$, and thus only even powers will contribute to the sum:

$$\langle e^{f(\hat{\phi}, \hat{\theta})} \rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \langle f(\hat{\phi}, \hat{\theta})^n \rangle = \sum_{n=0}^{\infty} \frac{1}{(2n)!} \langle f(\hat{\phi}, \hat{\theta})^{2n} \rangle, \quad (\text{B.33})$$

$$\langle f(\hat{\phi}, \hat{\theta})^{2n} \rangle = \langle f(\hat{\phi}, \hat{\theta}) f(\hat{\phi}, \hat{\theta}) f(\hat{\phi}, \hat{\theta}) \cdots f(\hat{\phi}, \hat{\theta}) \rangle. \quad (\text{B.34})$$

In the last expected value the operators are repeated $2n$ times, or a pair is repeated n times. At this point is where the bosonic nature of the fields is important since Wick's theorem can be used without the introduction of Grassmann variables [28]. According to the theorem, the $2n$ products of operators can be rewritten into normal ordered terms and contractions (the contraction pairs operators as $\langle f(\hat{\phi}, \hat{\theta}) f(\hat{\phi}, \hat{\theta}) \rangle$, and multiply all possible pairings), so the full expected value is the sum of terms that are the product of n contractions. Such pairing can be done in many ways, but as all the contractions are equal, the term repeats $\mathcal{C}$ times, so:

$$\langle f(\hat{\phi}, \hat{\theta})^{2n} \rangle = \mathcal{C} \langle f(\hat{\phi}, \hat{\theta}) f(\hat{\phi}, \hat{\theta}) \rangle^n. \quad (\text{B.35})$$

The problem now is to find $\mathcal{C}$, which is equivalent to finding in how many ways the $2n$ operators can be paired given that there are n pairs. This is a combinatorics problem, where that number is equal to [9, 76]:

$$\mathcal{C} = \frac{(2n)!}{2^n n!}. \quad (\text{B.36})$$

Replacing the latter relations, the matrix element $\langle f(\hat{\phi}, \hat{\theta})^{2n} \rangle$ is written as:

$$\langle f(\hat{\phi}, \hat{\theta})^{2n} \rangle = \frac{(2n)!}{2^n n!} \langle f(\hat{\phi}, \hat{\theta}) f(\hat{\phi}, \hat{\theta}) \rangle^n = \frac{(2n)!}{2^n n!} \langle f(\hat{\phi}, \hat{\theta})^2 \rangle^n = \frac{(2n)!}{n!} \left\langle \frac{1}{2} f(\hat{\phi}, \hat{\theta})^2 \right\rangle^n. \quad (\text{B.37})$$

In the end, replacing the matrix element into the expansion of equation B.33, the identity is proved:

$$\begin{aligned} \langle e^{f(\hat{\phi}, \hat{\theta})} \rangle &= \sum_{n=0}^{\infty} \frac{1}{(2n)!} \frac{(2n)!}{n!} \left\langle \frac{1}{2} f(\hat{\phi}, \hat{\theta})^2 \right\rangle^n \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \left\langle \frac{1}{2} f(\hat{\phi}, \hat{\theta})^2 \right\rangle^n \\ &= e^{\frac{1}{2} \langle f(\hat{\phi}, \hat{\theta})^2 \rangle}. \blacksquare \end{aligned} \quad (\text{B.38})$$

These two identities are widely exploited to calculate the correlations, as they are repetitively used. In that direction, one last previous step is to calculate three fundamental correlations that appear in the argument of the exponential function, which are presented in the next section.

B.2 Fundamental correlations

There are three fundamental correlations that appear in all the others to calculate:

$$\left\langle \left(\hat{\phi}(x, \tau) - \hat{\phi}(y, \tau') \right)^2 \right\rangle, \quad (\text{B.39})$$

$$\left\langle \left(\hat{\theta}(x, \tau) - \hat{\theta}(y, \tau') \right)^2 \right\rangle, \quad (\text{B.40})$$

$$\langle \hat{\phi}(x) \hat{\theta}(y) \rangle, \quad (\text{B.41})$$

which are defined for a Hamiltonian of equation 1.13 (without quantum numbers, as the index only refers to the respective Fermi velocity and Luttinger parameter). For this calculation it is supposed the velocity is ν_F and the Luttinger parameter K , so the fields transform as $\hat{\tilde{\phi}}(x, t) = \hat{\phi}(x, t)/\sqrt{K}$ and $\hat{\tilde{\theta}}(x, t) = \hat{\theta}(x, t)\sqrt{K}$. The last correlation is done without time evolution since the full form is not trivial, but also is not necessary to argue anything else in that direction [25]. On the other hand, τ is an imaginary time according to the path integral, where the mapping $\tau = it$ is considered with t the real time. The first step in the calculation of the first correlation is the following:

$$\begin{aligned} \left\langle \left(\hat{\phi}(x, \tau) - \hat{\phi}(y, \tau') \right)^2 \right\rangle &= \langle \hat{\phi}(x, \tau) \hat{\phi}(x, \tau) - 2\hat{\phi}(x, \tau) \hat{\phi}(y, \tau') + \hat{\phi}(y, \tau') \hat{\phi}(y, \tau') \rangle \\ &= \langle \hat{\phi}(x, \tau) \hat{\phi}(x, \tau) \rangle - 2 \langle \hat{\phi}(x, \tau) \hat{\phi}(y, \tau') \rangle + \langle \hat{\phi}(y, \tau') \hat{\phi}(y, \tau') \rangle, \end{aligned} \quad (\text{B.42})$$

so only the correlation $\langle \hat{\phi}(x, \tau) \hat{\phi}(y, \tau') \rangle$ is needed. As the calculation is at zero temperature, the bracket is the ground state and thus the tilde fields satisfy the mode expansion in equation A.59. Next, that mode expansion is replaced:

$$\begin{aligned} \langle \hat{\phi}(x, \tau) \hat{\phi}(y, \tau') \rangle &= \langle \sqrt{K} \hat{\tilde{\phi}}(x, \tau) \sqrt{K} \hat{\tilde{\phi}}(y, \tau') \rangle = K \langle \hat{\tilde{\phi}}(x, \tau) \hat{\tilde{\phi}}(y, \tau') \rangle \\ &= K \langle \left[\int \frac{dp}{\sqrt{4\pi|p|}} \left[e^{ipx} e^{-i|p|\nu_F\tau} \hat{b}(p) + e^{-ipx} e^{i|p|\nu_F\tau} \hat{b}^\dagger(p) \right] \right. \\ &\quad \left. \int \frac{dq}{\sqrt{4\pi|q|}} \left[e^{iqy} e^{-i|q|\nu_F\tau'} \hat{b}(q) + e^{-iqy} e^{i|q|\nu_F\tau'} \hat{b}^\dagger(q) \right] \right| \rangle \\ &= \frac{K}{4\pi} \int \frac{dpdq}{\sqrt{|pq|}} \left\langle e^{ipx} e^{-i|p|\nu_F\tau} e^{-iqy} e^{i|q|\nu_F\tau'} \hat{b}(p) \hat{b}^\dagger(q) + e^{-ipx} e^{i|p|\nu_F\tau} e^{iqy} e^{-i|q|\nu_F\tau'} \hat{b}^\dagger(p) \hat{b}(q) \right\rangle, \end{aligned} \quad (\text{B.43})$$

where the cross products are not considered because the correlations $\langle \hat{b}^{(\dagger)}(p) \hat{b}^{(\dagger)}(q) \rangle$ vanish because the effect of the creation (annihilation) operators could never be equal in the bra

and the ket for any momentum. For the remaining terms, only the normal ordered terms contribute, this is:

$$\begin{aligned}\langle \hat{b}(p)\hat{b}^\dagger(q) \rangle &= \langle \hat{b}(p)\hat{b}^\dagger(q) - \langle \hat{b}(p)\hat{b}^\dagger(q) \rangle \rangle = \langle \hat{b}(p)\hat{b}^\dagger(q) \rangle - \langle \hat{b}(p)\hat{b}^\dagger(q) \rangle = 0, \\ \langle \hat{b}^\dagger(p)\hat{b}(q) \rangle &= \langle \hat{b}^\dagger(p)\hat{b}(q) \rangle = \delta(p-q).\end{aligned}\tag{B.44}$$

So, the correlations takes the form:

$$\begin{aligned}\langle \hat{\phi}(x, \tau)\hat{\phi}(y, \tau') \rangle &= \frac{K}{4\pi} \int \frac{dpdq}{\sqrt{|pq|}} e^{-ipx} e^{i|p|\nu_F\tau} e^{iqy} e^{-i|q|\nu_F\tau'} \delta(p-q) \\ &= \frac{K}{4\pi} \int \frac{dp}{|p|} e^{-ipx} e^{i|p|\nu_F\tau} e^{ipy} e^{-i|p|\nu_F\tau'} \\ &= \frac{K}{4\pi} \int \frac{dp}{|p|} e^{-ipx} e^{-|p|\nu_F t} e^{ipy} e^{i|p|\nu_F t'} \\ &= \lim_{\delta \rightarrow 0^+} \frac{K}{4\pi} \int \frac{dp}{|p|} e^{-\delta|p|} e^{-ip(x-y)} e^{-|p|\nu_F(t-t')},\end{aligned}\tag{B.45}$$

where the mapping $\tau = it$ is replaced. Although the cutoff is introduced, this does not mean that the integral converges, and it is due to the time exponential. The integral converges if $t - t' > 0$, and this is the same as considering that the correlation is time ordered, but it could happen that it is not. That is why the time ordering is introduced, and for bosonic operators is given as [26]:

$$\langle \hat{\phi}(x, \tau)\hat{\phi}(y, \tau') \rangle \rightarrow \langle \mathcal{T}\hat{\phi}(x, \tau)\hat{\phi}(y, \tau') \rangle = Y(t-t') \langle \hat{\phi}(x, \tau)\hat{\phi}(y, \tau') \rangle + Y(t'-t) \langle \hat{\phi}(y, \tau')\hat{\phi}(x, \tau) \rangle,\tag{B.46}$$

The correlations is calculated for the time ordered operators instead of the operators without the order. This could have been prevented in the start, as it is expected in a quantum field theory, but the latter shows the necessity of these tools, as without them the results are plagued with divergences. Now, the integral to calculate is:

$$\begin{aligned}\langle \mathcal{T}\hat{\phi}(x, \tau)\hat{\phi}(y, \tau') \rangle &= \lim_{\delta \rightarrow 0^+} Y(t-t') \frac{K}{4\pi} \int \frac{dp}{|p|} e^{-\delta|p|} e^{-ip(x-y)} e^{-|p|\nu_F(t-t')} \\ &\quad + Y(t'-t) \frac{K}{4\pi} \int \frac{dp}{|p|} e^{-\delta|p|} e^{-ip(y-x)} e^{-|p|\nu_F(t'-t)} \\ &= \lim_{\delta \rightarrow 0^+} Y(t-t') \frac{K}{4\pi} \int \frac{dp}{|p|} e^{-\delta|p|} e^{-ip(x-y)} e^{-|p|\nu_F|t-t'|} \\ &\quad + Y(t'-t) \frac{K}{4\pi} \int \frac{dp}{|p|} e^{-\delta|p|} e^{-ip(x-y)} e^{-|p|\nu_F|t'-t|} \\ &= \lim_{\delta \rightarrow 0^+} \frac{K}{4\pi} \int \frac{dp}{|p|} e^{-|p|(\nu_F|t-t'|+\delta)} e^{-ip(x-y)},\end{aligned}\tag{B.47}$$

where it was used that in the term with $Y(t-t')$, then $t-t' = |t-t'|$ and in the term with $Y(t'-t)$, then $t'-t = |t'-t| = |t-t'|$, in the second integral a change $p \rightarrow -p$ is made so the two integrals are equal (the integrated function preserves its + sign), and $Y(t-t') + Y(t'-t) = 1$. The integral can be solved using a similar approach to the differentiation under the integral sign, but with an integral (integral under the integral sign), because it is satisfied:

$$\frac{1}{|p|} e^{-\delta|p|} = - \int d\delta e^{-\delta|p|}.\tag{B.48}$$

When the latter is replaced, the integral is calculated directly:

$$\begin{aligned}
\langle \mathcal{T} \hat{\phi}(x, \tau) \hat{\phi}(y, \tau') \rangle &= - \lim_{\delta \rightarrow 0^+} \frac{K}{4\pi} \int d\delta \int dp e^{-|p|(\nu_F|t-t'|+\delta)} e^{-ip(x-y)} \\
&= - \lim_{\delta \rightarrow 0^+} \frac{K}{4\pi} \int d\delta \int dp e^{-|p|(\nu_F|t-t'|+\delta)} \cos[p(x-y)] \\
&= - \lim_{\delta \rightarrow 0^+} \frac{K}{2\pi} \int d\delta \int_0^\infty dp e^{-p(\nu_F|t-t'|+\delta)} \cos[p(x-y)] \\
&= - \lim_{\delta \rightarrow 0^+} \frac{K}{2\pi} \int d\delta \frac{\nu_F|t-t'|+\delta}{(\nu_F|t-t'|+\delta)^2 + (x-y)^2} \\
&= - \frac{K}{4\pi} \ln [(x-y)^2 + (\nu_F|t-t'|+\delta)^2]. \tag{B.49}
\end{aligned}$$

The inner integral appears because the complex exponential is expanded with the Euler's formula and only the even part does not vanish, and it is solved with the identity in equation A.26. The last integral in δ is solved using the change of variable $z = (x-y)^2 + (\nu_F|t-t'|+\delta)^2$, so it takes the form of a logarithm. The limit $\delta \rightarrow 0^+$ is suppressed because the result is not convergent for every set of (x, y, t, t') if $\delta = 0$, again it is more convenient to leave the parameter as it could be tiny enough, but not zero. Now, replacing the latter result in the equation B.42, the correlation is given as:

$$\begin{aligned}
\left\langle \left(\hat{\phi}(x, \tau) - \hat{\phi}(y, \tau') \right)^2 \right\rangle &= -\frac{K}{4\pi} \ln [\delta^2] + \frac{K}{2\pi} \ln [(x-y)^2 + (\nu_F|t-t'|+\delta)^2] - \frac{K}{4\pi} \ln [\delta^2] \\
&= \frac{K}{2\pi} \ln \left[\frac{(x-y)^2 + (\nu_F|t-t'|+\delta)^2}{\delta^2} \right]. \tag{B.50}
\end{aligned}$$

For the second correlation, it can be noticed from the Hamiltonian in equation B.10 that the charge part has an exchange symmetry, this is, to change $\hat{\phi}(x) \leftrightarrow \hat{\theta}(x)$ and $K \leftrightarrow 1/K$ results in the same Hamiltonian. Thus, the correlation for $\hat{\theta}(x)$ fields is the same as of $\hat{\phi}(x)$ but with $1/K$ instead of K , which gives:

$$\left\langle \left(\hat{\theta}(x, \tau) - \hat{\theta}(y, \tau') \right)^2 \right\rangle = \frac{1}{2\pi K} \ln \left[\frac{(x-y)^2 + (\nu_F|t-t'|+\delta)^2}{\delta^2} \right]. \tag{B.51}$$

The third correlation has to be calculated explicitly in the following way:

$$\begin{aligned}
\langle \hat{\phi}(x) \hat{\theta}(y) \rangle &= \left\langle \sqrt{K} \hat{\phi}(x) \frac{1}{\sqrt{K}} \hat{\theta}(y) \right\rangle = \langle \hat{\phi}(x) \hat{\theta}(y) \rangle \\
&= -\frac{1}{4\pi} \int \frac{dp dq}{\sqrt{|pq|}} \text{sgn}(q) \left\langle e^{ipx} e^{-iqy} \hat{b}(p) \hat{b}^\dagger(q) + e^{-ipx} e^{iqy} \hat{b}^\dagger(p) \hat{b}(q) \right\rangle \\
&= -\frac{1}{4\pi} \int \frac{dp}{|p|} \text{sgn}(p) e^{-ip(x-y)} = -\frac{1}{4\pi} \int \frac{dp}{p} e^{-ip(x-y)}. \tag{B.52}
\end{aligned}$$

When doing the change $p \rightarrow -p$ and recovering the integral form of the sign function used in equation A.79, the correlation is:

$$\langle \hat{\phi}(x) \hat{\theta}(y) \rangle = \frac{1}{4\pi} \int \frac{dp}{p} e^{ip(x-y)} = \frac{i}{4} \text{sgn}(x-y). \tag{B.53}$$

The form of the latter fundamental correlations are used in all the other correlations of this appendix. It is pretty important to notice that those correlations only depend on the spatial and time separation, which is a consequence of the space-time invariance of the Hamiltonian (Lorentz invariant [72, 73]). In the following sections the specific correlations are calculated.

B.3 Charge density correlation

To start with the calculations, in addition to the latter fundamental relations there are some tiny details which are explained during these earlier calculations to make the last ones clearer. The first correlation is the charge density one, given as:

$$\begin{aligned}
\langle \hat{\rho}(x, \tau) \hat{\rho}(y, \tau') \rangle &= \langle | \sqrt{\frac{2}{\pi}} \partial_x \hat{\phi}_c(x, \tau) + \frac{2}{\pi \delta} \cos [\sqrt{2\pi} \hat{\phi}_c(x, \tau) + 2k_F x] \cos [\sqrt{2\pi} \hat{\phi}_s(x, \tau)] \\
&\quad \sqrt{\frac{2}{\pi}} \partial_y \hat{\phi}_c(y, \tau') + \frac{2}{\pi \delta} \cos [\sqrt{2\pi} \hat{\phi}_c(y, \tau') + 2k_F y] \cos [\sqrt{2\pi} \hat{\phi}_s(y, \tau')] | \rangle \\
&= \frac{2}{\pi} \langle \partial_x \hat{\phi}_c(x, \tau) \partial_y \hat{\phi}_c(y, \tau') \rangle \\
&\quad + \left(\frac{2}{\pi \delta} \right)^2 \langle | \cos [\sqrt{2\pi} \hat{\phi}_c(x, \tau) + 2k_F x] \cos [\sqrt{2\pi} \hat{\phi}_s(x, \tau)] \\
&\quad \cos [\sqrt{2\pi} \hat{\phi}_c(y, \tau') + 2k_F y] \cos [\sqrt{2\pi} \hat{\phi}_s(y, \tau')] | \rangle \\
&= \frac{2}{\pi} \langle \partial_x \hat{\phi}_c(x, \tau) \partial_y \hat{\phi}_c(y, \tau') \rangle \\
&\quad + \left(\frac{2}{\pi \delta} \right)^2 \langle \cos [\sqrt{2\pi} \hat{\phi}_c(x, \tau) + 2k_F x] \cos [\sqrt{2\pi} \hat{\phi}_c(y, \tau') + 2k_F y] \rangle \\
&\quad \langle \cos [\sqrt{2\pi} \hat{\phi}_s(x, \tau)] \cos [\sqrt{2\pi} \hat{\phi}_s(y, \tau')] \rangle. \tag{B.54}
\end{aligned}$$

The crossed products vanish because they contain odd powers of bosonic operators. Although the first term would contribute to the correlation (the derivatives of the fields), it will not contribute to the phase analysis under the LRT as it decays in an universal square power law [25, 26] (this argument is clearer when the susceptibilities are calculated). Then, the correlation reduces to:

$$\begin{aligned}
\langle \hat{\rho}(x, \tau) \hat{\rho}(y, \tau') \rangle &= \left(\frac{2}{\pi \delta} \right)^2 \langle \cos [\sqrt{2\pi} \hat{\phi}_c(x, \tau) + 2k_F x] \cos [\sqrt{2\pi} \hat{\phi}_c(y, \tau') + 2k_F y] \rangle \\
&\quad \langle \cos [\sqrt{2\pi} \hat{\phi}_s(x, \tau)] \cos [\sqrt{2\pi} \hat{\phi}_s(y, \tau')] \rangle. \tag{B.55}
\end{aligned}$$

The cosine correlations has to be calculated, but notice it is enough to calculate the first, as the second is a reduction of it (set $c \rightarrow s$ and $k_F = 0$). First, the product is separated using

$2 \cos(\alpha) \cos(\beta) = \cos(\alpha + \beta) + \cos(\alpha - \beta)$, this is:

$$\begin{aligned}
& \left\langle \cos \left[\sqrt{2\pi} \hat{\phi}_c(x, \tau) + 2k_F x \right] \cos \left[\sqrt{2\pi} \hat{\phi}_c(y, \tau') + 2k_F y \right] \right\rangle \\
&= \frac{1}{2} \left\langle \cos \left[\sqrt{2\pi} \left(\hat{\phi}_c(x, \tau) + \hat{\phi}_c(y, \tau') \right) + 2k_F(x + y) \right] \right\rangle \\
&+ \frac{1}{2} \left\langle \cos \left[\sqrt{2\pi} \left(\hat{\phi}_c(x, \tau) - \hat{\phi}_c(y, \tau') \right) + 2k_F(x - y) \right] \right\rangle \\
&= \frac{1}{4} e^{i2k_F(x+y)} \left\langle e^{i\sqrt{2\pi}(\hat{\phi}_c(x,\tau)+\hat{\phi}_c(y,\tau'))} \right\rangle + \frac{1}{4} e^{-i2k_F(x+y)} \left\langle e^{-i\sqrt{2\pi}(\hat{\phi}_c(x,\tau)+\hat{\phi}_c(y,\tau'))} \right\rangle \\
&+ \frac{1}{4} e^{i2k_F(x-y)} \left\langle e^{i\sqrt{2\pi}(\hat{\phi}_c(x,\tau)-\hat{\phi}_c(y,\tau'))} \right\rangle + \frac{1}{4} e^{-i2k_F(x-y)} \left\langle e^{-i\sqrt{2\pi}(\hat{\phi}_c(x,\tau)-\hat{\phi}_c(y,\tau'))} \right\rangle \\
&= \frac{1}{4} e^{i2k_F(x+y)} e^{\frac{1}{2} \langle [i\sqrt{2\pi}(\hat{\phi}_c(x,\tau)+\hat{\phi}_c(y,\tau'))]^2 \rangle} + \frac{1}{4} e^{-i2k_F(x+y)} e^{\frac{1}{2} \langle [-i\sqrt{2\pi}(\hat{\phi}_c(x,\tau)+\hat{\phi}_c(y,\tau'))]^2 \rangle} \\
&+ \frac{1}{4} e^{i2k_F(x-y)} e^{\frac{1}{2} \langle [i\sqrt{2\pi}(\hat{\phi}_c(x,\tau)-\hat{\phi}_c(y,\tau'))]^2 \rangle} + \frac{1}{4} e^{-i2k_F(x-y)} e^{\frac{1}{2} \langle [-i\sqrt{2\pi}(\hat{\phi}_c(x,\tau)-\hat{\phi}_c(y,\tau'))]^2 \rangle} \\
&= \frac{1}{4} e^{i2k_F(x+y)} e^{-\pi \langle (\hat{\phi}_c(x,\tau)+\hat{\phi}_c(y,\tau'))^2 \rangle} + \frac{1}{4} e^{-i2k_F(x+y)} e^{-\pi \langle (\hat{\phi}_c(x,\tau)+\hat{\phi}_c(y,\tau'))^2 \rangle} \\
&+ \frac{1}{4} e^{i2k_F(x-y)} e^{-\pi \langle (\hat{\phi}_c(x,\tau)-\hat{\phi}_c(y,\tau'))^2 \rangle} + \frac{1}{4} e^{-i2k_F(x-y)} e^{-\pi \langle (\hat{\phi}_c(x,\tau)-\hat{\phi}_c(y,\tau'))^2 \rangle} \\
&= \frac{1}{2} \cos[2k_F(x+y)] e^{-\pi \langle (\hat{\phi}_c(x,\tau)+\hat{\phi}_c(y,\tau'))^2 \rangle} + \frac{1}{2} \cos[2k_F(x-y)] e^{-\pi \langle (\hat{\phi}_c(x,\tau)-\hat{\phi}_c(y,\tau'))^2 \rangle}.
\end{aligned} \tag{B.56}$$

This correlation, and for all the remaining, has a part that depends on the sum of the fields and another on the difference of the fields. The correlation of the difference is given in equation B.50, but the sum part can be shown to vanish by changing the $+$ sign to a $-$ sign of the middle term in the first line of that equation, so the correlation is:

$$\begin{aligned}
\left\langle \left(\hat{\phi}(x, \tau) + \hat{\phi}(y, \tau') \right)^2 \right\rangle &= -\frac{K}{4\pi} \ln[\delta^2] - \frac{K}{2\pi} \ln[(x-y)^2 + (\nu_F|t-t'| + \delta)^2] - \frac{K}{4\pi} \ln[\delta^2] \\
&= -\frac{K}{2\pi} \ln[\delta^2 ((x-y)^2 + (\nu_F|t-t'| + \delta)^2)] \rightarrow
\end{aligned} \tag{B.57}$$

$$e^{-\pi \langle (\hat{\phi}_c(x,\tau)+\hat{\phi}_c(y,\tau'))^2 \rangle} = \delta^K ((x-y)^2 + (\nu_F|t-t'| + \delta)^2)^{K/2} \rightarrow 0, \tag{B.58}$$

in the limit $\delta \rightarrow 0^+$. This shows that this kind of correlations can be safely discarded in the rest of correlations. Under the latter detail, the cosine correlation reduces to:

$$\begin{aligned}
& \left\langle \cos \left[\sqrt{2\pi} \hat{\phi}_c(x, \tau) + 2k_F x \right] \cos \left[\sqrt{2\pi} \hat{\phi}_c(y, \tau') + 2k_F y \right] \right\rangle \\
&= \frac{1}{2} \cos[2k_F(x-y)] e^{-\pi \langle (\hat{\phi}_c(x,\tau)-\hat{\phi}_c(y,\tau'))^2 \rangle} \\
&= \frac{1}{2} \cos[2k_F(x-y)] e^{-\pi \frac{K_c}{2\pi} \ln \left[\frac{(x-y)^2 + (\nu_c|t-t'| + \delta)^2}{\delta^2} \right]} \\
&= \frac{1}{2} \cos[2k_F(x-y)] \left(\frac{\delta}{\sqrt{(x-y)^2 + (\nu_c|t-t'| + \delta)^2}} \right)^{K_c}.
\end{aligned} \tag{B.59}$$

And thus, the cosine correlation for the spin part is:

$$\left\langle \cos \left[\sqrt{2\pi} \hat{\phi}_s(x, \tau) \right] \cos \left[\sqrt{2\pi} \hat{\phi}_s(y, \tau') \right] \right\rangle = \frac{1}{2} \left(\frac{\delta}{\sqrt{(x-y)^2 + (\nu_s |t-t'| + \delta)^2}} \right)^{K_s}. \quad (\text{B.60})$$

If $|x-y|, |t-t'| \gg \delta$ (the correlation is evaluated for separated points, not the same point), the cutoff in the squared root of equations B.59 and B.60 can be neglected, which means those functions define a space-time distance denoted as [72, 73]:

$$r_c = \sqrt{(x-y)^2 + \nu_c^2 |t-t'|^2}, \quad (\text{B.61})$$

$$r_s = \sqrt{(x-y)^2 + \nu_s^2 |t-t'|^2}. \quad (\text{B.62})$$

With the latter notation, the final form of the charge density correlation for separated points is:

$$\begin{aligned} \langle \hat{\rho}(x, \tau) \hat{\rho}(y, \tau') \rangle &= \left(\frac{2}{\pi \delta} \right)^2 \frac{1}{2} \cos [2k_F(x-y)] \left(\frac{\delta}{r_c} \right)^{K_c} \frac{1}{2} \left(\frac{\delta}{r_s} \right)^{K_s} \\ &= \frac{1}{(\pi \delta)^2} \cos [2k_F(x-y)] \left(\frac{\delta}{r_c} \right)^{K_c} \left(\frac{\delta}{r_s} \right)^{K_s}. \end{aligned} \quad (\text{B.63})$$

Notice that if the correlations are time-independent ($t = t' = 0$) then simply $r_c = r_s = r = |x-y|$ (an argument on using time-independent correlations was already given, that the system is Lorentz invariant, but it will be used in further analysis). One question that might arise in all the analysis is, if it was said that the sum of fields correlation in equation B.57 vanish due to the limit $\delta \rightarrow 0^+$, why does not the functional form δ/r vanish? This is because r can still approach zero so the behavior is not generally convergent when both δ and r approach zero. The latter correlation describes the possible charge excitations for the system under given values of U , which as said before it indeed changes the behavior of it by means of the Luttinger parameters.

B.4 Spin density correlations

B.4.1 Axial spin density correlation

The next correlation is for the axial spin density, given as:

$$\begin{aligned} \langle \hat{\sigma}^z(x, \tau) \hat{\sigma}^z(y, \tau') \rangle &= \langle \left| \sqrt{\frac{2}{\pi}} \partial_x \hat{\phi}_s(x, \tau) - \frac{2}{\pi \delta} \sin \left[\sqrt{2\pi} \hat{\phi}_c(x, \tau) + 2k_F x \right] \sin \left[\sqrt{2\pi} \hat{\phi}_s(x, \tau) \right] \right. \\ &\quad \left. \sqrt{\frac{2}{\pi}} \partial_y \hat{\phi}_s(y, \tau') - \frac{2}{\pi \delta} \sin \left[\sqrt{2\pi} \hat{\phi}_c(y, \tau') + 2k_F y \right] \sin \left[\sqrt{2\pi} \hat{\phi}_s(y, \tau') \right] \right| \rangle \\ &= \frac{2}{\pi} \left\langle \partial_x \hat{\phi}_s(x, \tau) \partial_y \hat{\phi}_s(y, \tau') \right\rangle \\ &\quad + \left(\frac{2}{\pi \delta} \right)^2 \left\langle \sin \left[\sqrt{2\pi} \hat{\phi}_c(x, \tau) + 2k_F x \right] \sin \left[\sqrt{2\pi} \hat{\phi}_c(y, \tau') + 2k_F y \right] \right\rangle \\ &\quad \left\langle \sin \left[\sqrt{2\pi} \hat{\phi}_s(x, \tau) \right] \sin \left[\sqrt{2\pi} \hat{\phi}_s(y, \tau') \right] \right\rangle. \\ &\approx \left(\frac{2}{\pi \delta} \right)^2 \left\langle \sin \left[\sqrt{2\pi} \hat{\phi}_c(x, \tau) + 2k_F x \right] \sin \left[\sqrt{2\pi} \hat{\phi}_c(y, \tau') + 2k_F y \right] \right\rangle \\ &\quad \left\langle \sin \left[\sqrt{2\pi} \hat{\phi}_s(x, \tau) \right] \sin \left[\sqrt{2\pi} \hat{\phi}_s(y, \tau') \right] \right\rangle. \end{aligned} \quad (\text{B.64})$$

The first term does not contribute to the susceptibilities, and the second term corresponds to correlations of sine functions, which could look different than the cosine correlations of the charge density, but as $2 \sin(\alpha) \sin(\beta) = \cos(\alpha - \beta) - \cos(\alpha + \beta)$, then the difference of fields term is the same, which means the correlation is exactly the same:

$$\langle \hat{\sigma}^z(x, \tau) \hat{\sigma}^z(y, \tau') \rangle = \frac{1}{(\pi\delta)^2} \cos[2k_F(x - y)] \left(\frac{\delta}{r_c} \right)^{K_c} \left(\frac{\delta}{r_s} \right)^{K_s}. \quad (\text{B.65})$$

B.4.2 Planar spin density correlations

The next is the correlation for the spin density in x direction, expanded in the following way:

$$\begin{aligned} & \langle \hat{\sigma}^x(x, \tau) \hat{\sigma}^x(y, \tau') \rangle \\ &= \left(\frac{2}{\pi\delta} \right)^2 \left\langle \cos \left[\sqrt{2\pi} \hat{\theta}_s(x, \tau) \right] \cos \left[\sqrt{2\pi} \hat{\phi}_s(x, \tau) \right] \cos \left[\sqrt{2\pi} \hat{\theta}_s(y, \tau') \right] \cos \left[\sqrt{2\pi} \hat{\phi}_s(y, \tau') \right] \right\rangle \\ &+ \left(\frac{2}{\pi\delta} \right)^2 \left\langle \cos \left[\sqrt{2\pi} \hat{\theta}_s(x, \tau) \right] \cos \left[\sqrt{2\pi} \hat{\phi}_s(x, \tau) \right] \cos \left[\sqrt{2\pi} \hat{\theta}_s(y, \tau') \right] \right\rangle \\ &\quad \left\langle \cos \left[\sqrt{2\pi} \hat{\phi}_c(y, \tau') + 2k_F y \right] \right\rangle \\ &+ \left(\frac{2}{\pi\delta} \right)^2 \left\langle \cos \left[\sqrt{2\pi} \hat{\theta}_s(x, \tau) \right] \cos \left[\sqrt{2\pi} \hat{\theta}_s(y, \tau') \right] \cos \left[\sqrt{2\pi} \hat{\phi}_s(y, \tau') \right] \right\rangle \\ &\quad \left\langle \cos \left[\sqrt{2\pi} \hat{\phi}_c(x, \tau) + 2k_F x \right] \right\rangle \\ &+ \left(\frac{2}{\pi\delta} \right)^2 \left\langle \cos \left[\sqrt{2\pi} \hat{\theta}_s(x, \tau) \right] \cos \left[\sqrt{2\pi} \hat{\theta}_s(y, \tau') \right] \right\rangle \\ &\quad \left\langle \cos \left[\sqrt{2\pi} \hat{\phi}_c(x, \tau) + 2k_F x \right] \cos \left[\sqrt{2\pi} \hat{\phi}_c(y, \tau') + 2k_F y \right] \right\rangle. \end{aligned} \quad (\text{B.66})$$

To calculate these correlations, it is important to look at the first cosine correlation, which contains four cosine functions and mix $\hat{\theta}(x)$ and $\hat{\phi}(x)$ fields. In the previous correlations, the cosine ones always had functions of only one field, but not both. Hence, since those fields commute with themselves, the identity $e^{\hat{A}} e^{\hat{B}} = e^{\hat{A} + \hat{B}}$ was used indiscriminately, according to the Baker–Campbell–Hausdorff formula in equation A.87. This is not the case for $\hat{\theta}(x)$ and $\hat{\phi}(x)$, so the commutator $[\hat{\phi}(x), \hat{\theta}(y)]$ is calculated as follows:

$$\begin{aligned} [\hat{\phi}(x), \hat{\theta}(y)] &= \left[\int \frac{dp}{\sqrt{4\pi|p|}} \left[e^{ipx} \hat{b}(p) + e^{-ipx} \hat{b}^\dagger(p) \right], - \int dq \frac{\text{sgn}(q)}{\sqrt{4\pi|q|}} \left[e^{iqy} \hat{b}(q) + e^{-iqy} \hat{b}^\dagger(q) \right] \right] \\ &= -\frac{1}{4\pi} \int \frac{dp dq}{\sqrt{|pq|}} \text{sgn}(q) \left(e^{ipx} e^{-iqy} [\hat{b}(p), \hat{b}^\dagger(q)] + e^{-ipx} e^{iqy} [\hat{b}^\dagger(p), \hat{b}(q)] \right) \\ &= -\frac{1}{4\pi} \int \frac{dp}{|p|} \text{sgn}(p) e^{ip(x-y)} + \frac{1}{4\pi} \int \frac{dp}{|p|} \text{sgn}(p) e^{-ip(x-y)} \\ &= -\frac{1}{4\pi} \int \frac{dp}{|p|} \frac{|p|}{p} e^{ip(x-y)} - \frac{1}{4\pi} \int \frac{dp}{|p|} \frac{|p|}{p} e^{ip(x-y)} \\ &= -\frac{1}{2\pi} \int \frac{dp}{p} e^{ip(x-y)} = -\frac{i}{2} \frac{1}{2\pi} \int dp \left(\frac{2}{ip} \right) e^{ip(x-y)} = -\frac{i}{2} \text{sgn}(x - y). \end{aligned} \quad (\text{B.67})$$

Using the latter commutation relation, the four-cosine correlation is calculated in the next form:

$$\begin{aligned}
& \left\langle \cos \left[\sqrt{2\pi} \hat{\theta}_s(x, \tau) \right] \cos \left[\sqrt{2\pi} \hat{\phi}_s(x, \tau) \right] \cos \left[\sqrt{2\pi} \hat{\theta}_s(y, \tau') \right] \cos \left[\sqrt{2\pi} \hat{\phi}_s(y, \tau') \right] \right\rangle \\
&= \frac{1}{4} \langle | \left(\cos \left[\sqrt{2\pi} \left(\hat{\phi}_s(x, \tau) + \hat{\theta}_s(x, \tau) \right) \right] + \cos \left[\sqrt{2\pi} \left(\hat{\phi}_s(x, \tau) - \hat{\theta}_s(x, \tau) \right) \right] \right) \right. \\
&\quad \left. \left(\cos \left[\sqrt{2\pi} \left(\hat{\phi}_s(y, \tau') + \hat{\theta}_s(y, \tau') \right) \right] + \cos \left[\sqrt{2\pi} \left(\hat{\phi}_s(y, \tau') - \hat{\theta}_s(y, \tau') \right) \right] \right) | \right\rangle \\
&= \frac{1}{4} \langle | \left(\frac{1}{2} \left[e^{i\sqrt{2\pi}(\hat{\phi}_s(x, \tau) + \hat{\theta}_s(x, \tau))} + e^{-i\sqrt{2\pi}(\hat{\phi}_s(x, \tau) + \hat{\theta}_s(x, \tau))} \right] \right. \right. \\
&\quad \left. \left. + \frac{1}{2} \left[e^{i\sqrt{2\pi}(\hat{\phi}_s(x, \tau) - \hat{\theta}_s(x, \tau))} + e^{-i\sqrt{2\pi}(\hat{\phi}_s(x, \tau) - \hat{\theta}_s(x, \tau))} \right] \right) \right. \\
&\quad \left(\frac{1}{2} \left[e^{i\sqrt{2\pi}(\hat{\phi}_s(y, \tau') + \hat{\theta}_s(y, \tau'))} + e^{-i\sqrt{2\pi}(\hat{\phi}_s(y, \tau') + \hat{\theta}_s(y, \tau'))} \right] \right. \\
&\quad \left. \left. + \frac{1}{2} \left[e^{i\sqrt{2\pi}(\hat{\phi}_s(y, \tau') - \hat{\theta}_s(y, \tau'))} + e^{-i\sqrt{2\pi}(\hat{\phi}_s(y, \tau') - \hat{\theta}_s(y, \tau'))} \right] \right) | \right\rangle \\
&= \frac{1}{16} \langle | \left(e^{i\sqrt{2\pi}(\hat{\phi}_s(x, \tau) + \hat{\theta}_s(x, \tau))} + e^{-i\sqrt{2\pi}(\hat{\phi}_s(x, \tau) + \hat{\theta}_s(x, \tau))} \right. \right. \\
&\quad \left. \left. + e^{i\sqrt{2\pi}(\hat{\phi}_s(x, \tau) - \hat{\theta}_s(x, \tau))} + e^{-i\sqrt{2\pi}(\hat{\phi}_s(x, \tau) - \hat{\theta}_s(x, \tau))} \right) \right. \\
&\quad \left(e^{i\sqrt{2\pi}(\hat{\phi}_s(y, \tau') + \hat{\theta}_s(y, \tau'))} + e^{-i\sqrt{2\pi}(\hat{\phi}_s(y, \tau') + \hat{\theta}_s(y, \tau'))} \right. \\
&\quad \left. \left. + e^{i\sqrt{2\pi}(\hat{\phi}_s(y, \tau') - \hat{\theta}_s(y, \tau'))} + e^{-i\sqrt{2\pi}(\hat{\phi}_s(y, \tau') - \hat{\theta}_s(y, \tau'))} \right) | \right\rangle \\
&\approx \frac{1}{16} \langle | \left(e^{i\sqrt{2\pi}(\hat{\phi}_s(x, \tau) + \hat{\theta}_s(x, \tau))} e^{-i\sqrt{2\pi}(\hat{\phi}_s(y, \tau') + \hat{\theta}_s(y, \tau'))} \right. \right. \\
&\quad + e^{-i\sqrt{2\pi}(\hat{\phi}_s(x, \tau) + \hat{\theta}_s(x, \tau))} e^{i\sqrt{2\pi}(\hat{\phi}_s(y, \tau') + \hat{\theta}_s(y, \tau'))} \\
&\quad + e^{i\sqrt{2\pi}(\hat{\phi}_s(x, \tau) - \hat{\theta}_s(x, \tau))} e^{-i\sqrt{2\pi}(\hat{\phi}_s(y, \tau') - \hat{\theta}_s(y, \tau'))} \\
&\quad \left. \left. + e^{-i\sqrt{2\pi}(\hat{\phi}_s(x, \tau) - \hat{\theta}_s(x, \tau))} e^{i\sqrt{2\pi}(\hat{\phi}_s(y, \tau') - \hat{\theta}_s(y, \tau'))} \right) | \right\rangle, \tag{B.68}
\end{aligned}$$

where in the last line only the products resulting in correlations of difference of fields are considered. The product of exponential functions in the latter expression can be operated using the Baker–Campbell–Hausdorff formula, where the commutator is:

$$\begin{aligned}
\left[i\sqrt{2\pi} \left(\hat{\phi}(x) \pm \hat{\theta}(x) \right), -i\sqrt{2\pi} \left(\hat{\phi}(y) \pm \hat{\theta}(y) \right) \right] &= 2\pi \left[\hat{\phi}(x) \pm \hat{\theta}(x), \hat{\phi}(y) \pm \hat{\theta}(y) \right] \\
&= \pm 2\pi \left(\left[\hat{\phi}(x), \hat{\theta}(y) \right] + \left[\hat{\theta}(x), \hat{\phi}(y) \right] \right) \\
&= \pm 2\pi \left(-\frac{i}{2} \text{sgn}(x - y) + \frac{i}{2} \text{sgn}(y - x) \right) \\
&= \mp i 2\pi \text{sgn}(x - y). \tag{B.69}
\end{aligned}$$

Hence, when using the formula into the product of exponential functions of equation B.68, the

term is now:

$$\begin{aligned}
& \left\langle \cos \left[\sqrt{2\pi} \hat{\theta}_s(x, \tau) \right] \cos \left[\sqrt{2\pi} \hat{\phi}_s(x, \tau) \right] \cos \left[\sqrt{2\pi} \hat{\theta}_s(y, \tau') \right] \cos \left[\sqrt{2\pi} \hat{\phi}_s(y, \tau') \right] \right\rangle \\
&= \frac{1}{16} \left\langle \left(e^{i\sqrt{2\pi}(\hat{\phi}_s(x, \tau) - \hat{\phi}_s(y, \tau') + \hat{\theta}_s(x, \tau) - \hat{\theta}_s(y, \tau'))} e^{-\frac{1}{2}i2\pi \text{sgn}(x-y)} \right. \right. \\
&\quad + e^{-i\sqrt{2\pi}(\hat{\phi}_s(x, \tau) - \hat{\phi}_s(y, \tau') + \hat{\theta}_s(x, \tau) - \hat{\theta}_s(y, \tau'))} e^{-\frac{1}{2}i2\pi \text{sgn}(x-y)} \\
&\quad + e^{i\sqrt{2\pi}(\hat{\phi}_s(x, \tau) - \hat{\phi}_s(y, \tau') - (\hat{\theta}_s(x, \tau) - \hat{\theta}_s(y, \tau')))} e^{\frac{1}{2}i2\pi \text{sgn}(x-y)} \\
&\quad \left. \left. + e^{-i\sqrt{2\pi}(\hat{\phi}_s(x, \tau) - \hat{\phi}_s(y, \tau') - (\hat{\theta}_s(x, \tau) - \hat{\theta}_s(y, \tau')))} e^{\frac{1}{2}i2\pi \text{sgn}(x-y)} \right) \right\rangle \\
&= \frac{1}{8} \left(e^{-\pi \langle [\hat{\phi}_s(x, \tau) - \hat{\phi}_s(y, \tau') + \hat{\theta}_s(x, \tau) - \hat{\theta}_s(y, \tau')]^2 \rangle} e^{-i\pi \text{sgn}(x-y)} \right. \\
&\quad \left. + e^{-\pi \langle [\hat{\phi}_s(x, \tau) - \hat{\phi}_s(y, \tau') - (\hat{\theta}_s(x, \tau) - \hat{\theta}_s(y, \tau'))]^2 \rangle} e^{i\pi \text{sgn}(x-y)} \right) \\
&= -\frac{1}{8} \left(e^{-\pi \langle [\hat{\phi}_s(x, \tau) - \hat{\phi}_s(y, \tau') + \hat{\theta}_s(x, \tau) - \hat{\theta}_s(y, \tau')]^2 \rangle} + e^{-\pi \langle [\hat{\phi}_s(x, \tau) - \hat{\phi}_s(y, \tau') - (\hat{\theta}_s(x, \tau) - \hat{\theta}_s(y, \tau'))]^2 \rangle} \right) \\
&\hspace{25em} \text{(B.70)} \\
&= -\frac{1}{8} e^{-\pi \langle [\hat{\phi}_s(x, \tau) - \hat{\phi}_s(y, \tau')]^2 \rangle} e^{-\pi \langle [\hat{\theta}_s(x, \tau) - \hat{\theta}_s(y, \tau')]^2 \rangle} \\
&\quad \left(e^{-\pi \langle (\hat{\phi}_s(x, \tau) - \hat{\phi}_s(y, \tau'))(\hat{\theta}_s(x, \tau) - \hat{\theta}_s(y, \tau')) + (\hat{\theta}_s(x, \tau) - \hat{\theta}_s(y, \tau'))(\hat{\phi}_s(x, \tau) - \hat{\phi}_s(y, \tau')) \rangle} \right. \\
&\quad \left. + e^{\pi \langle (\hat{\phi}_s(x, \tau) - \hat{\phi}_s(y, \tau'))(\hat{\theta}_s(x, \tau) - \hat{\theta}_s(y, \tau')) + (\hat{\theta}_s(x, \tau) - \hat{\theta}_s(y, \tau'))(\hat{\phi}_s(x, \tau) - \hat{\phi}_s(y, \tau')) \rangle} \right) \\
&= -\frac{1}{8} e^{-\pi \langle [\hat{\phi}_s(x, \tau) - \hat{\phi}_s(y, \tau')]^2 \rangle} e^{-\pi \langle [\hat{\theta}_s(x, \tau) - \hat{\theta}_s(y, \tau')]^2 \rangle} \\
&\quad \left(e^{\pi \langle \hat{\phi}_s(x, \tau) \hat{\theta}_s(y, \tau') \rangle} e^{\pi \langle \hat{\phi}_s(y, \tau') \hat{\theta}_s(x, \tau) \rangle} e^{\pi \langle \hat{\theta}_s(x, \tau) \hat{\phi}_s(y, \tau') \rangle} e^{\pi \langle \hat{\theta}_s(y, \tau') \hat{\phi}_s(x, \tau) \rangle} \right. \\
&\quad \left. + e^{-\pi \langle \hat{\phi}_s(x, \tau) \hat{\theta}_s(y, \tau') \rangle} e^{-\pi \langle \hat{\phi}_s(y, \tau') \hat{\theta}_s(x, \tau) \rangle} e^{-\pi \langle \hat{\theta}_s(x, \tau) \hat{\phi}_s(y, \tau') \rangle} e^{-\pi \langle \hat{\theta}_s(y, \tau') \hat{\phi}_s(x, \tau) \rangle} \right) \\
&= -\frac{1}{8} e^{-\pi \langle [\hat{\phi}_s(x, \tau) - \hat{\phi}_s(y, \tau')]^2 \rangle} e^{-\pi \langle [\hat{\theta}_s(x, \tau) - \hat{\theta}_s(y, \tau')]^2 \rangle} \\
&\quad \left(e^{2\pi \langle \hat{\phi}_s(x, \tau) \hat{\theta}_s(y, \tau') \rangle} e^{2\pi \langle \hat{\phi}_s(y, \tau') \hat{\theta}_s(x, \tau) \rangle} e^{i\frac{\pi}{2} \text{sgn}(y-x)} e^{i\frac{\pi}{2} \text{sgn}(x-y)} \right. \\
&\quad \left. + e^{-2\pi \langle \hat{\phi}_s(x, \tau) \hat{\theta}_s(y, \tau') \rangle} e^{-2\pi \langle \hat{\phi}_s(y, \tau') \hat{\theta}_s(x, \tau) \rangle} e^{-i\frac{\pi}{2} \text{sgn}(y-x)} e^{-i\frac{\pi}{2} \text{sgn}(x-y)} \right), \hspace{2em} \text{(B.71)}
\end{aligned}$$

where in the line 4 (equation B.70), $e^{\pm i\pi \text{sgn}(x-y)} = -1$ if $x \neq y$ is used. In the last line the commutator in equation B.67 is used to invert the correlations $\hat{\theta}\hat{\phi}$ into $\hat{\phi}\hat{\theta}$. Then, it can be realized that any of the contributions from the correlations $\langle \hat{\phi}(x)\hat{\theta}(y) \rangle$ has its inverse form, so they always cancel (due to the odd nature of the sign function). In the end, the full term in parenthesis of last line reduces to a constant of 2! This happens in all the four-cosine correlation that have terms of difference of fields, so from now on one should not bother with

the $\langle \hat{\phi}(x)\hat{\theta}(y) \rangle$ terms. At last, the four-cosine correlations takes the form:

$$\begin{aligned}
& \left\langle \cos \left[\sqrt{2\pi} \hat{\theta}_s(x, \tau) \right] \cos \left[\sqrt{2\pi} \hat{\phi}_s(x, \tau) \right] \cos \left[\sqrt{2\pi} \hat{\theta}_s(y, \tau') \right] \cos \left[\sqrt{2\pi} \hat{\phi}_s(y, \tau') \right] \right\rangle \\
&= -\frac{1}{4} e^{-\pi \langle [\hat{\phi}_s(x, \tau) - \hat{\phi}_s(y, \tau')]^2 \rangle} e^{-\pi \langle [\hat{\theta}_s(x, \tau) - \hat{\theta}_s(y, \tau')]^2 \rangle} \\
&= -\frac{1}{4} e^{-\pi \frac{K_s}{2\pi} \ln \left[\frac{(x-y)^2 + (v_s |t-t'| + \delta)^2}{\delta^2} \right]} e^{-\pi \frac{1}{2\pi K_s} \ln \left[\frac{(x-y)^2 + (v_s |t-t'| + \delta)^2}{\delta^2} \right]} \\
&= -\frac{1}{4} \left(\frac{\delta}{r_s} \right)^{K_s + \frac{1}{K_s}}, \tag{B.72}
\end{aligned}$$

for $|x - y|, |t - t'| \gg \delta$. Next, the correlations for the intermediate terms in the equation B.66 are computed. Although in this case all those functions have even powers of bosonic operators, the separation leaves a correlation of only one cosine. Those correlations of only one trigonometric function have the same form as the sum of fields, so they vanish exactly for the same reason:

$$\begin{aligned}
\left\langle \cos \left[\sqrt{2\pi} \hat{\phi}_c(x, \tau) + 2k_F x \right] \right\rangle &= \frac{1}{2} \left(e^{i2k_F x} e^{-\pi \langle \hat{\phi}_c(x, \tau) \hat{\phi}_c(x, \tau) \rangle} + e^{-i2k_F x} e^{-\pi \langle \hat{\phi}_c(x, \tau) \hat{\phi}_c(x, \tau) \rangle} \right) \\
&= \cos(2k_F x) e^{\pi \frac{K_c}{2\pi} \ln(\delta)} = \delta^{\frac{K_c}{2}} \cos(2k_F x) \rightarrow 0. \tag{B.73}
\end{aligned}$$

Hence, the other net contribution is from the last term in equation B.66, which can be solved directly since it is formed of two-cosine correlations already calculated before, that results in:

$$\begin{aligned}
& \left\langle \cos \left[\sqrt{2\pi} \hat{\theta}_s(x, \tau) \right] \cos \left[\sqrt{2\pi} \hat{\theta}_s(y, \tau') \right] \right\rangle \\
& \left\langle \cos \left[\sqrt{2\pi} \hat{\phi}_c(x, \tau) + 2k_F x \right] \cos \left[\sqrt{2\pi} \hat{\phi}_c(y, \tau') + 2k_F y \right] \right\rangle \\
&= \frac{1}{2} \left(\frac{\delta}{r_s} \right)^{\frac{1}{K_s}} \frac{1}{2} \cos[2k_F(x - y)] \left(\frac{\delta}{r_c} \right)^{K_c} \\
&= \frac{1}{4} \cos[2k_F(x - y)] \left(\frac{\delta}{r_s} \right)^{\frac{1}{K_s}} \left(\frac{\delta}{r_c} \right)^{K_c}, \tag{B.74}
\end{aligned}$$

for $|x - y|, |t - t'| \gg \delta$. Wrapping both terms, the planar spin density correlation in the x direction is:

$$\langle \sigma^x(x, \tau) \sigma^x(y, \tau') \rangle = \frac{1}{(\pi\delta)^2} \cos[2k_F(x - y)] \left(\frac{\delta}{r_c} \right)^{K_c} \left(\frac{\delta}{r_s} \right)^{\frac{1}{K_s}} - \frac{1}{(\pi\delta)^2} \left(\frac{\delta}{r_s} \right)^{K_s + \frac{1}{K_s}}. \tag{B.75}$$

In contrast with the previous correlations, there are two contributions to this correlation. In the stationary case, the two functions can be compared by their power laws, so one term could decay faster than the other in some cases, which allows us to discard one in further analysis [25].

The last correlation to calculate is for the planar spin density in the y direction, but it can be shown that it is equal to the correlation for the planar spin density in the x direction with the

following expansion:

$$\begin{aligned}
& \langle \hat{\sigma}^y(x, \tau) \hat{\sigma}^y(y, \tau') \rangle \\
&= \left(\frac{2}{\pi\delta} \right)^2 \left\langle \sin \left[\sqrt{2\pi} \hat{\theta}_s(x, \tau) \right] \cos \left[\sqrt{2\pi} \hat{\phi}_s(x, \tau) \right] \sin \left[\sqrt{2\pi} \hat{\theta}_s(y, \tau') \right] \cos \left[\sqrt{2\pi} \hat{\phi}_s(y, \tau') \right] \right\rangle \\
&+ \left(\frac{2}{\pi\delta} \right)^2 \left\langle \sin \left[\sqrt{2\pi} \hat{\theta}_s(x, \tau) \right] \sin \left[\sqrt{2\pi} \hat{\theta}_s(y, \tau') \right] \right\rangle \\
&\quad \left\langle \cos \left[\sqrt{2\pi} \hat{\phi}_c(x, \tau) + 2k_F x \right] \cos \left[\sqrt{2\pi} \hat{\phi}_c(y, \tau') + 2k_F y \right] \right\rangle. \tag{B.76}
\end{aligned}$$

So, using the commutator in equation B.67 and the identity $2 \sin(\alpha) \cos(\beta) = \sin(\alpha + \beta) + \sin(\alpha - \beta)$, the first term equals the four-cosine correlation of the same fields, while the last term equals the last in the correlation for the planar spin density in the x direction if the product of sine functions is expanded as well (is equal in terms of the difference of fields, as they do not vanish). Hence, the result is repeated:

$$\langle \sigma^y(x, \tau) \sigma^y(y, \tau') \rangle = \frac{1}{(\pi\delta)^2} \cos[2k_F(x - y)] \left(\frac{\delta}{r_c} \right)^{K_c} \left(\frac{\delta}{r_s} \right)^{\frac{1}{K_s}} - \frac{1}{(\pi\delta)^2} \left(\frac{\delta}{r_s} \right)^{K_s + \frac{1}{K_s}}. \tag{B.77}$$

The equality on the planar spin densities tells about the planar symmetry of the system, which is exploded in chapter 2 by selecting an arbitrary axis to describe the SOC.

These correlations are basic to describe the possible spin density orders of the system at low energies, where the system tries to order in this feature according to a transition, fundamental in the critical behavior described in all the analysis of the phases [26, 36, 40, 41]. The following is to calculate the pairing correlations.

B.5 Pairing correlations

According to the latter explanations no how to calculate the correlations, this section starts with the clarification of why the considered pairing operators are of that form, which come from the full on-site pairing operator $\hat{O}(x) = \hat{\psi}_\uparrow^\dagger(x) \hat{\psi}_\downarrow^\dagger(x)$, bosonized as:

$$\begin{aligned}
\hat{O}(x) &= \hat{\psi}_\uparrow^\dagger(x) \hat{\psi}_\downarrow^\dagger(x) = \left[e^{ik_F x} \hat{\psi}_{L\uparrow}^\dagger(x) + e^{-ik_F x} \hat{\psi}_{R\uparrow}^\dagger(x) \right] \left[e^{ik_F x} \hat{\psi}_{L\downarrow}^\dagger(x) + e^{-ik_F x} \hat{\psi}_{R\downarrow}^\dagger(x) \right] \\
&= e^{i2k_F x} \hat{\psi}_{L\uparrow}^\dagger(x) \hat{\psi}_{L\downarrow}^\dagger(x) + e^{-i2k_F x} \hat{\psi}_{R\uparrow}^\dagger(x) \hat{\psi}_{R\downarrow}^\dagger(x) + \hat{\psi}_{L\uparrow}^\dagger(x) \hat{\psi}_{R\downarrow}^\dagger(x) + \hat{\psi}_{R\uparrow}^\dagger(x) \hat{\psi}_{L\downarrow}^\dagger(x). \tag{B.78}
\end{aligned}$$

The first two terms are the non-zero momentum pairings that contribute to the full pairing, while the last two are of the form previously presented. The momentum of the pair is $-2k_F$ for a $L_\uparrow L_\downarrow$ pair, and $+2k_F$ for a $R_\uparrow R_\downarrow$ pair. The zero momentum pairings are the two last terms, which have no phase and are of opposite branches [25]. Now, using the bosonization

identity in addition, the operator takes the form:

$$\begin{aligned}
\hat{O}(x) &= e^{i2k_F x} \hat{\psi}_{L\uparrow}^\dagger(x) \hat{\psi}_{L\downarrow}^\dagger(x) + e^{-i2k_F x} \hat{\psi}_{R\uparrow}^\dagger(x) \hat{\psi}_{R\downarrow}^\dagger(x) + \hat{\psi}_{L\uparrow}^\dagger(x) \hat{\psi}_{R\downarrow}^\dagger(x) + \hat{\psi}_{R\uparrow}^\dagger(x) \hat{\psi}_{L\downarrow}^\dagger(x) \\
&= \frac{e^{i2k_F x}}{2\pi\delta} e^{i\sqrt{\pi}(\hat{\phi}_\uparrow(x) + \hat{\theta}_\uparrow(x))} e^{i\sqrt{\pi}(\hat{\phi}_\downarrow(x) + \hat{\theta}_\downarrow(x))} + \frac{e^{-i2k_F x}}{2\pi\delta} e^{-i\sqrt{\pi}(\hat{\phi}_\uparrow(x) - \hat{\theta}_\uparrow(x))} e^{-i\sqrt{\pi}(\hat{\phi}_\downarrow(x) - \hat{\theta}_\downarrow(x))} \\
&\quad + \frac{1}{2\pi\delta} e^{i\sqrt{\pi}(\hat{\phi}_\uparrow(x) + \hat{\theta}_\uparrow(x))} e^{-i\sqrt{\pi}(\hat{\phi}_\downarrow(x) - \hat{\theta}_\downarrow(x))} + \frac{1}{2\pi\delta} e^{-i\sqrt{\pi}(\hat{\phi}_\uparrow(x) - \hat{\theta}_\uparrow(x))} e^{i\sqrt{\pi}(\hat{\phi}_\downarrow(x) + \hat{\theta}_\downarrow(x))} \\
&= \frac{e^{i2k_F x}}{2\pi\delta} e^{i\sqrt{2\pi}\hat{\phi}_c(x)} e^{i\sqrt{2\pi}\hat{\theta}_c(x)} + \frac{e^{-i2k_F x}}{2\pi\delta} e^{-i\sqrt{2\pi}\hat{\phi}_c(x)} e^{i\sqrt{2\pi}\hat{\theta}_c(x)} \\
&\quad + \frac{1}{2\pi\delta} e^{i\sqrt{2\pi}\hat{\phi}_s(x)} e^{i\sqrt{2\pi}\hat{\theta}_c(x)} + \frac{1}{2\pi\delta} e^{-i\sqrt{2\pi}\hat{\phi}_s(x)} e^{i\sqrt{2\pi}\hat{\theta}_c(x)} \\
&= \frac{1}{\pi\delta} e^{i\sqrt{2\pi}\hat{\theta}_c(x)} \cos \left[\sqrt{2\pi}\hat{\phi}_c(x) + 2k_F x \right] + \frac{1}{\pi\delta} e^{i\sqrt{2\pi}\hat{\theta}_c(x)} \cos \left[\sqrt{2\pi}\hat{\phi}_s(x) \right], \tag{B.79}
\end{aligned}$$

To show that indeed the last term is more relevant than the first, the correlation is calculated, but not directly. Instead, as the structure is of $\langle \hat{O}^\dagger(x, \tau) \hat{O}(y, \tau') \rangle$ (correlation of creating and destroying a pair in two different points), the first resulting term of the product have the form of a four-cosine correlation, just that the part of $\hat{\theta}_c(x)$ fields is directly exponential, while the last term separates in two correlations, one for $\hat{\theta}_c(x)$ fields and the other for $\hat{\phi}_s(x)$ fields. As $\hat{\theta}$ fields contribute with the $1/K$ parameter and $\hat{\phi}$ with K , the correlation have the form:

$$\langle \hat{O}^\dagger(x, \tau) \hat{O}(y, \tau') \rangle \propto \left(\frac{\delta}{r_c} \right)^{K_c + \frac{1}{K_c}} + \left(\frac{\delta}{r_s} \right)^{K_s} \left(\frac{\delta}{r_c} \right)^{\frac{1}{K_c}}. \tag{B.80}$$

If the correlation is stationary, the first correlation decays as $K_c + 1/K_c$ and the second as $K_s + 1/K_c$, in which, if the interactions are attractive ($U > 0$), then $K_c > 1$ and $K_s < 1$, so $K_c > K_s$, and the non-zero momentum contribution decays faster. This is the argument behind discarding the non-zero momentum contributions, but it works generally to discard the own-branch pairings. With this clarification, the pairing correlations are calculated in the next sections, starting with the spin-singlet pairings.

B.5.1 Spin-singlet pairing correlation

The singlet pairing is defined according to the spin-singlet state $|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle$, with the operator in equation B.15 and its bosonized form in equation B.25, so the correlation takes the form:

$$\begin{aligned}
&\langle \hat{O}_{SS}^\dagger(x, \tau) \hat{O}_{SS}(y, \tau') \rangle \\
&= \left\langle \left(\frac{1}{\pi\delta} e^{-i\sqrt{2\pi}\hat{\theta}_c(x, \tau)} \cos \left[\sqrt{2\pi}\hat{\phi}_s(x, \tau) \right] \right) \left(\frac{1}{\pi\delta} e^{i\sqrt{2\pi}\hat{\theta}_c(y, \tau')} \cos \left[\sqrt{2\pi}\hat{\phi}_s(y, \tau') \right] \right) \right\rangle \\
&= \frac{1}{(\pi\delta)^2} \left\langle e^{-i\sqrt{2\pi}(\hat{\theta}_c(x, \tau) - \hat{\theta}_c(y, \tau'))} \right\rangle \left\langle \cos \left[\sqrt{2\pi}\hat{\phi}_s(x, \tau) \right] \cos \left[\sqrt{2\pi}\hat{\phi}_s(y, \tau') \right] \right\rangle \\
&= \frac{1}{(\pi\delta)^2} e^{-\pi \langle (\hat{\theta}_c(x, \tau) - \hat{\theta}_c(y, \tau'))^2 \rangle} \frac{1}{2} \left(\frac{\delta}{r_s} \right)^{K_s} \\
&= \frac{1}{(\pi\delta)^2} \left(\frac{\delta}{r_c} \right)^{\frac{1}{K_c}} \frac{1}{2} \left(\frac{\delta}{r_s} \right)^{K_s} = \frac{2}{(2\pi\delta)^2} \left(\frac{\delta}{r_s} \right)^{K_s} \left(\frac{\delta}{r_c} \right)^{\frac{1}{K_c}}, \tag{B.81}
\end{aligned}$$

As the form of the pairing operators are directly given in chiral fields, the calculation is more simple and allows to use all the previous results. The next correlation is for the axil spin-triplet pairings.

B.5.2 Axil spin-triplet pairing correlation

The axial triplet is defined according to the spin-triplet state in the S_z basis, which is $|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle$, with the operator in equation B.16 and its bosonized form in equation B.26, so the respective correlation is given as:

$$\begin{aligned}
& \langle \hat{O}_{TS}^{z\dagger}(x, \tau) \hat{O}_{TS}^z(y, \tau') \rangle \\
&= \left\langle \left(\frac{i}{\pi\delta} e^{-i\sqrt{2\pi}\hat{\theta}_c(x, \tau)} \sin \left[\sqrt{2\pi}\hat{\phi}_s(x, \tau) \right] \right) \left(-\frac{i}{\pi\delta} e^{i\sqrt{2\pi}\hat{\theta}_c(y, \tau')} \sin \left[\sqrt{2\pi}\hat{\phi}_s(y, \tau') \right] \right) \right\rangle \\
&= \frac{1}{(\pi\delta)^2} \left\langle e^{-i\sqrt{2\pi}(\hat{\theta}_c(x, \tau) - \hat{\theta}_c(y, \tau'))} \right\rangle \left\langle \sin \left[\sqrt{2\pi}\hat{\phi}_s(x, \tau) \right] \sin \left[\sqrt{2\pi}\hat{\phi}_s(y, \tau') \right] \right\rangle \\
&= \frac{2}{(2\pi\delta)^2} \left(\frac{\delta}{r_s} \right)^{K_s} \left(\frac{\delta}{r_c} \right)^{\frac{1}{K_c}}, \tag{B.82}
\end{aligned}$$

which is equal to the correlation of the spin-singlet pairing. This is similar to the equality between the correlations of charge density and axial spin density, but in fact the correlations can involve different orders although their functional form is equal [25]. The last correlations are for the planar spin-triplet pairings.

B.5.3 Planar spin-triplet pairing correlations

The planar spin-triplet pairings are defined according to the states $|\uparrow\uparrow\rangle$ and $|\downarrow\downarrow\rangle$, with operators given in equations B.17 and B.18, and bosonized form in equations B.27 and B.28, so the correlations have the form:

$$\begin{aligned}
& \langle \hat{O}_{TS}^{x\dagger}(x, \tau) \hat{O}_{TS}^x(y, \tau') \rangle \\
&= \frac{1}{(\pi\delta)^2} \left\langle e^{-i\sqrt{2\pi}(\hat{\theta}_c(x, \tau) - \hat{\theta}_c(y, \tau'))} \right\rangle \left\langle \cos \left[\sqrt{2\pi}\hat{\theta}_s(x, \tau) \right] \cos \left[\sqrt{2\pi}\hat{\theta}_s(y, \tau') \right] \right\rangle \\
&= \frac{2}{(2\pi\delta)^2} \left(\frac{\delta}{r_s} \right)^{\frac{1}{K_s}} \left(\frac{\delta}{r_c} \right)^{\frac{1}{K_c}}, \tag{B.83}
\end{aligned}$$

$$\begin{aligned}
& \langle \hat{O}_{TS}^{y\dagger}(x, \tau) \hat{O}_{TS}^y(y, \tau') \rangle \\
&= \frac{1}{(\pi\delta)^2} \left\langle e^{-i\sqrt{2\pi}(\hat{\theta}_c(x, \tau) - \hat{\theta}_c(y, \tau'))} \right\rangle \left\langle \sin \left[\sqrt{2\pi}\hat{\theta}_s(x, \tau) \right] \sin \left[\sqrt{2\pi}\hat{\theta}_s(y, \tau') \right] \right\rangle \\
&= \frac{2}{(2\pi\delta)^2} \left(\frac{\delta}{r_s} \right)^{\frac{1}{K_s}} \left(\frac{\delta}{r_c} \right)^{\frac{1}{K_c}}. \tag{B.84}
\end{aligned}$$

From the previous results, the planar symmetry is again manifested on the pairings.

To this point, all the correlations are calculated and their respective forms are power laws. The one-dimensional nature of the system emerges from these calculations as the correlations exhibit distinct decays given by the Luttinger parameters [25]. These correlations have a simpler functional form in the stationary case as discussed before, and according to the Lorentz invariance of the system, then a coordinate system change can be used, so under a scaling the spatial separation is only considered, where such scaling will not change the decay, the analysis

is completely equivalent with or without time separation [72, 73]. Also, that the correlations depend only on the separation means that one field can be evaluated at the origin and the other at any other point, so all the correlations of this work, with all the conditions previously discussed up to this point, are:

$$\langle \hat{\rho}(r) \hat{\rho}(0) \rangle = \frac{1}{(\pi\delta)^2} \cos(2k_F x) \left(\frac{\delta}{r} \right)^{K_c + K_s}, \quad (\text{B.85})$$

$$\langle \hat{\sigma}^z(r) \hat{\sigma}^z(0) \rangle = \frac{1}{(\pi\delta)^2} \cos(2k_F x) \left(\frac{\delta}{r} \right)^{K_c + K_s}, \quad (\text{B.86})$$

$$\langle \sigma^x(r) \sigma^x(0) \rangle = \frac{1}{(\pi\delta)^2} \left(\frac{\delta}{r} \right)^{K_s + \frac{1}{K_s}}, \quad (\text{B.87})$$

$$\langle \sigma^y(r) \sigma^y(0) \rangle = \frac{1}{(\pi\delta)^2} \left(\frac{\delta}{r} \right)^{K_s + \frac{1}{K_s}}, \quad (\text{B.88})$$

$$\langle \hat{O}_{SS}^\dagger(r) \hat{O}_{SS}(0) \rangle = \frac{2}{(2\pi\delta)^2} \left(\frac{\delta}{r} \right)^{K_s + \frac{1}{K_c}}, \quad (\text{B.89})$$

$$\langle \hat{O}_{TS}^{z\dagger}(r) \hat{O}_{TS}^z(0) \rangle = \frac{2}{(2\pi\delta)^2} \left(\frac{\delta}{r} \right)^{K_s + \frac{1}{K_c}}, \quad (\text{B.90})$$

$$\langle \hat{O}_{TS}^{x\dagger}(r) \hat{O}_{TS}^x(0) \rangle = \frac{2}{(2\pi\delta)^2} \left(\frac{\delta}{r} \right)^{\frac{1}{K_s} + \frac{1}{K_c}}, \quad (\text{B.91})$$

$$\langle \hat{O}_{TS}^{y\dagger}(r) \hat{O}_{TS}^y(0) \rangle = \frac{2}{(2\pi\delta)^2} \left(\frac{\delta}{r} \right)^{\frac{1}{K_s} + \frac{1}{K_c}}, \quad (\text{B.92})$$

with $r = |x|$, given $r \gg \delta$, and $K_c > K_s$. Notice that in the planar spin density correlations there is not the $e^{i\pi}$ phase that appeared in the equation B.72. That phase can be fixed by recalling the Klein factors and doing the anticommutations which will result in an extra $e^{i\pi}$. The behavior is regular in the sense that, as r grows the correlation tends to zero (points too separated), while it diverges close to $r = 0$ (points too close). The distinction of this behavior is the way they decay, and it is fundamental for the prediction of phases under the LRT. The direct observables considered for this system, and the system of chapter 2, are the susceptibilities, which are presented and calculated in the next section.

B.6 Susceptibilities

According to the LRT, the correlation is connected with the susceptibility via the equation 1.17. As the external force that perturbs the random order of the system could be of any kind, it is not specified in the calculation. Instead, it is said that the susceptibility is equal to the Fourier transform of the correlation, although it is not strictly true [25, 77]. Recalling the simile with the magnetic susceptibility done in the introduction, the order is established if the susceptibility diverges at the determined excitation momentum, it does not matter whether the force appears at the equation or not, a minimum introduction of it makes the susceptibility diverge [25, 26].

In this section, the Fourier transform of all the correlations calculated in the latter section is computed, but not in a complete calculation since that all the correlations have the same functional form, a power law. Definitely the resultant transforms are not exactly the same for

each property of the system, but the power law is the one that controls the decay and hence the behavior, more than any other feature. Thus, it is enough to evaluate how the transform of a power law is and how it behaves according to the divergence, which does not have to do anything with the constants, or external parameters outside the decay. That is why the only direct calculation is done for a function $f(r) = cr^{-\alpha}$, where $\alpha > 0$ is the power law and c is a constant.

For a power law function $f(r) = cr^{-\alpha}$, they satisfy $f(\lambda r) = \lambda^{-\alpha} f(r)$ with λ a constant. It can be shown that the Fourier transform of a power law satisfies the same property, so the Fourier transform of a power law is a power law too. First, the Fourier transform for the function $f(r)$ is written according to the equation 1.3 as follows:

$$\tilde{f}(\vec{p}) = \frac{1}{2\pi} \int d\vec{r} f(\vec{r}) e^{-i\vec{p} \cdot \vec{r}}, \quad (\text{B.93})$$

in which the elements are vectors in general, this is, $\vec{r} = (x, \nu t)$ and $\vec{p} = (p, \omega)$. Using the rotational symmetry $f(\vec{r}) = f(r)$, so the easier coordinate system to solve the integral is the polar system, where $\vec{p} \cdot \vec{r} = rp \cos \varphi$ and $d\vec{r} = r dr d\varphi$, with $p = |\vec{p}|$ the norm of the momentum vector. The Fourier transform is now:

$$\tilde{f}(\vec{p}) = \frac{1}{2\pi} \int_0^\infty \int_0^{2\pi} r dr d\varphi f(r) e^{-irp \cos \varphi} = \frac{1}{2\pi} \int_0^\infty r dr f(r) \int_0^{2\pi} d\varphi e^{-irp \cos \varphi} = \tilde{f}(p). \quad (\text{B.94})$$

Due to the rotation symmetry of the function $f(r)$, the integral separates into (r, ϕ) perfectly, and the remaining variable is p , so its Fourier transform is also rotational invariant in the reciprocal space [25, 29, 78]. Second, the condition to be a power law is proved by scaling the argument:

$$f(\tilde{\lambda} p) = \frac{1}{2\pi} \int_0^\infty \int_0^{2\pi} r dr d\varphi f(r) e^{-ir\lambda p \cos \varphi}. \quad (\text{B.95})$$

Next, a change of variable $u = \lambda r$ is set, so $u \in (0, \infty)$, $r = u/\lambda$, $dr = du/\lambda$ and $f(r) = f(u/\lambda) = (1/\lambda)^{-\alpha} f(u)$. Then, the latter integral becomes:

$$\begin{aligned} \tilde{f}(\lambda p) &= \frac{1}{2\pi} \int_0^\infty \int_0^{2\pi} \frac{u}{\lambda} \frac{du}{\lambda} d\varphi \frac{f(u)}{\lambda^{-\alpha}} e^{-iup \cos \varphi} = \frac{1}{\lambda^{2-\alpha}} \frac{1}{2\pi} \int_0^\infty \int_0^{2\pi} u du d\varphi f(u) e^{-iup \cos \varphi} \\ &= \lambda^{-(2-\alpha)} \tilde{f}(p). \end{aligned} \quad (\text{B.96})$$

It has been shown that the Fourier transform of a power law is also a power law with $\beta = 2 - \alpha > 0$ the power law in the momentum space [25, 29, 78]. This is used to find the behavior of the susceptibilities given the correlations, without any explicit calculation. To give an example, it can be done explicitly for the correlation of the planar spin density in the x direction, reviewing the equation B.87. In that case, the correlation is written as:

$$\langle \sigma^x(r) \sigma^x(0) \rangle = cr^{-\left(K_s + \frac{1}{K_s}\right)} \propto r^{-\left(K_s + \frac{1}{K_s}\right)}, \quad (\text{B.97})$$

where the susceptibilities are denoted with a sub index of the property and super index of its direction, and in this case $c = \delta^{K_s+1/K_s}/(\pi\delta)^2$. Notice that it does not matter if the constant

has the Luttinger parameters, the relevant form is the function of r . In fact, at the end the proportion is written because the constant is not relevant to the decay. In this correlation the power law is $\alpha = K_s + 1/K_s$, so the corresponding susceptibility is given as:

$$\chi_\sigma^x(p) \propto p^{-\left(2 - \left(K_s + \frac{1}{K_s}\right)\right)}, \quad (\text{B.98})$$

in which no constant is set, as the power law is the main feature for the decay, so in the following susceptibilities they are only given as proportions (see [78] for a direct derivation of the constant). The latter calculation is the same for many of the correlations which only have the functional form of a power law, but there are others that have also a cosine function, like for the charge density or axial spin density in equations B.85 and B.86 respectively. In that case, the cosine is expanded in its exponential parts so the Fourier transform has two contributions exponential contributions in product, like $e^{\pm i 2k_F x} r^{-\alpha}$, and the transform of such product is equal to the transform of the power law but displaced by the phase of the exponential [29]. The fact is that the exponential $e^{\pm i 2k_F x}$ does not depend explicitly on $|x| = r$, so the displacement is only on the momentum component. Considering the stationary correlations, in the Fourier space only the momenta is considered (not frequencies), so it can be applied in that part.

With all the latter said, the susceptibilities for the correlations are:

$$\chi_\rho(p) \propto \left(\frac{1}{|p - 2k_F|}\right)^{2 - (K_c + K_s)} + \left(\frac{1}{|p + 2k_F|}\right)^{2 - (K_c + K_s)}, \quad (\text{B.99})$$

$$\chi_\sigma^z(p) \propto \left(\frac{1}{|p - 2k_F|}\right)^{2 - (K_c + K_s)} + \left(\frac{1}{|p + 2k_F|}\right)^{2 - (K_c + K_s)}, \quad (\text{B.100})$$

$$\chi_\sigma^x(p) \propto \left(\frac{1}{|p|}\right)^{2 - \left(K_s + \frac{1}{K_s}\right)}, \quad (\text{B.101})$$

$$\chi_\sigma^y(p) \propto \left(\frac{1}{|p|}\right)^{2 - \left(K_s + \frac{1}{K_s}\right)}, \quad (\text{B.102})$$

$$\chi_{SS}(p) \propto \left(\frac{1}{|p|}\right)^{2 - \left(K_s + \frac{1}{K_c}\right)}, \quad (\text{B.103})$$

$$\chi_{TS}^z(p) \propto \left(\frac{1}{|p|}\right)^{2 - \left(K_s + \frac{1}{K_c}\right)}, \quad (\text{B.104})$$

$$\chi_{TS}^x(p) \propto \left(\frac{1}{|p|}\right)^{2 - \left(\frac{1}{K_s} + \frac{1}{K_c}\right)}, \quad (\text{B.105})$$

$$\chi_{TS}^y(p) \propto \left(\frac{1}{|p|}\right)^{2 - \left(\frac{1}{K_s} + \frac{1}{K_c}\right)}. \quad (\text{B.106})$$

With the given forms of the susceptibilities, they are analyzed to conclude the possible phases that could exist in the system under certain values of the parameters, which is the discussed matter in the next section.

B.7 Analysis of the possible phases

Recalling that the power law decay indicates that the system wants to order, hence the previous condition of $\beta = 2 - \alpha > 0$ is the one discussed in the introduction that makes the

susceptibilities diverge, as the power law is positive. For the latter susceptibilities, the possible orders and their conditions are:

- The system wants to order in charge density or axial spin density if $2 - (K_c + K_s) > 0 \rightarrow K_c + K_s < 2$.
- The system wants to order in planar spin density if $2 - (K_s + \frac{1}{K_s}) > 0 \rightarrow K_s + \frac{1}{K_s} < 2$.
- The system wants to order in spin-singlet pairs or axial spin-triplet pairs if $2 - (K_s + \frac{1}{K_c}) > 0 \rightarrow K_s + \frac{1}{K_c} < 2$.
- The system wants to order in planar spin-triplet pairs if $2 - (\frac{1}{K_s} + \frac{1}{K_c}) > 0 \rightarrow \frac{1}{K_s} + \frac{1}{K_c} < 2$.

This is the reason of why the behavior of the system depends on the values of the Luttinger parameters [25]. According to their definitions in the equation B.8, if the interactions are attractive $K_c > 1$ and $K_s < 1$, and if the interactions are repulsive $K_c < 1$ and $K_s > 1$. Notice then that there is a critical change around $U = 0$, which means there is a transition between such behaviors at zero temperature, this is the quantum phase transition of the system [25]. It can be seen that in the attractive case, the only susceptibility that diverges for any $U > 0$ is for the spin-singlet pairing and the axial spin-triplet pairing, the others will depend on specific values of U . The bosonization procedure shows the superconducting behavior of the one-dimensional system as with any other methods in higher dimensions [25, 34, 35]. But, one has to be careful, because that phases are possible which does not mean they indeed happen, for the true establishing of the phases there are still two analyses to do, that is to show how the divergence of the susceptibility implies the order, and how the orders of the same susceptibilities are distinguished.

The divergence implies a high response of the property associated with the susceptibility, so that property does not average to zero anymore. For example, in the case of the correlation for the charge density, its susceptibility diverges and so the charge wants to take a constant value. Using the definition of the charge density in terms of the fields in equation A.69, and using the transformation of charge and spin excitations in equations B.3 and B.4, it is shown that:

$$\hat{\phi}_c(x) = \sqrt{\frac{\pi}{2}} \int_{-\infty}^x dx' \hat{\rho}(x'). \quad (\text{B.107})$$

This means that the charge field of the theory evaluated in the point x is the total charge of the system from the left end of the one-dimensional gas up to the point x . As the charge wants to order, this field takes a constant value a (the charge deviates from the mean charge), so $\hat{\phi}_c(x) = a$. If this condition is satisfied, then one must revisit the bosonized operators, that in this case gives:

$$\begin{aligned} \hat{\rho}(x) &= \sqrt{\frac{2}{\pi}} \partial_x \hat{\phi}_c(x) + \frac{2}{\pi\delta} \cos \left[\sqrt{2\pi} \hat{\phi}_c(x) + 2k_F x \right] \cos \left[\sqrt{2\pi} \hat{\phi}_s(x) \right] \\ &= \frac{2}{\pi\delta} \cos \left[\sqrt{2\pi} a + 2k_F x \right] \cos \left[\sqrt{2\pi} \hat{\phi}_s(x) \right] \\ &\propto \cos [2k_F x + \alpha], \end{aligned} \quad (\text{B.108})$$

where the last form is possible considering that if the charge field is constant then $\sqrt{2\pi} a = \alpha$ is just a phase, and as the spin field is not ordered, it just introduces some fluctuations to the main functional form of the operator. The final form of the operator means that the charge is

oscillating around the average with an inverse wavelength of $2k_F$ as illustrated in figure B.1, in which is said that the system enters in a *charge density wave* phase, denoted as a CDW phase [25].

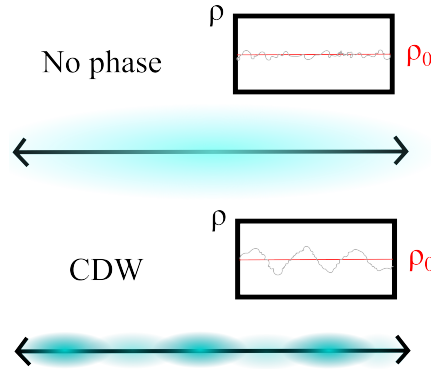


Figure B.1: Illustration of the CDW phase. The top figure, where there is no phase, shows that the charge is equally distributed (cyan contour), so the box has a graph where there are quantum fluctuations of the charge around the average ρ_0 . In the bottom figure, there is a CDW phase, so the charge is greater and minor in some regions according to the oscillating form of the box, which also has little fluctuations from the $\hat{\phi}_s(x)$ field.

In the case of the correlation of the axial spin density, the resultant ordering is the same but in terms of the spin fluctuations, and it is said that the system enters an axial *spin density wave*, which is denoted as a SDW phase [25].

Taking advantage of the latter two correlations discussed, it was shown that they have the same form, so the remaining question is: how the system finally orders in one or the other? When the system is not ordered, then there is a translational symmetry for those properties, this is, no matter what region is, the average charge and spin in that region is the mean charge density and spin density multiplied by the size of the region, respectively. When there is a CDW phase, that translational symmetry would be broken, there is more charge in one part than others, and if there is a SDW phase there is more spin $\uparrow$ in some regions and more spin $\downarrow$ in other regions. The problem with these orders are that symmetries have to be spontaneously broken by the transition, but **that cannot happen in one-dimensional systems at finite temperature** according to the Mermin-Wagner theorem [25–28], the perfect order is disrupted by the strong quantum fluctuations of these systems. Then, if one wants one of the orders to really happen, that symmetry has to be broken from the start or to be discrete, otherwise the order is called a *quasi-order*. The translation spin density symmetry could be broken by introducing modulated magnetic fields, and the charge density symmetry with a modulated chemical potential, for example. Also, the broken spin symmetry in terms of the oscillations implies that the charge density symmetry is preserved, so the system would tend to go from one phase to the other depending on the strength of the couplings that break the symmetry. This is how the system really enters some phases, although the susceptibilities are equal.

Finally, after all the previous analysis for the correlations and susceptibilities, the second and last analysis is the real order of the system by means of its interaction term, that so far is incomplete because the cosine term in the Hamiltonian of equation B.10 has not been included. It is time to discuss the way of introducing it for the purpose of this work in the

next section, and hence revealing the real phase transition of the system by means of the interaction term.

B.8 Strong interaction limit

According to previously mentioned, the cosines are introduced in two limits: the weak interaction limit and the strong interaction limit.

The weak interaction limit considers that the amplitude of the cosine is infinitesimal, that is, $U \rightarrow 0$, and the cosine contribution of the Hamiltonian is expanded up to first or second order in the correlations from the approach of the Path Integral in the equation B.31. Hence, the correlations calculated before are corrected by perturbative terms that change the Luttinger parameters K to others that are said effective, K_{eff} . This procedure resumes the renormalization group method, which is not explored in this work for the phase diagram of the main system, and thus not developed.

The strong interaction limit considers the opposite, that the amplitude of the cosine tends to infinity, $U \rightarrow +\infty$, so the field of the cosine gets trapped in one its minima [25]. Then, an expansion of the cosine around one of those minima can be done as follows:

$$\begin{aligned} -\frac{2U}{(2\pi\delta)^2} \int dx \cos \left[\sqrt{8\pi} \hat{\phi}_s(x) \right] &\approx -\frac{2U}{(2\pi\delta)^2} \int dx \left[1 - \frac{1}{2} \left(\sqrt{8\pi} \hat{\phi}_s(x) \right)^2 \right] \\ &= \frac{2\pi U}{(\pi\delta)^2} \int dx \hat{\phi}_s(x)^2 + c, \end{aligned} \quad (\text{B.109})$$

where the constant c is just part of the expansion that does not depend on the field. Notice that now there is a quadratic contribution of the field to the Hamiltonian, so it takes the form:

$$\begin{aligned} \hat{H} &\approx \frac{\nu_c}{2} \int dx \left[\frac{1}{K_c} \left(\partial_x \hat{\phi}_c(x) \right)^2 + K_c \left(\partial_x \hat{\theta}_c(x) \right)^2 \right] \\ &+ \frac{\nu_s}{2} \int dx \left[\frac{1}{K_s} \left(\partial_x \hat{\phi}_s(x) \right)^2 + K_s \left(\partial_x \hat{\theta}_s(x) \right)^2 + \frac{4\pi U}{\nu_s(\pi\delta)^2} \hat{\phi}_s(x)^2 \right] \\ &= \frac{\nu_c}{2} \int dx \left[\left(\partial_x \hat{\phi}_c(x) \right)^2 + \left(\partial_x \hat{\theta}_c(x) \right)^2 \right] \\ &+ \frac{\nu_s}{2} \int dx \left[\left(\partial_x \hat{\phi}_s(x) \right)^2 + \left(\partial_x \hat{\theta}_s(x) \right)^2 + \frac{4\pi K_s U}{\nu_s(\pi\delta)^2} \hat{\phi}_s(x)^2 \right]. \end{aligned} \quad (\text{B.110})$$

Hence, it can be seen that the spin part of the Hamiltonian acquires a *gap*, that is the meaning of the quadratic contribution [25]. This is because from the point of view of the Klein-Gordon equation, the quadratic term introduces a mass in the spectra since the Hamiltonian is $\hat{H} = \int dx [(\nabla \hat{\theta}(x))^2 + (\nabla \hat{\phi}(x))^2 + m^2 \hat{\phi}(x)^2]$ [26–28]. The mass term would be in this case:

$$M_s = \sqrt{\frac{4\pi K_s U}{\nu_s(\pi\delta)^2}} = \frac{2}{\pi\delta} \sqrt{\frac{\pi K_s U}{\nu_s}}. \quad (\text{B.111})$$

This means that, under strong attractive interactions, the system enters a massive phase, it makes a phase transition from a massless phase (repulsive interactions, renormalization solution), to a massive phase (attractive interactions, strong interaction limit solution) [25].

In this scheme, the correlations change by expanding the cosine around zero, which is a maximum of the cosine, not a minimum. This is done because the amplitude in $U \rightarrow +\infty$ is negative (it has a global minus sign), so there is a gap if the cosine takes a positive extreme value, which is 1 (the expansion depends on the specific global sign of the term). Hence, the field in the cosine gets pinned to the values that make it to be 1, which are:

$$\cos \left[\sqrt{8\pi} \hat{\phi}_s(x) \right] = 1 \rightarrow \sqrt{8\pi} \hat{\phi}_s(x) = 0 + 2\pi n \rightarrow \hat{\phi}_s(x) = \frac{2\pi n}{\sqrt{8\pi}}, \quad U \rightarrow +\infty, \quad (\text{B.112})$$

for any integer n . As the $\hat{\phi}_s(x)$ field is constant, then its reciprocal field $\hat{\theta}_s(x)$ oscillates, so the expected values of their trigonometric functions satisfy:

$$\left\langle \cos \left[\sqrt{2\pi} \hat{\phi}_s(x) \right] \right\rangle = b, \quad \left\langle \sin \left[\sqrt{2\pi} \hat{\phi}_s(x) \right] \right\rangle = 0, \quad \left\langle \cos \left[a \hat{\theta}_s(x) \right] \right\rangle = \left\langle \sin \left[a \hat{\theta}_s(x) \right] \right\rangle = 0, \quad (\text{B.113})$$

where b is a constant and a could be any constant parameter in the trigonometric functions of the fields. These expected values are replaced while calculating the correlations. For example, any term that has trigonometric functions of $\hat{\theta}_s(x)$ vanish, and only cosine functions of $\hat{\phi}_s(x)$ fields contribute. In fact, also the power law changes because $\hat{\phi}_s(x)$ does not contribute with its Luttinger parameter K_s , since its expected value is just a constant that does not change the decay behavior of the other fields ($\hat{\phi}_c(x)$ and $\hat{\theta}_c(x)$) [25].

So, if one checks all the correlations, it can be seen that:

- The correlations for the spin densities in equations B.64, B.66 and B.76, and the spin-triplet pairings in equations B.82, B.83 and B.84, vanish due to the direct vanishing of either the sine function of the $\hat{\phi}_s(x)$ field or the trigonometric functions of the $\hat{\theta}_s$.
- The only correlations that decay as power laws are for the charge density and the spin-singlet pairing.

Thus, in the massive region of the phase diagram there can only be a charge density order or a one-dimensional superconducting order. The two phases could overlap in the phase diagram, but as there is a BCS channel from the interaction term (the cosine function), then the system will prefer to be in the superconducting state, so it becomes a real phase (not quasi-order) [25]. It also must be mentioned that in the case where $U \rightarrow -\infty$, the minimum is reached if the field is trapped in one of the minima of the cosine function, so the constant value of the field is:

$$\hat{\phi}_s(x) = \frac{\pi}{\sqrt{8\pi}} + \frac{2\pi n}{\sqrt{8\pi}}, \quad (\text{B.114})$$

for any integer n . In this case the averages are given as:

$$\left\langle \cos \left[\sqrt{2\pi} \hat{\phi}_s(x) \right] \right\rangle = 0, \quad \left\langle \sin \left[\sqrt{2\pi} \hat{\phi}_s(x) \right] \right\rangle = b, \quad \left\langle \cos \left[a \hat{\theta}_s(x) \right] \right\rangle = \left\langle \sin \left[a \hat{\theta}_s(x) \right] \right\rangle = 0, \quad (\text{B.115})$$

which means the behavior is exchanged, the system prefer to be in spin density orders and axial spin-triplet pairings, while the charge density and spin-singlet correlations average to zero. This is the real phase transition of this system we want to get for the system of chapter 2, where all the previous analyses in this appendix are used to make the calculations quicker and easier to understand that system. This appendix and the previous appendix A summarize most of the fundamental analysis that has to be done when starting to solve and characterize a system using bosonization.

Appendix C

Extended calculations

This appendix presents the extended or full calculations of many results presented along the document, where it refers to the equation and then shows the complete procedure.

C.1 Determination of the dispersion relation of the coupled Fermi gas

The first procedure is from the result of equation 3.1, achieved as follows:

$$\begin{aligned}
\hat{H} &= \int dx \hat{\Psi}_{cs}^\dagger \left(-\frac{\partial_x^2}{2m} - \mu + \left(-i\alpha\partial_x + \frac{h}{2} \right) \sigma^z \right) \hat{\Psi}_{cs} \\
&= \int dx \left[\hat{\psi}_c^\dagger(x) \left(-\frac{\partial_x^2}{2m} - \mu + \frac{h}{2} \right) \hat{\psi}_c(x) + \hat{\psi}_s^\dagger(x) \left(-\frac{\partial_x^2}{2m} - \mu - \frac{h}{2} \right) \hat{\psi}_s(x) \right. \\
&\quad \left. - i\alpha\hat{\psi}_c^\dagger(x)\partial_x\hat{\psi}_c(x) + i\alpha\hat{\psi}_s^\dagger(x)\partial_x\hat{\psi}_s(x) \right] \\
&= \int dx \left[\frac{1}{\sqrt{2\pi}} \int dk e^{ikx} \hat{\psi}_c^\dagger(k) \left(-\frac{\partial_x^2}{2m} - \mu + \frac{h}{2} \right) \frac{1}{\sqrt{2\pi}} \int dq e^{-iqx} \hat{\psi}_c(q) \right. \\
&\quad + \frac{1}{\sqrt{2\pi}} \int dk e^{ikx} \hat{\psi}_s^\dagger(k) \left(-\frac{\partial_x^2}{2m} - \mu - \frac{h}{2} \right) \frac{1}{\sqrt{2\pi}} \int dq e^{-iqx} \hat{\psi}_s(q) \\
&\quad - i\alpha \frac{1}{\sqrt{2\pi}} \int dk e^{ikx} \hat{\psi}_c^\dagger(k) \partial_x \frac{1}{\sqrt{2\pi}} \int dq e^{-iqx} \hat{\psi}_c(q) \\
&\quad \left. + i\alpha \frac{1}{\sqrt{2\pi}} \int dk e^{ikx} \hat{\psi}_s^\dagger(k) \partial_x \frac{1}{\sqrt{2\pi}} \int dq e^{-iqx} \hat{\psi}_s(q) \right] \\
&= \int dx \left[\frac{1}{2\pi} \int dk dq e^{ikx} \left(-\frac{\partial_x^2}{2m} - i\alpha\partial_x - \mu + \frac{h}{2} \right) e^{-iqx} \hat{\psi}_c^\dagger(k) \hat{\psi}_c(q) \right. \\
&\quad \left. + \frac{1}{2\pi} \int dk dq e^{ikx} \left(-\frac{\partial_x^2}{2m} + i\alpha\partial_x - \mu - \frac{h}{2} \right) e^{-iqx} \hat{\psi}_s^\dagger(k) \hat{\psi}_s(q) \right]
\end{aligned}$$

$$\begin{aligned}
&= \int dx \left[\frac{1}{2\pi} \int dk dq e^{ix(k-q)} \left(\frac{q^2}{2m} + \alpha q - \mu + \frac{h}{2} \right) \hat{\psi}_c^\dagger(k) \hat{\psi}_c(q) \right. \\
&\quad \left. + \frac{1}{2\pi} \int dk dq e^{ix(k-q)} \left(\frac{q^2}{2m} - \alpha q - \mu - \frac{h}{2} \right) \hat{\psi}_s^\dagger(k) \hat{\psi}_s(q) \right] \\
&= \int dk dq \left[\left(\frac{1}{2\pi} \int dx e^{ix(k-q)} \right) \left(\frac{q^2}{2m} + \alpha q - \mu + \frac{h}{2} \right) \hat{\psi}_c^\dagger(k) \hat{\psi}_c(q) \right. \\
&\quad \left. + \left(\frac{1}{2\pi} \int dx e^{ix(k-q)} \right) \left(\frac{q^2}{2m} - \alpha q - \mu - \frac{h}{2} \right) \hat{\psi}_s^\dagger(k) \hat{\psi}_s(q) \right] \\
&= \int dk dq \delta(k-q) \left[\left(\frac{q^2}{2m} + \alpha q - \mu + \frac{h}{2} \right) \hat{\psi}_c^\dagger(k) \hat{\psi}_c(q) \right. \\
&\quad \left. + \left(\frac{q^2}{2m} - \alpha q - \mu - \frac{h}{2} \right) \hat{\psi}_s^\dagger(k) \hat{\psi}_s(q) \right] \\
&= \int dk \left[\left(\frac{k^2 + 2m\alpha k}{2m} - \mu + \frac{h}{2} \right) \hat{\psi}_c^\dagger(k) \hat{\psi}_c(k) \right. \\
&\quad \left. + \left(\frac{k^2 - 2m\alpha k}{2m} - \mu - \frac{h}{2} \right) \hat{\psi}_s^\dagger(k) \hat{\psi}_s(k) \right] \\
&= \int dk \left[\frac{(k + m\alpha)^2}{2m} - \frac{1}{2}m\alpha^2 - \mu + \frac{h}{2} \right] \hat{\psi}_c^\dagger(k) \hat{\psi}_c(k) + \\
&\quad + \int dk \left[\frac{(k - m\alpha)^2}{2m} - \frac{1}{2}m\alpha^2 - \mu - \frac{h}{2} \right] \hat{\psi}_s^\dagger(k) \hat{\psi}_s(k) \\
&= \int dk (\epsilon_c(k) - \mu_c) \hat{\psi}_c^\dagger(k) \hat{\psi}_c(k) + \int dk (\epsilon_s(k) - \mu_s) \hat{\psi}_s^\dagger(k) \hat{\psi}_s(k). \tag{C.1}
\end{aligned}$$

The Fermi momenta of the coupled Fermi gas in equations 3.4 and 3.5 are derived equaling the energy of the dispersion relation with the Fermi energy, which are the chemical potentials. When doing it, due to the root square there are two solutions, so every curve has two points, which in total are four different points (in contrast with the free system that has two different points). The explicit calculation is the following:

$$\epsilon_c(k) = \mu_c \rightarrow \frac{(k + k_0)^2}{2m} = \mu + \frac{1}{2}m\alpha^2 - \frac{h}{2} \rightarrow k + k_0 = \pm \sqrt{2m\mu + m^2\alpha^2 - mh} = \pm k_{F_c},$$

$$\epsilon_s(k) = \mu_s \rightarrow \frac{(k - k_0)^2}{2m} = \mu + \frac{1}{2}m\alpha^2 + \frac{h}{2} \rightarrow k - k_0 = \pm \sqrt{2m\mu + m^2\alpha^2 + mh} = \pm k_{F_s},$$

that leads to the same definition of k_{R_c} , k_{L_c} , k_{R_s} and k_{L_s} in chapter 3.

C.2 Bosonization of the fermionic-field densities

The C and S densities are mapped by expanding the bosonization map of the fermionic fields in equations 3.6 and 3.7, where the resultant expansions are shown in equations 3.17 and 3.18.

The extended calculation is:

$$\begin{aligned}
\hat{\psi}_c^\dagger(x)\hat{\psi}_c(x) &= \\
&\left[e^{-i(-k_{F_c}-k_0)}\hat{\psi}_{L_c}^\dagger(x) + e^{-i(k_{F_c}-k_0)}\hat{\psi}_{R_c}^\dagger(x) \right] \left[e^{i(-k_{F_c}-k_0)}\hat{\psi}_{L_c}(x) + e^{i(k_{F_c}-k_0)}\hat{\psi}_{R_c}(x) \right] \\
&= \hat{\psi}_{R_c}^\dagger(x)\hat{\psi}_{R_c}(x) + \hat{\psi}_{L_c}^\dagger(x)\hat{\psi}_{L_c}(x) + e^{-i2k_{F_c}x}\hat{\psi}_{R_c}^\dagger(x)\hat{\psi}_{L_c}(x) + e^{i2k_{F_c}x}\hat{\psi}_{L_c}^\dagger(x)\hat{\psi}_{R_c}(x) \\
&= \frac{1}{\sqrt{\pi}}\partial_x\hat{\phi}_{R_c}(x) + \frac{1}{\sqrt{\pi}}\partial_x\hat{\phi}_{L_c}(x) \\
&\quad + \frac{e^{-i2k_{F_c}x}}{2\pi\delta}e^{-2\sqrt{\pi}\hat{\phi}_{R_c}(x)}e^{-2\sqrt{\pi}\hat{\phi}_{L_c}(x)} + \frac{e^{i2k_{F_c}x}}{2\pi\delta}e^{2\sqrt{\pi}\hat{\phi}_{L_c}(x)}e^{2\sqrt{\pi}\hat{\phi}_{R_c}(x)} \\
&= \frac{1}{\sqrt{\pi}}\partial_x\hat{\phi}_c(x) + \frac{e^{-i2k_{F_c}x}}{2\pi\delta}e^{-2\sqrt{\pi}\hat{\phi}_c(x)} + \frac{e^{i2k_{F_c}x}}{2\pi\delta}e^{2\sqrt{\pi}\hat{\phi}_c(x)} \\
&= \frac{1}{\sqrt{\pi}}\partial_x\hat{\phi}_c(x) + \frac{1}{\pi\delta}\cos\left[2\sqrt{\pi}\hat{\phi}_c(x) + 2k_{F_c}x\right], \tag{C.2}
\end{aligned}$$

$$\hat{\psi}_s^\dagger(x)\hat{\psi}_s(x) = \frac{1}{\sqrt{\pi}}\partial_x\hat{\phi}_s(x) + \frac{1}{\pi\delta}\cos\left[2\sqrt{\pi}\hat{\phi}_s(x) + 2k_{F_s}x\right], \tag{C.3}$$

where the normal ordered densities are replaced, given in terms of the fields in the equation A.75. Notice the similarity with the calculation done in the equation A.95 of the appendix A, the quantum numbers of the fields and phases simply are changed, but the form is the same. On the other hand, for the cross densities (products of C and S fermionic fields) in equations 3.19 and 3.20, the complete expansion is:

$$\begin{aligned}
\hat{\psi}_c^\dagger(x)\hat{\psi}_s(x) &= \\
&\left[e^{-i(k_{F_c}+k_0)x}\hat{\psi}_{L_c}^\dagger(x) + e^{-i(k_{F_c}-k_0)x}\hat{\psi}_{R_c}^\dagger(x) \right] \left[e^{i(-k_{F_s}+k_0)x}\hat{\psi}_{L_s}(x) + e^{i(k_{F_s}+k_0)x}\hat{\psi}_{R_s}(x) \right] \\
&= e^{i(k_{F_c}-k_{F_s})x}e^{i2k_0x}\hat{\psi}_{L_c}^\dagger(x)\hat{\psi}_{L_s}(x) + e^{i(k_{F_c}+k_{F_s})x}e^{i2k_0x}\hat{\psi}_{L_c}^\dagger(x)\hat{\psi}_{R_s}(x) \\
&\quad + e^{-i(k_{F_c}+k_{F_s})x}e^{i2k_0x}\hat{\psi}_{R_c}^\dagger(x)\hat{\psi}_{L_s}(x) + e^{-i(k_{F_c}-k_{F_s})x}e^{i2k_0x}\hat{\psi}_{R_c}^\dagger(x)\hat{\psi}_{R_s}(x) \\
&= \frac{1}{\pi\delta}e^{i2k_0x}e^{i\sqrt{\pi}(\hat{\theta}_c(x)-\hat{\theta}_s(x))}\cos\left[\sqrt{\pi}\left(\hat{\phi}_c(x)-\hat{\phi}_s(x)\right) + (k_{F_c}-k_{F_s})x\right] \\
&\quad + \frac{1}{\pi\delta}e^{i2k_0x}e^{i\sqrt{\pi}(\hat{\theta}_c(x)-\hat{\theta}_s(x))}\cos\left[\sqrt{\pi}\left(\hat{\phi}_c(x)+\hat{\phi}_s(x)\right) + (k_{F_c}+k_{F_s})x\right], \tag{C.4}
\end{aligned}$$

$$\begin{aligned}
\hat{\psi}_s^\dagger(x)\hat{\psi}_c(x) &= \frac{1}{\pi\delta}e^{-i2k_0x}e^{-i\sqrt{\pi}(\hat{\theta}_c(x)-\hat{\theta}_s(x))}\cos\left[\sqrt{\pi}\left(\hat{\phi}_c(x)-\hat{\phi}_s(x)\right) + (k_{F_c}-k_{F_s})x\right] \\
&\quad + \frac{1}{\pi\delta}e^{-i2k_0x}e^{-i\sqrt{\pi}(\hat{\theta}_c(x)-\hat{\theta}_s(x))}\cos\left[\sqrt{\pi}\left(\hat{\phi}_c(x)+\hat{\phi}_s(x)\right) + (k_{F_c}+k_{F_s})x\right]. \tag{C.5}
\end{aligned}$$

C.3 Refermionization of the cosine terms

In chapter 3 the resultant cosine terms of the bosonized Hamiltonian are refermionized to understand the processes they introduced to the model which separates the solutions for the system from the harmonic oscillator. The extended calculation of that refermionization is given in the next equations. The refermionization of the cosine in equation 3.37 is done as follows:

$$\begin{aligned}
& \cos \left[\sqrt{2\pi} \left(\hat{\phi}_-(x) + \hat{\theta}_-(x) \right) + (k_{F_c} - k_{F_s} + 2k_0)x \right] \\
&= \frac{1}{2} e^{i(k_{F_c} - k_{F_s} + 2k_0)x} e^{i\sqrt{2\pi}(\hat{\phi}_-(x) + \hat{\theta}_-(x))} + h.c \\
&= \frac{1}{2} e^{i(k_{F_c} - k_{F_s} + 2k_0)x} e^{i\sqrt{\pi}(\hat{\phi}_c(x) - \hat{\phi}_s(x) + \hat{\theta}_c(x) - \hat{\theta}_s(x))} + h.c \\
&= \frac{1}{2} e^{i(k_{F_c} - k_{F_s} + 2k_0)x} e^{i2\sqrt{\pi}(\hat{\phi}_{L_c}(x) - \hat{\phi}_{L_s}(x))} + h.c \\
&= \frac{2\pi\delta}{2} e^{i(k_{F_c} - k_{F_s} + 2k_0)x} \frac{1}{\sqrt{2\pi\delta}} e^{i2\sqrt{\pi}\hat{\phi}_{L_c}(x)} \frac{1}{\sqrt{2\pi\delta}} e^{-i2\sqrt{\pi}\hat{\phi}_{L_s}(x)} + h.c \\
&= \pi\delta e^{i(k_{F_c} - k_{F_s} + 2k_0)x} \hat{\psi}_{L_c}^\dagger(x) \hat{\psi}_{L_s}(x) + h.c \\
&= \frac{\delta}{2} e^{i(k_{F_c} - k_{F_s} + 2k_0)x} \int_{\substack{p \sim -k_{F_c} - k_0 \\ q \sim -k_{F_s} + k_0}} dp dq \hat{\psi}_{L_c}^\dagger(p) \hat{\psi}_{L_s}(q) + h.c. \tag{C.6}
\end{aligned}$$

For the case of the cosine term in equation 3.38, the extended calculation is:

$$\begin{aligned}
& \cos \left[\sqrt{2\pi} \left(\hat{\phi}_-(x) - \hat{\theta}_-(x) \right) + (k_{F_c} - k_{F_s} - 2k_0)x \right] \\
&= \frac{1}{2} e^{i(k_{F_c} - k_{F_s} - 2k_0)x} e^{i\sqrt{2\pi}(\hat{\phi}_-(x) - \hat{\theta}_-(x))} + h.c \\
&= \frac{1}{2} e^{i(k_{F_c} - k_{F_s} - 2k_0)x} e^{i\sqrt{\pi}(\hat{\phi}_c(x) - \hat{\phi}_s(x) - \hat{\theta}_c(x) + \hat{\theta}_s(x))} + h.c \\
&= \frac{1}{2} e^{i(k_{F_c} - k_{F_s} - 2k_0)x} e^{i2\sqrt{\pi}(\hat{\phi}_{R_c}(x) - \hat{\phi}_{R_s}(x))} + h.c \\
&= \frac{2\pi\delta}{2} e^{i(k_{F_c} - k_{F_s} - 2k_0)x} \frac{1}{\sqrt{2\pi\delta}} e^{-i2\sqrt{\pi}\hat{\phi}_{R_s}(x)} \frac{1}{\sqrt{2\pi\delta}} e^{i2\sqrt{\pi}\hat{\phi}_{R_c}(x)} + h.c \\
&= \pi\delta e^{i(k_{F_c} - k_{F_s} - 2k_0)x} \hat{\psi}_{R_s}^\dagger(x) \hat{\psi}_{R_c}(x) + h.c \\
&= \frac{\delta}{2} e^{i(k_{F_c} - k_{F_s} - 2k_0)x} \int_{\substack{p \sim k_{F_s} + k_0 \\ q \sim k_{F_c} - k_0}} dp dq \hat{\psi}_{R_s}^\dagger(p) \hat{\psi}_{R_c}(q) + h.c. \tag{C.7}
\end{aligned}$$

It can be noticed from the two last examples, that what it is done is to replace the exponential form of the cosine, taking care of the possible commutations of the fields, then re-arranging the terms to write the fermionic fields and then at last the Fourier expansion of the fields is replaced, so the term is written as the collection of the different excitations the term introduce, which is what is called a process. The same is done for the remaining cosine terms.

In the case of the cosine term in equation 3.39, the full calculation is:

$$\begin{aligned}
& \cos \left[\sqrt{2\pi} \left(\hat{\phi}_+(x) + \hat{\theta}_-(x) \right) + (k_{F_c} + k_{F_s} + 2k_0)x \right] \\
&= \frac{1}{2} e^{i(k_{F_c} + k_{F_s} + 2k_0)x} e^{i\sqrt{2\pi}(\hat{\phi}_+(x) + \hat{\theta}_-(x))} + h.c \\
&= \frac{1}{2} e^{i(k_{F_c} + k_{F_s} + 2k_0)x} e^{i\sqrt{\pi}(\hat{\phi}_c(x) + \hat{\phi}_s(x) + \hat{\theta}_c(x) - \hat{\theta}_s(x))} + h.c \\
&= \frac{1}{2} e^{i(k_{F_c} + k_{F_s} + 2k_0)x} e^{i2\sqrt{\pi}(\hat{\phi}_{L_c}(x) + \hat{\phi}_{R_s}(x))} + h.c \\
&= \frac{2\pi\delta}{2} e^{i(k_{F_c} + k_{F_s} + 2k_0)x} \frac{1}{\sqrt{2\pi\delta}} e^{i2\sqrt{\pi}\hat{\phi}_{L_c}(x)} \frac{1}{\sqrt{2\pi\delta}} e^{i2\sqrt{\pi}\hat{\phi}_{R_s}(x)} + h.c \\
&= \pi\delta e^{i(k_{F_c} + k_{F_s} + 2k_0)x} \hat{\psi}_{L_c}^\dagger(x) \hat{\psi}_{R_s}(x) + h.c \\
&= \frac{\delta}{2} e^{i(k_{F_c} + k_{F_s} + 2k_0)x} \int_{\substack{p \sim -k_{F_c} - k_0 \\ q \sim k_{F_s} + k_0}} dp dq \hat{\psi}_{L_c}^\dagger(p) \hat{\psi}_{R_s}(q) + h.c. \tag{C.8}
\end{aligned}$$

For the cosine term in equation 3.40, the calculation is done in the following way:

$$\begin{aligned}
& \cos \left[\sqrt{2\pi} \left(\hat{\phi}_+(x) - \hat{\theta}_-(x) \right) + (k_{F_c} + k_{F_s} - 2k_0)x \right] \\
&= \frac{1}{2} e^{i(k_{F_c} + k_{F_s} - 2k_0)x} e^{i\sqrt{2\pi}(\hat{\phi}_+(x) - \hat{\theta}_-(x))} + h.c \\
&= \frac{1}{2} e^{i(k_{F_c} + k_{F_s} - 2k_0)x} e^{i\sqrt{\pi}(\hat{\phi}_c(x) + \hat{\phi}_s(x) - \hat{\theta}_c(x) + \hat{\theta}_s(x))} + h.c \\
&= \frac{1}{2} e^{i(k_{F_c} + k_{F_s} - 2k_0)x} e^{i2\sqrt{\pi}(\hat{\phi}_{R_c}(x) + \hat{\phi}_{L_s}(x))} + h.c \\
&= \frac{2\pi\delta}{2} e^{i(k_{F_c} + k_{F_s} - 2k_0)x} \frac{1}{\sqrt{2\pi\delta}} e^{i2\sqrt{\pi}\hat{\phi}_{L_s}(x)} \frac{1}{\sqrt{2\pi\delta}} e^{i2\sqrt{\pi}\hat{\phi}_{R_c}(x)} + h.c \\
&= \pi\delta e^{i(k_{F_c} + k_{F_s} - 2k_0)x} \hat{\psi}_{L_s}^\dagger(x) \hat{\psi}_{R_c}(x) + h.c \\
&= \frac{\delta}{2} e^{i(k_{F_c} + k_{F_s} - 2k_0)x} \int_{\substack{p \sim -k_{F_s} + k_0 \\ q \sim k_{F_c} - k_0}} dp dq \hat{\psi}_{L_s}^\dagger(p) \hat{\psi}_{R_c}(q) + h.c. \tag{C.9}
\end{aligned}$$

At last, the cosine term in equation 3.41 is done as:

$$\begin{aligned}
& \cos \left[\sqrt{8\pi} \hat{\phi}_-(x) + (k_{F_c} - k_{F_s})x \right] \\
&= \frac{1}{2} e^{i(k_{F_c} - k_{F_s})x} e^{i\sqrt{8\pi}\hat{\phi}_-(x)} + h.c \\
&= \frac{1}{2} e^{i(k_{F_c} - k_{F_s})x} e^{i\sqrt{4\pi}(\hat{\phi}_c(x) - \hat{\phi}_s(x))} + h.c \\
&= \frac{1}{2} e^{i(k_{F_c} - k_{F_s})x} e^{i2\sqrt{\pi}(\hat{\phi}_{R_c}(x) + \hat{\phi}_{L_c}(x) - \hat{\phi}_{R_s}(x) - \hat{\phi}_{L_s}(x))} + h.c \\
&= 2\pi^2\delta^2 e^{i(k_{F_c} - k_{F_s})x} \hat{\psi}_{L_c}^\dagger(x) \hat{\psi}_{R_s}^\dagger(x) \hat{\psi}_{R_c}(x) \hat{\psi}_{L_s}(x) + h.c \\
&= \frac{\delta^2}{2} e^{i(k_{F_c} - k_{F_s})x} \int_{\substack{p \sim -k_{F_c} - k_0 \\ q \sim k_{F_s} + k_0 \\ l \sim k_{F_c} - k_0 \\ u \sim -k_{F_s} + k_0}} dp dq dl du \hat{\psi}_{L_c}^\dagger(p) \hat{\psi}_{R_s}^\dagger(q) \hat{\psi}_{R_c}(l) \hat{\psi}_{L_s}(u) + h.c. \tag{C.10}
\end{aligned}$$

C.4 Hamiltonian in the 0-band and 2-band basis

In the chapter 4 it is analyzed that for the correct introduction of the cosine terms in the correlations, another change of basis needs to be applied. The transformation in equation 4.2 is replaced in the Hamiltonian of equation 3.45, which results in:

$$\begin{aligned}
\hat{H} &= \frac{\nu_+}{2} \int dx \left[\frac{1}{K_+} \left(\partial_x \left(\frac{1}{\sqrt{2}} \hat{\phi}_\square(x) \right) \right)^2 + K_+ \left(\partial_x \left(\frac{1}{\sqrt{2}} \left(\hat{\theta}_\blacksquare(x) + 2\hat{\theta}_\square(x) \right) \right) \right)^2 \right] \\
&+ \frac{\nu_-}{2} \int dx \left[\frac{1}{K_-} \left(\partial_x \left(-\frac{1}{\sqrt{2}} \hat{\theta}_\blacksquare(x) \right) \right)^2 + K_- \left(\partial_x \left(\frac{1}{\sqrt{2}} \left(\hat{\phi}_\square(x) - 2\hat{\phi}_\blacksquare(x) \right) \right) \right)^2 \right] \\
&- g_B \int dx \cos \left[\sqrt{2\pi} \sqrt{2} \hat{\phi}_\blacksquare(x) \right] - g_U \int dx \cos \left[\sqrt{8\pi} \left(-\frac{1}{\sqrt{2}} \hat{\theta}_\blacksquare(x) \right) \right] \\
&= \frac{\nu_+}{2} \int dx \left[\frac{1}{2K_+} \left(\partial_x \hat{\phi}_\square(x) \right)^2 + \frac{K_+}{2} \left(\partial_x \hat{\theta}_\blacksquare(x) + 2\partial_x \hat{\theta}_\square(x) \right)^2 \right] \\
&+ \frac{\nu_-}{2} \int dx \left[\frac{1}{2K_-} \left(\partial_x \hat{\theta}_\blacksquare(x) \right)^2 + \frac{K_-}{2} \left(\partial_x \hat{\phi}_\square(x) - 2\partial_x \hat{\phi}_\blacksquare(x) \right)^2 \right] \\
&- g_B \int dx \cos \left[\sqrt{4\pi} \hat{\phi}_\blacksquare(x) \right] - g_U \int dx \cos \left[\sqrt{4\pi} \hat{\theta}_\blacksquare(x) \right] \\
&= \frac{\nu_+}{2} \int dx \left[\frac{1}{2K_+} \left(\partial_x \hat{\phi}_\square(x) \right)^2 + \frac{K_+}{2} \left(\partial_x \hat{\theta}_\blacksquare(x) \right)^2 \right. \\
&\quad \left. + 2K_+ \partial_x \hat{\theta}_\square(x) \partial_x \hat{\theta}_\blacksquare(x) + 2K_+ \left(\partial_x \hat{\theta}_\square(x) \right)^2 \right] \\
&+ \frac{\nu_-}{2} \int dx \left[\frac{1}{2K_-} \left(\partial_x \hat{\theta}_\blacksquare(x) \right)^2 + \frac{K_-}{2} \left(\partial_x \hat{\phi}_\square(x) \right)^2 \right. \\
&\quad \left. - 2K_- \partial_x \hat{\phi}_\square(x) \partial_x \hat{\phi}_\blacksquare(x) + 2K_- \left(\partial_x \hat{\phi}_\blacksquare(x) \right)^2 \right] \\
&- g_B \int dx \cos \left[\sqrt{4\pi} \hat{\phi}_\blacksquare(x) \right] - g_U \int dx \cos \left[\sqrt{4\pi} \hat{\theta}_\blacksquare(x) \right] \\
&= \int dx \left[\left(\frac{\nu_+}{4K_+} + \frac{\nu_- K_-}{4} \right) \left(\partial_x \hat{\phi}_\square(x) \right)^2 + \nu_+ K_+ \left(\partial_x \hat{\theta}_\square(x) \right)^2 \right] \\
&+ \int dx \left[\nu_- K_- \left(\partial_x \hat{\phi}_\blacksquare(x) \right)^2 + \left(\frac{\nu_+ K_+}{4} + \frac{\nu_-}{4K_-} \right) \left(\partial_x \hat{\theta}_\blacksquare(x) \right)^2 \right] \\
&+ \int dx \left[\nu_+ K_+ \partial_x \hat{\theta}_\square(x) \partial_x \hat{\theta}_\blacksquare(x) - \nu_- K_- \partial_x \hat{\phi}_\square(x) \partial_x \hat{\phi}_\blacksquare(x) \right] \\
&- g_B \int dx \cos \left[\sqrt{4\pi} \hat{\phi}_\blacksquare(x) \right] - g_U \int dx \cos \left[\sqrt{4\pi} \hat{\theta}_\blacksquare(x) \right]. \tag{C.11}
\end{aligned}$$

C.5 Correlations for dominant magnetic field

The results of the equations 4.23, 4.24, 4.25 and 4.26 are computed in this section of the appendix, which are the specific functional form of the relevant correlations for $g_B > g_U$. First, the corresponding bosonized operators are rewritten to replace the transformation of the equation 4.2, done as follows:

$$\begin{aligned}
\hat{\sigma}_{cs}^x(x) &= \frac{2}{\pi\delta} \cos \left[\sqrt{2\pi} \hat{\theta}_-(x) + 2k_0 x \right] \cos \left[\sqrt{2\pi} \hat{\phi}_+(x) + 2k_0 x \right] \\
&= \frac{1}{\pi\delta} \cos \left[\sqrt{2\pi} \left(\hat{\theta}_-(x) + \hat{\phi}_+(x) \right) + 4k_0 x \right] + \frac{1}{\pi\delta} \cos \left[\sqrt{2\pi} \left(\hat{\theta}_-(x) - \hat{\phi}_+(x) \right) \right] \\
&= \frac{1}{2\pi\delta} e^{i\sqrt{2\pi} \left(\frac{2}{\sqrt{2}} \hat{\phi}_\square(x) - \frac{2}{\sqrt{2}} \hat{\phi}_\blacksquare(x) \right)} e^{i4k_0 x} + \frac{1}{2\pi\delta} e^{-i\sqrt{2\pi} \left(\frac{2}{\sqrt{2}} \hat{\phi}_\square(x) - \frac{2}{\sqrt{2}} \hat{\phi}_\blacksquare(x) \right)} e^{-i4k_0 x} \\
&+ \frac{1}{2\pi\delta} e^{i\sqrt{2\pi} \left(\frac{1}{\sqrt{2}} \hat{\phi}_\square(x) - \frac{1}{\sqrt{2}} \hat{\phi}_\square(x) + \frac{2}{\sqrt{2}} \hat{\phi}_\blacksquare(x) \right)} + \frac{1}{2\pi\delta} e^{-i\sqrt{2\pi} \left(\frac{1}{\sqrt{2}} \hat{\phi}_\square(x) - \frac{1}{\sqrt{2}} \hat{\phi}_\square(x) + \frac{2}{\sqrt{2}} \hat{\phi}_\blacksquare(x) \right)} \\
&= \frac{1}{2\pi\delta} e^{i\sqrt{4\pi} \hat{\phi}_\square(x)} e^{-i\sqrt{4\pi} \hat{\phi}_\blacksquare(x)} e^{i4k_0 x} + \frac{1}{2\pi\delta} e^{-i\sqrt{4\pi} \hat{\phi}_\square(x)} e^{i\sqrt{4\pi} \hat{\phi}_\blacksquare(x)} e^{-i4k_0 x} \\
&+ \frac{1}{2\pi\delta} e^{i\sqrt{4\pi} \hat{\phi}_\blacksquare(x)} + \frac{1}{2\pi\delta} e^{-i\sqrt{4\pi} \hat{\phi}_\blacksquare(x)} \\
&= \frac{1}{2\pi\delta} e^{i\sqrt{4\pi} \hat{\phi}_\square(x)} e^{-i\sqrt{4\pi} \hat{\phi}_\blacksquare(x)} e^{i4k_0 x} + \frac{1}{2\pi\delta} e^{-i\sqrt{4\pi} \hat{\phi}_\square(x)} e^{i\sqrt{4\pi} \hat{\phi}_\blacksquare(x)} e^{-i4k_0 x} \\
&+ \frac{1}{\pi\delta} \cos \left[\sqrt{4\pi} \hat{\phi}_\blacksquare(x) \right]. \tag{C.12}
\end{aligned}$$

$$\begin{aligned}
\hat{O}_{HSS}(x) &= \frac{1}{2\pi\delta} e^{i\sqrt{2\pi} \hat{\theta}_+(x)} e^{i\sqrt{2\pi} \hat{\phi}_-(x)} = \frac{1}{2\pi\delta} e^{i\sqrt{2\pi} (\hat{\theta}_+(x) + \hat{\phi}_-(x))} \\
&= \frac{1}{2\pi\delta} e^{i\sqrt{2\pi} \left(\frac{1}{\sqrt{2}} \hat{\theta}_\blacksquare(x) + \frac{2}{\sqrt{2}} \hat{\theta}_\square(x) - \frac{1}{\sqrt{2}} \hat{\theta}_\blacksquare(x) \right)} \\
&= \frac{1}{2\pi\delta} e^{i\sqrt{4\pi} \hat{\theta}_\square(x)}, \tag{C.13}
\end{aligned}$$

where the operators $\hat{\sigma}_{cs}^y$ and $\hat{O}_{HTS}^z$ are not calculated explicitly since both planar helical spin density correlations are equal, and the spin-singlet and axial spin-triplet correlations are equal, according to the results in the appendix B. The calculation of the correlations are done as follows:

$$\begin{aligned}
\langle \hat{O}_{HSS}^\dagger(x) \hat{O}_{HSS}(y) \rangle &= \left\langle \frac{1}{2\pi\delta} e^{-i\sqrt{4\pi} \hat{\theta}_\square(x)} \frac{1}{2\pi\delta} e^{i\sqrt{4\pi} \hat{\theta}_\square(y)} \right\rangle \\
&= \frac{1}{(2\pi\delta)^2} \left\langle e^{-i\sqrt{4\pi} (\hat{\theta}_\square(x) - \hat{\theta}_\square(y))} \right\rangle \\
&= \frac{1}{(2\pi\delta)^2} e^{-2\pi \frac{1}{K_\square \pi} \ln \left(\frac{|x-y|}{\delta} \right)} \\
&= \frac{1}{(2\pi\delta)^2} \left(\frac{\delta}{|x-y|} \right)^{\frac{2}{K_\square}}, \tag{C.14}
\end{aligned}$$

$$\langle \hat{O}_{HTS}^{\dagger z}(x) \hat{O}_{HTS}^z(y) \rangle = \frac{1}{(2\pi\delta)^2} \left(\frac{\delta}{|x-y|} \right)^{\frac{2}{K_\square}}, \tag{C.15}$$

For the non-vanishing pairing correlations, using the identity in equation B.32 of the appendix B. For the planar helical spin densities, separating the terms and using the same identity, the

correlations are:

$$\begin{aligned}
\langle \hat{\sigma}_{cs}^x(x) \hat{\sigma}_{cs}^x(y) \rangle &= \langle | \left(\frac{1}{2\pi\delta} e^{i\sqrt{4\pi}\hat{\phi}_{\square}(x)} e^{-i\sqrt{4\pi}\hat{\phi}_{\blacksquare}(x)} e^{i4k_0x} + \frac{1}{2\pi\delta} e^{-i\sqrt{4\pi}\hat{\phi}_{\square}(x)} e^{i\sqrt{4\pi}\hat{\phi}_{\blacksquare}(x)} e^{-i4k_0x} \right. \\
&\quad \left. + \frac{1}{\pi\delta} \cos[\sqrt{4\pi}\hat{\phi}_{\blacksquare}(x)] \right) \\
&\quad \left(\frac{1}{2\pi\delta} e^{i\sqrt{4\pi}\hat{\phi}_{\square}(y)} e^{-i\sqrt{4\pi}\hat{\phi}_{\blacksquare}(y)} e^{i4k_0y} + \frac{1}{2\pi\delta} e^{-i\sqrt{4\pi}\hat{\phi}_{\square}(y)} e^{i\sqrt{4\pi}\hat{\phi}_{\blacksquare}(y)} e^{-i4k_0y} \right. \\
&\quad \left. + \frac{1}{\pi\delta} \cos[\sqrt{4\pi}\hat{\phi}_{\blacksquare}(y)] \right) | \rangle \\
&= \frac{e^{i4k_0(x-y)}}{(2\pi\delta)^2} \langle e^{i\sqrt{4\pi}\hat{\phi}_{\square}(x)} e^{-i\sqrt{4\pi}\hat{\phi}_{\blacksquare}(x)} e^{-i\sqrt{4\pi}\hat{\phi}_{\square}(y)} e^{i\sqrt{4\pi}\hat{\phi}_{\blacksquare}(y)} \rangle \\
&\quad + \frac{e^{-i4k_0(x-y)}}{(2\pi\delta)^2} \langle e^{-i\sqrt{4\pi}\hat{\phi}_{\square}(x)} e^{i\sqrt{4\pi}\hat{\phi}_{\blacksquare}(x)} e^{i\sqrt{4\pi}\hat{\phi}_{\square}(y)} e^{-i\sqrt{4\pi}\hat{\phi}_{\blacksquare}(y)} \rangle \\
&\quad + \frac{1}{(\pi\delta)^2} \langle \cos[\sqrt{4\pi}\hat{\phi}_{\blacksquare}(x)] \cos[\sqrt{4\pi}\hat{\phi}_{\blacksquare}(y)] \rangle \\
&= \frac{A^2 e^{i4k_0(x-y)}}{(2\pi\delta)^2} \langle e^{i\sqrt{4\pi}(\hat{\phi}_{\square}(x) - \hat{\phi}_{\square}(y))} \rangle + \frac{A^2 e^{-i4k_0(x-y)}}{(2\pi\delta)^2} \langle e^{-i\sqrt{4\pi}(\hat{\phi}_{\square}(x) - \hat{\phi}_{\square}(y))} \rangle \\
&\quad + \frac{A^2}{(\pi\delta)^2} \\
&= \frac{2A^2}{(2\pi\delta)^2} \cos[4k_0(x-y)] e^{-2\pi\langle(\hat{\phi}_{\square}(x) - \hat{\phi}_{\square}(y))^2\rangle} + \left(\frac{A}{2\pi\delta}\right)^2 \\
&= \frac{2A^2}{(2\pi\delta)^2} \cos[4k_0(x-y)] e^{-2\pi\frac{K_{\square}}{\pi} \ln\left(\frac{|x-y|}{\delta}\right)} + \left(\frac{A}{2\pi\delta}\right)^2 \\
&= \frac{2A^2}{(2\pi\delta)^2} \cos[4k_0(x-y)] \left(\frac{\delta}{|x-y|}\right)^{2K_{\square}} + \left(\frac{A}{2\pi\delta}\right)^2, \\
\langle \hat{\sigma}_{cs}^y(x) \hat{\sigma}_{cs}^y(y) \rangle &= \frac{2A^2}{(2\pi\delta)^2} \cos[4k_0(x-y)] \left(\frac{\delta}{|x-y|}\right)^{2K_{\square}} + \left(\frac{A}{2\pi\delta}\right)^2. \tag{C.16}
\end{aligned}$$

C.6 Edge points of the Majorana fermions in the wire

To calculate the edge point of the topological regions one must know that the Zeeman coupling split the bands vertically in $B/2$, so the gap is between the energies $[\mu_{\alpha} - B/2, \mu_{\alpha} + B/2]$ according to the figure 4.5. Equating the upper and lower energy to the space-varying chemical potential, the points are:

$$\begin{aligned}
\mu - \frac{m\omega^2 x^2}{2} &= \frac{1}{2} m\alpha^2 \pm \frac{B}{2} \\
m\omega^2 x^2 &= 2\mu - m\alpha^2 \mp B \\
x &= \pm \sqrt{\frac{2\mu - m\alpha^2 \mp B}{m\omega^2}}, \tag{C.17}
\end{aligned}$$

where the four different solutions are illustrated in figure 4.5 and presented in equations 4.30 and 4.31.

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